Numerical assessment of time-domain methods for the estimation of local arterial pulse wave speed
(supplementary material)

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This supplementary material expands the description of the visco-elastic tube law (Section 1), details the numerical solution of the visco-elastic 1-D formulation (Section 2), and provides all the physiological data used in the 55-artery models (Section 3).

1. Visco-elastic tube law

The nonlinear 1-D equations of incompressible and axisymmetric flow in Voigt-type visco-elastic vessels considered in this study are an extension of the 1-D formulation developed by Sherwin et al. (2003) and Alastruey (2006). Here the dynamics of the arterial wall were modelled using a generalised string model of the form (Quarteroni et al., 2000; Formaggia et al., 2003)

$$P = P_{ext} + \frac{\beta}{A_0}\left(\sqrt{A} - \sqrt{A_0}\right) + \Gamma\frac{\partial A}{\partial t} + m\frac{\partial^2 A}{\partial t^2} - a\frac{\partial^2}{\partial x^2}\left(\sqrt{A} - \sqrt{A_0}\right),$$

$$\beta(x) = \frac{4}{3}\sqrt{\pi}Eh, \quad \Gamma(x) = \frac{\gamma}{2\sqrt{\pi A_0}}, \quad m(x) = \frac{\rho_w h}{2\sqrt{\pi A_0}}, \quad a(x) = \frac{\widetilde{a}}{\sqrt{\pi}}.$$

The assumptions and parameters of this model are the same as those described in Section 2.1. In addition, P_{ext} is the external pressure, $\rho_w(x)$ the wall mass density and $\widetilde{a}(x)$ a coefficient related to the longitudinal pre-stress state of the vessel. Assuming $P_{ext} = 0$ and neglecting the inertia and longitudinal pre-stress terms yield the visco-elastic tube law shown in Eq. (1).

2. Numerical solution

A discontinuous Galerkin scheme was used to solve the equations in (1) in the 55-artery network in Fig. 1. This is a convenient scheme for high-order discretisation of convection-dominated flows (Cockburn and Shu, 1998), such as arterial flows. It allows us to propagate waves of different frequencies without suffering from excessive dispersion and diffusion errors.

The equations in (1) can be written in the following conservative form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}_U,$$

with

$$\mathbf{U} = \begin{bmatrix} A \\ U \end{bmatrix}, \quad \mathbf{S}_U = \begin{bmatrix} 0 \\ \frac{f}{\rho A} \end{bmatrix},$$

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_v = \begin{bmatrix} AU \\ \frac{U^2}{2} + \frac{P_e}{\rho} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{\Gamma}{\rho}\frac{\partial (AU)}{\partial x} \end{bmatrix},$$

and $P_e = \frac{\beta}{A_0}(\sqrt{A} - \sqrt{A_0})$ being the elastic component of pressure. The flux $\mathbf{F}$ was separated into an elastic ($\mathbf{F}_e$) and a viscous ($\mathbf{F}_v$) term, and $\frac{\partial A}{\partial t} = -\frac{\partial (AU)}{\partial x}$ was used for the viscous term in the tube law.

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The discrete form of this conservative form in a domain $\Omega = [a, b]$ discretised into a mesh of N_{el} elemental non-overlapping regions $\Omega_e = [x_e^l, x_e^u]$, such that $x_e^u = x_{e+1}^l$ for $e = 1, \ldots, N_{el} - 1$, and $\bigcup_{e=1}^{N_{el}} \Omega_e = \Omega$, is given by (Alastruey, 2006)

$$\sum_{e=1}^{N_{el}} \left[\left(\frac{\partial \mathbf{U}^\delta}{\partial t}, \boldsymbol{\psi}^\delta \right)_{\Omega_e} + \left(\frac{\partial \mathbf{F}(\mathbf{U}^\delta)}{\partial x}, \boldsymbol{\psi}^\delta \right)_{\Omega_e} + \left[\boldsymbol{\psi}^\delta \cdot \left\{ \mathbf{F}^u - \mathbf{F}(\mathbf{U}^\delta) \right\} \right]_{x_e^l}^{x_e^u} \right] = \sum_{e=1}^{N_{el}} \left(\mathbf{S}_U^\delta, \boldsymbol{\psi}^\delta \right)_{\Omega_e},$$

for all $\boldsymbol{\psi}^\delta$ in $\mathbf{V}^\delta$, where $(\mathbf{u}, \mathbf{v})_\Omega = \int_\Omega \mathbf{u}\mathbf{v}\, \mathrm{d}x$ is the standard $\mathbf{L}^2(\Omega)$ inner product, $\mathbf{U}^\delta$ and $\boldsymbol{\psi}^\delta$ denote the approximation of $\mathbf{U}$ and test functions $\boldsymbol{\psi}$, respectively, in the finite space $\mathbf{V}^\delta$ of piecewise polynomial vector functions (they may be discontinuous across inter-element boundaries), and $\mathbf{F}^u = \mathbf{F}_e^u + \mathbf{F}_v^u$ is the approximation of the flux at the interface.

