Appendix B. Derivation of Formula 5.2.1

Given assumptions *B.4* of the text, Eq. **5.1.3** becomes

$$\delta'^{(n)}\Delta\langle V_c\rangle = \frac{a^2}{4\pi^2\varepsilon_{sc}}\int_{\mathrm{MIR}} d\omega\, Im\,\{\eta(\omega)K_o(\omega)J_n(\omega)\}, \quad \textbf{[B.1]}$$

where

$$J_n(\omega) \equiv \int_o^\infty q^2 dq\left\{n^{-1}\sum_{i=1}^{n}\frac{f_i(q)}{(1+qf_i(q)K_o(\omega/\varepsilon_{sc})^2)}\right\}. \quad \textbf{[B.2]}$$

Let us suppose that the ith acoustic eigenvalue (see section 5.1) tends to $\alpha_i qd$ for $qd \ll 1$ ($i = 1, \ldots n-1$). To evaluate expression **B.2**, we invoke assumption *B.7* of the text and assume that n is not too large; then, the inequality $|K_o(\omega)| \gg nd\varepsilon_{sc}$ is fulfilled for most of the relevant range of ω. This allows us to split the integral in Eq. **B.2** into two parts, J_1 and J_2, corresponding to $0 \le q \le q_c$ and $q_c \le q \le \infty$, where we choose the cutoff q_c so that $nq_cd \ll 1$ but $q_c^2|K_o|d/\varepsilon_{sc} \gg |$, and to approximate the ith term in the integrand of J_1 by $n^{-1}\alpha_i dq^3/(1 + \alpha_i dq^2K_o(\omega)/\varepsilon_{sc})^2$ ($i = 1 \ldots n-1$) and in the integrand of J_2 by $\varepsilon_{sc}^{-2}K_o^{-2}\{n^{-1}f_i^{-1}(q)\}$ ($i = 1 \ldots n$). The single optical term ($i = n$) contributes to J_1 only a term of relative order q_cd, which may consistently be neglected (through compare section 6).

The crucial observation, now, is that[j]

$$n^{-1}\sum_{j=1}^{n} f_i^{-1}(q) - 1 = 2\left(1-\frac{1}{n}\right)(\exp(2qd)-1)^{-1}, \quad \textbf{[B.3]}$$

from which it follows, taking the limit $q \to 0$, that $\Sigma_{i=1}^{n-1}\,\alpha_i^{-1} = n-1$. Explicit evaluation of J_1 and J_2 using Eq. **B.3** then leads to the result (to within terms of relative order $\zeta^{-1} \equiv \varepsilon_{sc}d/|K|$)

$$J_n(\omega) = \left(1-\frac{1}{n}\right)\cdot\frac{\varepsilon_{sc}^2}{2dK_o^2(\omega)}\cdot\{\ln(\mathrm{K_o}(\omega)/4\mathrm{e}^{1/2}\mathrm{d}\varepsilon_{sc}) - \beta_n\}, \quad \textbf{[B.4]}$$

where

$$\beta_n \equiv \frac{1}{2}(n-1)\sum_{i=1}^{n-1}\alpha_i^{-1}\ln\alpha_i \quad \textbf{[B.5]}$$

The term β_n is numerically small ($\beta_n = 0$, 0.13, 0.18 … 0.3, … for $n = 2, 3, 4 \ldots \infty$). If we neglect it, we immediately obtain from Eq. **B.4** formula **5.2.1** of the text.

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[j]It is straightforward to prove Eq. **B.3** case by case for $n \le 6$ and $n = \infty$. I am indebted to Misha Turlakov for a proof in the general case.