Appendix I: Estimation and Inference of the Concordance Measure

Let $\{A_i(t) = I(Y_i \geq t; t \geq 0)\}$ be the at-risk process, and $\{B_i(t) = I(Y_i \leq t, \delta_i = 0); t \geq 0\}$ be the counting process for the censoring time. Let $\Lambda_C(t) = -\log S_C(t)$ be the cumulative hazard function for the censoring time. Define $M_i(t) = B_i(t) - \int_0^t A_i(s) d\Lambda_C(s)$. Let $W_i^j = (Z_i^j, Y_i, \delta_i)$.

The consistency of $\hat{\tau}$ follows from

$$\begin{aligned} n^{-2} \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\delta_i}{S_C^2(Y_i)} I(Z_i^j < Z_k^j, Y_i < Y_k) &\xrightarrow{p} E\{I(Z_i^j < Z_k^j, Y_i < Y_k)\}, \\ n^{-2} \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\delta_i}{S_C^2(Y_i)} I(Y_i < Y_k) &\xrightarrow{p} E\{I(Y_i < Y_k)\}, \\ \sup_u |\hat{S}_C(u) - S_C(u)| &\rightarrow 0. \end{aligned}$$

Define

$$\varphi_i^j = h_{i1}^j + h_{i2}^j,$$

$$h_{i1}^j = \int \frac{\left\{ q_1^j(u) - q_2(u)\tau^j \right\}}{E\left\{A_i(u)\right\} E\left\{I(T_i < T_k)\right\}} dM_i(u),$$

$$h_{i2}^j = \frac{q_3^j(W_i^j)}{E\left\{I(T_i < T_j)\right\}} - \frac{\tau^j q_4(W_i^j)}{E\left\{I(T_i < T_j)\right\}},$$

$$q_1^j(u) = E\left\{ \frac{2\delta_i I(Z_i^j < Z_k^j, Y_i < Y_k)}{S_C^2(Y_i)} I(Y_i \geq u) \right\},$$

$$q_2(u) = E\left\{ \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{2\delta_i I(Y_i < Y_k)}{S_C^2(Y_i)} I(Y_i \geq u) \right\},$$

$$\begin{aligned} q_3^j(W_i^j) &= E\Bigg\{ \frac{\delta_i}{S_C^2(Y_i)} I(Z_i^j < Z_k^j, Y_i < Y_k) + \frac{\delta_k}{S_C^2(Y_k)} I(Z_i^j < Z_k^j, Y_k < Y_i) \\ &\qquad - 2E\left\{ I(Z_i^j < Z_k^j, T_i < T_k) \right\} |W_i^j, i \neq k \Bigg\}, \\ q_4(W_i^j) &= E\Bigg\{ \frac{\delta_i}{S_C^2(Y_i)} I(Y_i < Y_k) + \frac{\delta_j}{S_C^2(Y_k)} I(Y_k < Y_i) - 2E\left\{I(T_i < T_k)\right\} |W_i^j, i \neq k \Bigg\}. \end{aligned}$$

Note that $n^{1/2}(\hat{\tau}^j - \tau^j)$ can be written as $I_1^j + I_2^j$, where $I_s^j = I_{s1}^j + I_{s2}^j$ for $s = 1, 2$, and

$$\begin{aligned}
I_{11}^j &= \frac{n^{-3/2}\sum_i \sum_k \frac{\delta_i}{\hat{S}^2(Y_i)} I(Z_i^j < Z_k^j, Y_i < Y_k)}{n^{-2}\sum_i \sum_k \frac{\delta_i}{\hat{S}^2(Y_i)} I(Y_i < Y_k)} - \frac{n^{-3/2}\sum_i \sum_k \frac{\delta_i}{\hat{S}^2(Y_i)} I(Z_i^j < Z_k^j, Y_i < Y_k)}{n^{-2}\sum_i \sum_k \frac{\delta_i}{S^2(Y_i)} I(Y_i < Y_k)}, \\
I_{12}^j &= \frac{n^{-3/2}\sum_i \sum_k \frac{\delta_i}{\hat{S}^2(Y_i)} I(Z_i^j < Z_k^j, Y_i < Y_j)}{n^{-2}\sum_i \sum_k \frac{\delta_i}{S^2(Y_i)} I(Y_i < Y_j)} - \frac{n^{-3/2}\sum_i \sum_k \frac{\delta_i}{S^2(Y_i)} I(Z_i^j < Z_k^j, Y_i < Y_j)}{n^{-2}\sum_i \sum_k \frac{\delta_i}{S^2(Y_i)} I(Y_i < Y_k)}, \\
I_{21}^j &= \frac{n^{-3/2}\sum_i \sum_k \frac{\delta_i}{S^2(Y_i)} I(Z_i^j < Z_k^j, Y_i < Y_k)}{n^{-2}\sum_i \sum_k \frac{\delta_i}{S^2(Y_i)} I(Y_i < Y_k)} - \frac{n^{1/2} P(Z_i^j < Z_k^j, T_i < T_k)}{n^{-2}\sum_i \sum_k \frac{\delta_i}{S^2(Y_i)} I(Y_i < Y_k)}, \\
I_{22}^j &= \frac{n^{1/2} P(Z_i^j < Z_k^j, T_i < T_k)}{n^{-2}\sum_i \sum_k \frac{\delta_i}{S^2(Y_i)} I(Y_i < Y_k)} - \frac{n^{1/2} P(Z_i^j < Z_k^j, T_i < T_k)}{P(Z_i^j < Z_k^j)}.
\end{aligned}$$