The term $\mathbf{F}_e^u$ was treated through the solution of a Riemann problem, as described by Alastruey (2006). The term $\mathbf{F}_v^u$ requires a different treatment. Various ways of dealing with this term were analysed by Zienkiewicz et al. (2003). Here, $\mathbf{F}_v^u$ at the inter-element boundaries was approximated as

$$\mathbf{F}_v^u\Big|_{x_e^u} = \mathbf{F}_v^u\Big|_{x_{e+1}^l} = \frac{1}{2}\left(\mathbf{F}_v|_{x_e^u} + \mathbf{F}_v|_{x_{e+1}^l} \right), \qquad e = 1, \ldots, N_{el} - 1,$$

with $\mathbf{F}_v^u\Big|_{x_1^l} = \mathbf{F}_v|_{x_1^l}$ at the inlet of the domain and $\mathbf{F}_v^u\Big|_{x_{N_{el}}^u} = \mathbf{F}_v|_{x_{N_{el}}^u}$ at the outlet, so that $\mathbf{F}_v^u - \mathbf{F}_v(\mathbf{U}^\delta) = 0$ at both boundaries.

The expansion bases were selected to be a polynomial space of order P and the solution was expanded on each region Ω_e in terms of Legendre polynomials $L_p(\xi)$; i.e.

$$\mathbf{U}^\delta\Big|_{\Omega_e}(x_e(\xi), t) = \sum_{p=0}^{P} L_p(\xi) \widehat{\mathbf{U}}_e^p,$$

where $\widehat{\mathbf{U}}_e^p(t)$ are the expansion coefficients. Following standard finite element techniques, the following elemental affine mapping was introduced,

$$x_e(\xi) = x_e^l \frac{(1-\xi)}{2} + x_e^u \frac{(1+\xi)}{2},$$

with ξ in the reference element $\Omega_{st} = \{-1 \le \xi \le 1\}$.

The choice of discontinuous discrete solution and test functions allows us to decouple the problem on each element, the only link coming through the boundary fluxes. Legendre polynomials are particularly convenient because the basis is orthogonal with respect to the $\mathrm{L}^2(\Omega_e)$ inner product. Visco-elasticity was neglected at the boundary conditions of the network and the junctions, which were implemented as described in Alastruey (2006) for both the purely elastic and visco-elastic models. The discretisation in time was performed by a second-order Adams-Bashforth scheme.

3. Physiological data

The physiological data of the 55 arteries in Fig. 1 were taken from Alastruey (2010) and are shown in Tables 1 to 2. A constant $\gamma = 0.1$ MPa s m^{-1} was assumed in all the arteries based on data in Armentano et al. (1995). The 'well-matched model' is a version of the 55-artery model with zero reflection coefficients for forward-travelling waves at the arterial junctions. For each of the three edges a, b and c connected at a junction, the reflection coefficients R_f^j ($j = a, b, c$) are defined as the ratio of the change of pressure across the reflected wave to the change of pressure in the incident wave. Using a linearised version of the purely elastic 1-D equations, they can be expressed as a function of the characteristic admittance of the edge $Y_0^j = A_0^j / \rho c^j$ ($j = a, b, c$), with $c = \sqrt{\frac{\beta}{2\rho}} A_0^{-1/4}$ (Alastruey et al., 2009),

$$R_f^a = \frac{Y_0^a - Y_0^b - Y_0^c}{Y_0^a + Y_0^b + Y_0^c}, \quad R_f^b = \frac{Y_0^b - Y_0^c - Y_0^a}{Y_0^b + Y_0^c + Y_0^a}, \quad R_f^c = \frac{Y_0^c - Y_0^a - Y_0^b}{Y_0^c + Y_0^a + Y_0^b}.$$

The last two columns of Table 1 show these coefficients at the inlet and outlet of segments connected to junctions.

