SUPPLEMENTAL FILE

5 MATHEMATICAL NOTATION

In this paper, we consider real variables. We use $\mathbb{R}$ to denote the set of real numbers. Scalars are denoted by lower case letters, e.g., s, t. Vectors in $\mathbb{R}^n$ are denoted by either bold letters and numbers, or lower-case Greek letters, e.g., $\mathbf{1}, \mathbf{x}, \pi$, where $\mathbf{1}$ denotes a vector all of whose components are equal to one. $\mathbf{x}^t$ denotes the transpose of the vector $\mathbf{x}$. The notation x_i refers to the ith component of the vector $\mathbf{x}$. Matrices in $\mathbb{R}^{m\times n}$ are denoted by either capital letters or upper-case Greek letters, e.g., C, P, Λ. I stands for the identity matrix. If $P \in \mathbb{R}^{m\times n}$, then $vec(P)$ transforms P into an nm-dimensional vector by stacking the columns. The equality and inequality symbols, $=, \leq$ and $\geq$ denote component-wise equality and inequality, respectively, for arrays of the same size. For example, if C is an $m \times n$ matrix, then $C \geq 0$ denotes the mn inequalities: each element of the matrix C is nonnegative. The curled inequality symbols, $\preceq, \prec, \succeq, \succ$, denote generalized matrix inequalities associated with the positive semi-definite cone. That is, if $A, B \in \mathbb{R}^{n\times n}$, then $A \succeq B$ (resp., $A \succ B$) means that $A - B$ is positive-semi-definite (resp., positive definite); and $A \preceq B$ (resp., $A \prec B$) means that $A - B$ is negative-semi-definite (resp., negative definite). We recall that a matrix $A \in \mathbb{R}^{n\times n}$ is called positive-semi-definite (resp., positive definite) if $\mathbf{x}^t A\mathbf{x} \geq 0$ (resp., $\mathbf{x}^t A\mathbf{x} > 0$) for all $\mathbf{x} \in \mathbb{R}^n$ (resp., $\mathbf{x} \neq 0$). If $-A$ is positive semi-definite (resp., positive definite), then A is called negative semi-definite (resp., negative definite).

PROOF OF PROPOSITION 1.

$$
\begin{aligned}
& \pi_d^t = \pi_0^t (I - CZ_0)^{-1} \\
\iff\ & \pi_d^t (I - CZ_0) - \pi_0^t = 0 \\
\iff\ & [\pi_d^t (I - CZ_0) - \pi_0^t] Z_0^{-1} = 0 && (27) \\
\iff\ & \pi_d^t (I - P_0 + P_0^*) - \pi_d^t C - \pi_0^t (I - P_0 + P_0^*) = 0 \\
\iff\ & (\pi_d^t - \pi_d^t P_0 + \pi_0^t) - \pi_d^t C - (\pi_0^t - \pi_0^t + \pi_0^t) && (28) \\
\iff\ & \pi_d^t = \pi_d^t (P_0 + C),
\end{aligned}
$$

where Eq. (27) follows from the fact that Z_0 is invertible, and Eq. (28) follows from the properties: $\pi_d^t P_0^* = \pi_d^t \mathbf{1} \pi_0^t = \pi_0^t$; $\pi_0^t P_0 = \pi_0^t$; and $\pi_0^t P_0^* = \pi_0^t$.

PROOF OF PROPOSITION 2. It is straightforward to check that the perturbation matrix $C = (\mathbf{1}\pi_d^t - P_0) \in \mathcal{D}$. That is we have $(i)\ \pi_d^t = \pi_d^t (P_0 + C)$; $(ii)\ C\mathbf{1} = \mathbf{0}$; $(iii)\ P_0 + C \geq 0$. Moreover, the perturbed matrix $P_0 + C = \mathbf{1}\pi_d^t$ is a stochastic matrix with rank one. Therefore, it has a simple eigenvalue 1 corresponding to eigenvector $\mathbf{1}$, and eigenvalue 0 with multiplicity $n - 1$. Hence, its SLEM = 0.

