Supporting Information

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SI Text

The process $\mathbf{x}(s)$ defined in Eq. [12] is a Markov process. The equation of motion for the probability density function (pdf) $P_M(x,s|x_0,0)$ is the regular Fokker–Planck equation, i.e., Eq. [10] with $\alpha = 1$. The solution for the initial condition $P_M(x,0|x_0,0) = \delta(x - x_0)$ is then obtained using the eigenfunctions and eigenvalues of the Fokker–Planck operator $L_{\mathrm{FP}}\phi_n(x) = -\gamma_n \phi_n(x)$ ($L_{\mathrm{FP}} = -(\partial/\partial x)[F(x)/(k_B T)] + \partial^2/\partial x^2$) (1). A confining potential causes a discrete eigenvalue spectrum $0 < \gamma_1 < \gamma_2 \ldots$ and a stationary solution $L_{\mathrm{FP}}\phi_0(x) = 0$. The pdf $P_M(x,s|x_0,0)$ is (1)

$$P_M(x,s|x_0,0) = \sum_{n=0}^{\infty} e^{-\gamma_n K s}\hat{\phi}_n(x_0)\phi_n(x), \qquad \text{[S1]}$$

with coefficients $\hat{\phi}_n(x_0) = e^{U(x_0)/k_B T}\phi_n(x_0)$, and the orthonormality relation $\int_{-\infty}^{\infty} e^{U(x)/k_B T}\phi_n\phi_m dx = \delta_{nm}$. Orthonormality relation $\int_{-\infty}^{\infty} e^{\Phi(x)}\phi_n\phi_m dx = \delta_{nm}$ ($\Phi(x) = \ln(D) - \frac{1}{D}\int F(x')dx'$) $P_M(x_2,s|x_1,0)$ is obtained from Eq. **S1** by substituting $x \to x_2$ and $x_0 \to x_1$. The Laplace transform of $P_M(x,s|x_0,0)$, $\tilde{P}_M(x,\lambda|x_0,0) = \int_0^{\infty} e^{-s\lambda}P_M(x,s|x_0,0)ds$, is given by

$$\tilde{P}_M(x,\lambda|x_0,0) = \frac{e^{-U(x)/k_B T}}{\lambda Z} + \sum_{n=1}^{\infty}\frac{\hat{\phi}_n(x_0)\phi_n(x)}{\lambda + \gamma_n K}, \qquad \text{[S2]}$$

where we used the Boltzmann equilibrium pdf for the ground state solution of Fokker–Planck equation, and $Z = \int_{-\infty}^{\infty} e^{-U(x)/k_B T}dx$ is the normalizing partition function. We are now ready to compute the correlation function (CF) $\langle x(t_1)x(t_2)\rangle$. Using the definition for the Laplace transform of the two-time CF

$$\langle x(\lambda_2)x(\lambda_1)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_2,\lambda_2;x_1,\lambda_1)dx_1 dx_2, \qquad \text{[S3]}$$

and applying Eq. **20** and Eq. **S2**, we obtain

$$\begin{aligned}
\langle x(\lambda_2)x(\lambda_1)\rangle = {} & \frac{\lambda_1^\alpha - \Lambda^\alpha + \lambda_2^\alpha}{\lambda_1\lambda_2}\left[\frac{\langle x^2\rangle_B}{\Lambda^\alpha} + \sum_{n_1=1}^{\infty}\frac{\hat{\phi}_{n_1}(x_0)C_{n_1}^{(1)}}{\Lambda^\alpha + \gamma_{n_1}K_\alpha}\right] \\
& + \frac{(\lambda_2^\alpha)(\Lambda^\alpha - \lambda_2^\alpha)}{\lambda_1\lambda_2}\left[\frac{\langle x\rangle_B^2}{\lambda_2^\alpha\Lambda^\alpha} + \frac{\langle x\rangle_B}{\lambda_2^\alpha}\sum_{n_2=1}^{\infty}\frac{\hat{\phi}_{n_2}(x_0)C_{n_2}^{(2)}}{\Lambda^\alpha + \gamma_{n_2}K_\alpha}\right. \\
& + \sum_{n_3=1}^{\infty}\frac{C_{n_3}^{(3)}C_{n_3}^{(2)}}{\Lambda^\alpha(\lambda_2^\alpha + \gamma_{n_3}K_\alpha)} \\
& \left. + \sum_{n_4=1}^{\infty}\sum_{n_5=1}^{\infty}\frac{C_{n_4,n_5}^{(4)}C_{n_4}^{(2)}\hat{\phi}_{n_5}(x_0)}{(\lambda_2^\alpha + \gamma_{n_4}K_\alpha)(\Lambda^\alpha + \gamma_{n_5}K_\alpha)}\right] \\
& + \frac{(\lambda_1^\alpha)(\Lambda^\alpha - \lambda_1^\alpha)}{\lambda_1\lambda_2}\left[\frac{\langle x\rangle_B^2}{\lambda_1^\alpha\Lambda^\alpha} + \frac{\langle x\rangle_B}{\lambda_1^\alpha}\sum_{n_2'=1}^{\infty}\frac{\hat{\phi}_{n_2'}(x_0)C_{n_2'}^{(2)}}{\Lambda^\alpha + \gamma_{n_2'}K_\alpha}\right. \\
& + \sum_{n_3'=1}^{\infty}\frac{C_{n_3'}^{(3)}C_{n_3'}^{(2)}}{\Lambda^\alpha(\lambda_1^\alpha + \gamma_{n_3'}K_\alpha)} \\
& \left. + \sum_{n_4'=1}^{\infty}\sum_{n_5'=1}^{\infty}\frac{C_{n_4',n_5'}^{(4)}C_{n_4'}^{(2)}\hat{\phi}_{n_5'}(x_0)}{(\lambda_1^\alpha + \gamma_{n_4'}K_\alpha)(\Lambda^\alpha + \gamma_{n_5'}K_\alpha)}\right],
\end{aligned} \qquad \text{[S4]}$$

