A APPENDIX

We prove that if all error estimators are unbiased, then $\lim_{K\to\infty} B(m,n,K) = 0$.

LEMMA A.1. *If all error estimators are unbiased, then* $B(m,n,K) \leq 0$.

PROOF. Define the set $\mathcal{S}_n = \{\mathcal{S}_n^1, \mathcal{S}_n^2, \ldots, \mathcal{S}_n^K\}$, where $\mathcal{S}_n^k$, $k = 1, 2, \ldots, K$ is a random sample taken from the distribution F_k for $k = 1, 2, \ldots, K$. Also, we can rewrite Equation (9) as

$$\varepsilon_{\text{est}}^{\min}(K) = \min_{i,j} \left\{ \frac{1}{K} \sum_{k=1}^{K} \varepsilon_{\text{est}}^{i,j,k} \right\}, \tag{A.1}$$

where $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$. Owing to the unbiasedness of the error estimators, $E_{\mathcal{S}_n^k}[\varepsilon_{\text{est}}^{i,j,k}] = E_{\mathcal{S}_n^k}[\varepsilon_{\text{true}}^{i,k}]$. Referring to Equations (10) and (A.1), we have

$$\begin{aligned}
&B(m,n,K) \\
&= E_{\mathcal{S}_n}[\varepsilon_{\text{est}}^{\min}(K)] - \frac{1}{K}\sum_{k=1}^{K} E_{\mathcal{S}_n^k}\left[\varepsilon_{\text{true}}^{i_{\min},k}\right] \\
&= E_{\mathcal{S}_n}\left[\min_{i,j}\left\{\frac{1}{K}\sum_{k=1}^{K}\varepsilon_{\text{est}}^{i,j,k}\right\}\right] - \frac{1}{K}\sum_{k=1}^{K} E_{\mathcal{S}_n^k}\left[\varepsilon_{\text{true}}^{i_{\min},k}\right] \\
&\leq \min_{i,j}\left\{E_{\mathcal{S}_n}\left[\frac{1}{K}\sum_{k=1}^{K}\varepsilon_{\text{est}}^{i,j,k}\right]\right\} - \frac{1}{K}\sum_{k=1}^{K} E_{\mathcal{S}_n^k}\left[\varepsilon_{\text{true}}^{i_{\min},k}\right] \\
&= \min_{i,j}\left\{\frac{1}{K}\sum_{k=1}^{K}E_{\mathcal{S}_n}\left[\varepsilon_{\text{est}}^{i,j,k}\right]\right\} - \frac{1}{K}\sum_{k=1}^{K} E_{\mathcal{S}_n^k}\left[\varepsilon_{\text{true}}^{i_{\min},k}\right] \\
&= \min_{i,j}\left\{\frac{1}{K}\sum_{k=1}^{K}E_{\mathcal{S}_n^k}\left[\varepsilon_{\text{est}}^{i,j,k}\right]\right\} - \frac{1}{K}\sum_{k=1}^{K} E_{\mathcal{S}_n^k}\left[\varepsilon_{\text{true}}^{i_{\min},k}\right] \\
&= \min_{i}\left\{\frac{1}{K}\sum_{k=1}^{K}E_{\mathcal{S}_n^k}\left[\varepsilon_{\text{true}}^{i,k}\right]\right\} - \frac{1}{K}\sum_{k=1}^{K} E_{\mathcal{S}_n^k}\left[\varepsilon_{\text{true}}^{i_{\min},k}\right] \\
&\leq 0.
\end{aligned} \tag{A.2}$$

where the relations in the third and sixth lines result from Jensen's inequality and unbiasedness of the error estimators, respectively. □

LEMMA A.2. *If all error estimators are unbiased, then* $\lim_{K\to\infty} B(m,n,K) \geq 0$.