PROOF OF PROPOSITION 3. For any vector $\mathbf{f}$, we introduce its unique direct sum decomposition $\mathbf{f} = \alpha_f \mathbf{1} + \mathbf{f}^\perp$, where $\alpha_f = \pi_d^t \mathbf{f}$ and $\mathbf{f}^\perp \perp \pi_d$. It is easy to check that $\mathbf{f}^\perp$ is proportional to $\mathbf{1}$ if and only if $\mathbf{f}^\perp = \mathbf{0}$.

Let $\psi = \alpha_\psi \mathbf{1} + \psi^\perp$ be a non-trivial eigenvector (i.e., $\psi^\perp \neq \mathbf{0}$) of P_E^* with eigenvalue μ. We will look for a vector ϕ, in the form $\phi = \psi + c\mathbf{1}$, that satisfies $P(s)\phi = (1 - s)\mu\phi$. We have

$$
\begin{aligned}
P(s)\phi &= P(s)\psi + c\mathbf{1} && (29) \\
&= (1 - s)\mu\psi + s\alpha_\psi \mathbf{1} + c\mathbf{1} && (30) \\
&= (1 - s)\mu\phi + (s\alpha_\psi + c - (1 - s)\mu c)\mathbf{1}, && (31)
\end{aligned}
$$

where Eq. (29) follows from the fact that $P(s)$ is stochastic, i.e., $P(s)\mathbf{1} = \mathbf{1}$, and Eq. (30) is obtained by replacing $P(s)$ by its expression in Eq. (15). Therefore, if we chose $c = \frac{s\alpha_\psi}{(1-s)\mu - 1}$, we obtain $P(s)\phi = (1 - s)\mu\phi$.

Let now $\phi = \alpha_\phi \mathbf{1} + \phi^\perp$ be a non trivial eigenvector of $P(s)$ with eigenvalue λ. In particular, $\phi^\perp \neq \mathbf{0}$. We first show that $\lambda \neq (1 - s)$. From Eq. (15), we have

$$
\begin{aligned}
P(s)\phi &= (1 - s)P_E^*\phi + s\alpha_\phi \mathbf{1} && (32) \\
&= (1 - s)P_E^*\phi^\perp + (1 - s)\alpha_\phi \mathbf{1} + s\alpha_\phi \mathbf{1}. && (33)
\end{aligned}
$$

On the other hand, if $\lambda = 1 - s$, then we would have

$$
\begin{aligned}
P(s)\phi &= (1 - s)\phi && (34) \\
&= (1 - s)\alpha_\phi \mathbf{1} + (1 - s)\phi^\perp. && (35)
\end{aligned}
$$

By equating Eqs. (33) and (35), we obtain

$$
P_E^*\phi^\perp = \phi^\perp - \frac{s\alpha_\phi}{1 - s}\mathbf{1}. \quad (36)
$$

It follows that, for any positive integer j we have

$$
P_E^{*j}\phi^\perp = P_E^{*(j-1)}\phi^\perp - \frac{s\alpha_\phi}{1 - s}\mathbf{1}. \quad (37)
$$

Taking the limit as $j \longrightarrow \infty$, and because P_E^* is ergodic, we get

$$
P_E^{*\infty}\phi^\perp = P_E^{*\infty}\phi^\perp - \frac{s\alpha_\phi}{1 - s}\mathbf{1}. \quad (38)
$$

Thus, $s\alpha_\phi = 0$, which implies, from Eq. (36), that $P_E^*\phi^\perp = \phi^\perp$. Hence, $\phi^\perp$ is an eigenvector of P_E^* corresponding to eigenvalue 1. Therefore, $\phi^\perp$ must be proportional to $\mathbf{1}$. We recall that $\phi^\perp$ is proportional to $\mathbf{1}$ if and only if $\phi^\perp = \mathbf{0}$. This results in a contradiction because of the fact that $\phi^\perp \neq \mathbf{0}$. Therefore, we conclude that $\lambda \neq 1 - s$.