Using a martingale representation of $n^{1/2}(\hat{S} - S)$ [34] and the uniform convergence of

$$q_{2n}(u) = n^{-2} \sum_i \sum_k \frac{2\delta_i I(Y_i < Y_k)}{S^2(Y_i)} I(Y_i \geq u)$$

to $q_2(u)$, we can show that

$$\begin{aligned}
I_{11}^j &= -\tau^j n^{-3/2} \sum_i \sum_k \frac{2\delta_i I(Y_i < Y_j)}{S^2(Y_i)} \left\{ \frac{\hat{S}(Y_i) - S(Y_i)}{\hat{S}(Y_i)} \right\} + o_p(1) \\
&= -\tau^j n^{-1} \sum_r \int \frac{q_2(u) dM_r(u)}{E\{A_r(u)\}} + o_p(1),
\end{aligned}$$

Similarly, we can show that

$$I_{12}^j = \frac{1}{P(T_i < T_k)} n^{-1} \sum_r \int \frac{q_1^j(u) dM_r(u)}{E\{A_r(u)\}} + o_p(1).$$

Using the properties of U-statistics [35], we can show that

$$\begin{aligned}
I_{21}^j &= -\frac{1}{P(T_i < T_k)} n^{-1/2} \sum_i q_4(W_i^j) + o_p(1), \\
I_{22}^j &= \frac{\tau^j}{P(T_i < T_k)} n^{-1/2} \sum_i q_3^j(W_i^j) + o_p(1).
\end{aligned}$$

Therefore

$$n^{1/2}\left(\hat{\tau}^j - \tau^j\right) = n^{-1/2}\sum_{i=1}^{n}\varphi_i^j + o_p(1). \tag{1}$$

When estimating the variance of $\hat{\tau}^{L_1,L_2}$, we have the following definition:

$$\hat{\varphi}_i^j = \hat{h}_{i1}^j + \hat{h}_{i2}^j,$$

$$\hat{h}_{i1}^j = \int \frac{\left\{\hat{q}_1^j(u) - \tau^j \hat{q}_2(u)\right\}}{n^{-1}\sum_{k=1}^{n} A_k(u)\xi} d\hat{M}_k(u),$$

$$\hat{h}_{i2}^j = \frac{q_3^j(W_i^j) - \tau^j q_4(W_i^j)}{\xi},$$

$$\xi = n^{-2}\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\delta_i I(Y_i < Y_k)}{S^2(Y_i)},$$

$$\hat{q}_1^j(u) = n^{-2}\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{2\delta_i I(Z_i^j < Z_k^j, Y_i < Y_k)}{S^2(Y_i)} I(Y_i \geq u),$$

$$\hat{q}_2(u) = n^{-2}\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{2\delta_i I(Y_i < Y_k)}{S^2(Y_i)} I(Y_i \geq u),$$

$$\begin{aligned}\hat{q}_3^j(W_i^j) &= n^{-1}\sum_{k=1}^{n}\left\{\frac{\delta_i}{S_C^2(Y_i)} I(Z_i^j < Z_k^j, Y_i < Y_k) + \frac{\delta_j}{S_C^2(Y_k)} I(Z_i^j < Z_k^j, Y_k < Y_i)\right\} \\ &\quad - 2n^{-2}\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{\delta_r}{S_C^2(Y_r)} I(Z_r^j < Z_s^j, Y_r < Y_s),\end{aligned}$$

$$\hat{q}_4(W_i^j) = n^{-1}\sum_{j=1}^{n}\left\{\frac{\delta_i}{S_C^2(Y_i)} I(Y_i < Y_k) + \frac{\delta_k}{S_C^2(Y_k)} I(Y_k < Y_i)\right\} - 2n^{-2}\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{\delta_r}{S_C^2(Y_r)} I(Y_r < Y_s),$$

and $d\hat{M}_i(u) = dB_i(u) - A_i(u)d\hat{\Lambda}_C(u)$ with

$$\hat{\Lambda}(u) = \sum_{i=1}^{n}\int_0^u \frac{dB_i(s)}{\sum_{k=1}^{n} A_k(s)}$$

being the Nelson-Aalen estimator of $\Lambda_C(u)$.