PROOF. Let

$$A^{i,j} = \frac{1}{K}\sum_{k=1}^{K}\varepsilon_{\text{est}}^{i,j,k}, \qquad T^i = \frac{1}{K}\sum_{k=1}^{K}E_{\mathcal{S}_n}\left[\varepsilon_{\text{true}}^{i,k}\right]. \tag{A.3}$$

Owing to the unbiasedness of the error estimators, $E_{\mathcal{S}_n}[A^{i,j}] = T^i \leq 1$. Without loss of generality, we assume $T^1 \leq T^2 \leq \ldots \leq T^r$. To avoid cumbersome notation, we will further assume that $T^1 < T^2$ (with some adaptation, the proof goes through without this assumption). Let $2\delta = T^2 - T^1$ and

$$\begin{aligned}
B_\delta = &\left(\bigcap_{j=1}^{s}\left(T^1 - \delta \leq A^{1,j} \leq T^1 + \delta\right)\right) \\
&\bigcap\left(\min_{i\neq 1,j}\left\{A^{i,j}\right\} > T^1 + \delta\right).
\end{aligned} \tag{A.4}$$

Because $|\varepsilon_{\text{est}}^{i,j,k}| \leq 1$, $\text{Var}_{\mathcal{S}_n}[A^{i,j}] \leq 1/K$. hence, for $\tau > 0$, there exists $K_{\delta,\tau}$ such that $K \geq K_{\delta,\tau}$ implies $P(B_\delta(K)) > 1 - \tau$. Hence, referring to Equation (10), for $K \geq K_{\delta,\tau}$,

$$\begin{aligned}
E_{\mathcal{S}_n}\left[\varepsilon_{\text{est}}^{\min}(K)\right] &= E_{\mathcal{S}_n}\left[\varepsilon_{\text{est}}^{\min}(K) \mid B_\delta\right]P(B_\delta) \\
&\quad + E_{\mathcal{S}_n}\left[\varepsilon_{\text{est}}^{\min}(K) \mid B_\delta^c\right]P(B_\delta^c) \\
&\geq E_{\mathcal{S}_n}\left[\varepsilon_{\text{est}}^{\min}(K) \mid B_\delta\right]P(B_\delta) \\
&= E_{\mathcal{S}_n}\left[\min_j A^{1,j}\right]P(B_\delta) \\
&\geq (T^1 - \delta)(1 - \tau).
\end{aligned} \tag{A.5}$$

Again referring to Equation (10) and recognizing that $i_{\min} = 1$ in B_δ, for $K \geq K_{\delta,\tau}$,

$$\begin{aligned}
\frac{1}{K}\sum_{k=1}^{K}E_{\mathcal{S}_n}\left[\varepsilon_{\text{true}}^{i_{\min},k}\right] &= \frac{1}{K}\sum_{k=1}^{K}\Big(E_{\mathcal{S}_n}\left[\varepsilon_{\text{true}}^{i_{\min},k} \mid B_\delta\right]P(B_\delta) \\
&\quad + E_{\mathcal{S}_n}\left[\varepsilon_{\text{true}}^{i_{\min},k} \mid B_\delta^c\right]P(B_\delta^c)\Big) \\
&\leq \frac{1}{K}\sum_{k=1}^{K}\Big(E_{\mathcal{S}_n}\left[\varepsilon_{\text{true}}^{1,k} \mid B_\delta\right]P(B_\delta) \\
&\quad + P(B_\delta^c)\Big) \\
&\leq T^1 + \tau.
\end{aligned} \tag{A.6}$$

Putting Equations (A.5) and (A.6) together and referring to Equation (10) yields, for $K \geq K_{\delta,\tau}$,

$$B(m,n,K) \geq (T^1 - \delta)(1 - \tau) - T^1 - \tau \geq -(2\tau + \delta) \tag{A.7}$$

Since δ and τ are arbitrary positive numbers, this implies that for any $\eta > 0$, there exists K_η such that $K \geq K_\eta$ implies $\lim_{K\to\infty} B(m,n,K) \geq 0$, which is precisely what we want to prove. □

Combining Lemmas A.1 and A.2, we have proven that $\lim_{K\to\infty} B(m,n,K) = 0$ under the assumption that all the error estimators are unbiased.