Now, we consider $\lambda \neq 1 - s$, we will find ψ in the form $\psi = \phi + c\mathbf{1}$, that satisfies $P_E^*\psi = \frac{\lambda}{1-s}\psi$. We have

$$
\begin{aligned}
P_E^*\psi &= P_E^*\phi + c\mathbf{1} && (39) \\
&= \frac{\lambda}{1 - s}\phi - \frac{s\alpha_\phi}{1 - s}\mathbf{1} + c\mathbf{1} && (40) \\
&= \frac{\lambda}{1 - s}\psi + (-\frac{\lambda}{1 - s}c - \frac{s\alpha_\phi}{1 - s} + c)\mathbf{1}, && (41)
\end{aligned}
$$

where Eq. (39) follows from the stochasticity of P_E^* and Eq. (40) is obtained by replacing P_E^* by its expression in Eq. (15) and using the fact that ϕ is an eigenvector of $P(s)$ with eigenvalue λ. Finally, Eq. (41) follows by replacing $\phi = \psi - c\mathbf{1}$. Therefore, if we chose $c = \frac{s\alpha_\phi}{1-s-\lambda}$, we obtain $P_E^*\psi = \frac{\lambda}{1-s}\psi$.

PROOF OF PROPOSITION 4. $\|C(s)\|_2$ is a convex function in s, which reaches its minimum at $s = 0$. Therefore, it must be increasing for $s \geq 0$ (Boyd and Vandenberghe, 2003).

We also provide an alternative proof as follows: Let $A = P_E^* - P_0$ and $B = \mathbf{1}\pi_d^t - P_E^*$. Then, from Eq. (16),

$$C(s) = A + Bs. \tag{42}$$

By construction, we have for all $s \geq 0$,

$$\|C(s)\|_2 \geq \|P_E^* - P_0\|_2 = \|C(0)\|_2 = \|A\|_2 \Longleftrightarrow \tag{43}$$
$$\max_{\mathbf{x}:\|\mathbf{x}\|=1} < (A+sB)^t(A+sB)\mathbf{x}, \mathbf{x} > \geq \max_{\mathbf{x}:\|\mathbf{x}\|=1} < A^tA\mathbf{x}, \mathbf{x} >,$$

where the right hand side equivalence follows from the definition of the spectral norm given in Eq. (8). Let $\mathbf{x}_s$ be such that $\|\mathbf{x}_s\| = 1$ and

$$\max_{\mathbf{x}:\|\mathbf{x}\|=1} < (A+sB)^t(A+sB)\mathbf{x}, \mathbf{x} > = < (A+sB)^t(A+sB)\mathbf{x}_s, \mathbf{x}_s > . \tag{44}$$

Then,

$$< (A+sB)^t(A+sB)\mathbf{x}_s, \mathbf{x}_s > \ \geq \ < A^tA\mathbf{x}_s, \mathbf{x}_s >, \tag{45}$$

which means

$$< (A^tB + B^tA + sB^tB)\mathbf{x}_s, \mathbf{x}_s > \geq 0. \tag{46}$$

Let $\tilde{s} \geq s$. We need to show that

$$\max_{\mathbf{x}:\|\mathbf{x}\|=1} < (A+\tilde{s}B)^t(A+\tilde{s}B)\mathbf{x}, \mathbf{x} > \geq < (A+sB)^t(A+sB)\mathbf{x}_s, \mathbf{x}_s > . \tag{47}$$

It is sufficient to show that

$$< (A+\tilde{s}B)^t(A+\tilde{s}B)\mathbf{x}_s, \mathbf{x}_s > \geq < (A+sB)^t(A+sB)\mathbf{x}_s, \mathbf{x}_s > . \tag{48}$$

But,

$$\begin{aligned} &< (A+\tilde{s}B)^t(A+\tilde{s}B)\mathbf{x}_s, \mathbf{x}_s > - < (A+sB)^t(A+sB)\mathbf{x}_s, \mathbf{x}_s > \\ &= (\tilde{s} - s) < (A^tB + B^tA + sB^tB + \tilde{s}B^tB)\mathbf{x}_s, \mathbf{x}_s >, \end{aligned}$$

which is positive because of Eq. (46).