where $\Lambda \equiv \lambda_1 + \lambda_2$ and

$$\begin{aligned}
C_n^{(1)} &= \int_{-\infty}^{\infty} x^2\phi_n(x)dx \qquad C_n^{(2)} = \int_{-\infty}^{\infty} x\phi_n(x)dx \\
C_n^{(3)} &= \frac{1}{Z}\int_{-\infty}^{\infty} x\hat{\phi}_n(x)\exp[-U(x)/k_B T]dx \\
C_{n,n'}^{(4)} &= \int_{-\infty}^{\infty} x\hat{\phi}_n(x)\phi_{n'}(x)dx.
\end{aligned} \qquad \text{[S5]}$$

$C_n^{(3)} = \int_{-\infty}^{\infty} x\tilde{\phi}_n(x)\exp[-U(x)/k_B T]/Z dx$, and the symbol $\langle\ldots\rangle_B$ denotes an average over the Boltzmann distribution, e.g.,

$$\langle x^2\rangle_B = \frac{1}{Z}\int_{-\infty}^{\infty} x^2\exp[-U(x)/k_B T]dx. \qquad \text{[S6]}$$

All terms of Eq. **S4** that are γ_n dependent ($n > 0$) are equivalent to the transient part of the CF for a normally diffusing particle $\alpha \to 1$. Although the γ_n dependent components could be inverted into time-domain explicitly, for our purpose here it suffices to use Tauberian theorems (2) for $\lambda_1 \to 0$ and $\lambda_2 \to 0$, in the limit $\lambda_1^\alpha \ll \gamma_n K_\alpha$ and $\lambda_2^\alpha \ll \gamma_n K_\alpha$ the leading order for the power series of the transient part is of the form $\lambda^{\alpha-1}$ and therefore all transient terms decay to zero at least as $t^{-\alpha}$ for $t_2 \gg 1/(K_\alpha\gamma_1)^{1/\alpha}$, $t_1 \gg 1/(K_\alpha\gamma_1)^{1/\alpha}$, where γ_1 is the smallest nonzero eigenvalue. The remaining part

$$\frac{\lambda_1^\alpha - \Lambda^\alpha + \lambda_2^\alpha}{\lambda_1\lambda_2}\frac{\langle x^2\rangle_B}{\Lambda^\alpha} + \frac{(\lambda_2^\alpha)(\Lambda^\alpha - \lambda_2^\alpha)}{\lambda_1\lambda_2}\frac{\langle x\rangle_B^2}{\lambda_2^\alpha\Lambda^\alpha} + \frac{(\lambda_1^\alpha)(\Lambda^\alpha - \lambda_1^\alpha)}{\lambda_1\lambda_2}\frac{\langle x\rangle_B^2}{\lambda_1^\alpha\Lambda^\alpha} \qquad \text{[S7]}$$

is equivalent to the stationary behavior of the normally diffusing particle (i.e., independent of all the nonzero eigenvalues of L_{FP}) and depends only on equilibrium quantities. **S7** is transformed into the time domain using the Laplace inversion

$$\mathcal{L}^{-1}\{1/(\lambda_2\lambda_1^{1-\alpha}\Lambda^\alpha)\} = \theta(t_2 - t_1) + \theta(t_1 - t_2)B(t_2/t_1,\alpha,1-\alpha)/[\Gamma(\alpha)\Gamma(1-\alpha)], \qquad \text{[S8]}$$

where $B(z,a,b) = \int_0^z y^{a-1}(1-y)^{b-1}dy$ is the incomplete beta function (3). For the two -time CF we obtain

$$\langle x(t_2)x(t_1)\rangle \sim (\langle x^2\rangle_B - \langle x\rangle_B^2)\frac{B(t_1/t_2,\alpha,1-\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)} + \langle x\rangle_B^2 \qquad \text{[S9]}$$

in the limit $t_2 \geq t_1 \gg (1/K_\alpha\gamma_1)^{1/\alpha}$.

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