PROOF OF PROPOSITION 5. For $0 \leq s \leq 1$, we have the following three properties

$$\pi_d(s)^tQ(s) = \pi_d(s)^t \tag{49}$$
$$\pi_d(s)\mathbf{1} = 1 \tag{50}$$
$$\pi_d(s) \geq 0. \tag{51}$$

Because $Q(s)$ is ergodic, we know that such $\pi_d(s)$ exists and is unique. Let

$$\phi = \pi_d(s) - ((1-s)\pi_0 + s\pi_d). \tag{52}$$

Then, we have

$$\begin{aligned} \phi^tQ(s) &= \pi_d(s)^t - (1-s)^2\pi_0^t - \qquad (53) \\ &\quad s(1-s)\pi_d^tP_0 - s(1-s)\pi_d^t - s^2\pi_d^t \\ &= \phi^t + s(1-s)(\pi_0^t - \pi_d^tP_0) \qquad (54) \\ &= \phi^t + s(1-s)(\pi_0^t - \pi_d^t)P_0, \qquad (55) \end{aligned}$$

where Eq. (55) follows from the fact that $\pi_0^tP_0 = \pi_0^t$. Next, we notice that $\phi^t\mathbf{1} = 0$. Thus, from Eq. (20), we obtain

$$\phi^tQ(s) = (1-s)\phi^tP_0. \tag{56}$$

By equating Eqs. (55) and (56), we obtain

$$\phi^t[I - (1-s)P_0] = s(1-s)(\pi_d^t - \pi_0^t)P_0. \tag{57}$$

Observe that for $s > 0$, 1 is not an eigenvalue of $(1-s)P_0$. Hence, $I - (1-s)P_0$ is invertible, and we have

$$\phi = s(1-s)[I - (1-s)P_0^t]^{-1}P_0^t(\pi_d - \pi_0). \tag{58}$$

From Eq. (52), we have

$$\pi_d(s) - \pi_d = \phi + (1-s)(\pi_0 - \pi_d). \tag{59}$$

Replacing ϕ by its expression in Eq. (58), Eq. (59) can be written as

$$\pi_d(s) - \pi_d = (1-s)\left(I - s(I-(1-s)P_0^t)^{-1}P_0^t\right)(\pi_0 - \pi_d). \tag{60}$$

That is, by factoring by $(I - (1-s)P_0^t)^{-1}$,

$$\pi_d(s) - \pi_d = (1-s)(I-(1-s)P_0^t)^{-1}(I - P_0^t)(\pi_0 - \pi_d). \tag{61}$$

If P_0 is symmetric, then by the spectral theorem we have $\|P_0\|_2 = \lambda_{\max}(P_0) = 1$, and by the triangle inequality,

$$\|(I-(1-s)P_0^t)^{-1}(I - P_0^t)\|_2 \leq \frac{2}{2-s},$$

and thus

$$\|\pi_d(s) - \pi_d\| \leq \frac{2(1-s)}{2-s}\|\pi_0 - \pi_d\|.$$

In the case of a non-symmetric matrix P_0, we use geometric progression:

$$[I - (1-s)P_0^t]^{-1} = \sum_{k=0}^{\infty}(1-s)^kP_0^t. \tag{62}$$

We note that the last series is convergent for any $0 < s \leq 1$ because P_0^k has a limit as $k \to \infty$. Equation (60) becomes then

$$\pi_d(s) - \pi_d = (1-s)\left(I - s\sum_{k=0}^{\infty}(1-s)^k(P_0^t)^{k+1}\right)(\pi_0 - \pi_d). \tag{63}$$

By noting that $\sup_{k\geq 1}\|P_0^k\|_2 = \sup_{k\geq 1}\|(P_0^t)^k\|_2$ is finite, we have the desired upper bound on $\|\pi_d(s) - \pi_d\|$.