Table 1: Length, initial radius and wave speed of each arterial segment in the 55-artery network in Fig. 1. The last two columns show the reflection coefficients at the inlet and outlet of segments connected to junctions. The radii and wave speeds in brackets yield well-matched junctions for forward-travelling waves (with $R_f = 0.000$ at the outlets of internal segments).

Arterial segment	Length (cm)	Radius (mm)	c (m s^{-1})	R_f inlet	R_f outlet
1. Ascending aorta	4.0	14.5 (14.7)	4.0 (6.2)	-	0.1
2. Aortic arch I	2.0	11.2 (12.6)	4.0 (5.8)	−0.3	0.0
3. Brachiocephalic	3.4	6.2 (7.0)	4.3 (6.3)	−0.8	0.2
4. R. subclavian	3.4	4.2 (5.4)	4.8 (6.5)	−0.5	0.2
5. R. common carotid	17.7	3.7 (4.7)	4.9 (6.8)	−0.6	0.8
6. R. vertebral	14.8	1.9 (2.4)	8.3 (11.3)	−0.9	-
7. R. brachial	42.2	3.2 (5.2)	5.5 (6.7)	−0.4	0.4
8. R. radial	23.5	1.6 (3.7)	8.8 (9.0)	−0.8	-
9. R. ulnar I	6.7	2.2 (4.5)	7.8 (8.3)	−0.6	0.1
10. R. interosseous	7.9	0.9 (1.9)	13.2 (14.1)	−0.9	-
11. R. ulnar II	17.1	1.9 (4.3)	8.2 (8.5)	−0.2	-
12. R. internal carotid	17.7	1.3 (3.8)	9.9 (9.0)	−0.9	-
13. R. external carotid	17.7	1.3 (3.8)	9.6 (8.7)	−0.9	-
14. Aortic arch II	3.9	10.7 (11.9)	3.9 (5.7)	−0.1	0.0
15. L. common carotid	20.8	3.7 (4.2)	4.9 (7.2)	−0.9	0.8
16. L. internal carotid	17.7	1.3 (3.4)	9.9 (9.6)	−0.9	-
17. L. external carotid	17.7	1.3 (3.4)	9.6 (9.2)	−0.9	-
18. Thoracic aorta I	5.2	10.0 (11.2)	4.0 (5.8)	−0.1	0.4
19. L. subclavian	3.4	4.2 (4.7)	4.8 (7.0)	−0.9	0.2
20. L. vertebral	14.8	1.9 (2.0)	8.3 (12.5)	−0.9	-
21. L. brachial	42.2	3.2 (4.6)	5.5 (7.1)	−0.4	0.4
22. L. radial	23.5	1.6 (3.2)	8.8 (9.5)	−0.8	-
23. L. ulnar I	6.7	2.2 (4.0)	7.8 (8.9)	−0.6	0.1
24. L. interosseous	7.9	0.9 (1.7)	13.2 (14.9)	−0.9	-
25. L. ulnar II	17.1	1.9 (3.8)	8.2 (9.1)	−0.2	-
26. Intercostals	8.0	1.8 (3.1)	6.3 (7.3)	−1.0	-
27. Thoracic aorta II	10.4	6.6 (10.7)	4.6 (5.7)	−0.5	−0.1
28. Abdominal aorta I	5.3	6.1 (9.2)	4.6 (5.8)	−0.2	−0.2
29. Celiac I	1.0	3.9 (5.9)	4.8 (6.1)	−0.7	0.4
30. Celiac II	1.0	2.0 (4.0)	6.8 (7.4)	−0.7	−0.5
31. Hepatic	6.6	2.2 (4.6)	5.6 (6.1)	−0.6	-
32. Gastric	7.1	1.8 (2.2)	6.0 (8.3)	−0.6	-
33. Splenic	6.3	2.8 (3.4)	5.3 (7.3)	0.1	-
34. Superior mesenteric	5.9	4.4 (4.1)	4.8 (7.6)	−0.6	-
35. Abdominal aorta II	1.0	6.0 (8.4)	4.4 (5.7)	−0.2	−0.1
36. L. renal	3.2	2.6 (3.5)	5.4 (7.2)	−0.9	-
37. Abdominal aorta III	1.0	5.9 (7.9)	4.4 (5.9)	−0.1	0.0
38. R. renal	3.2	2.6 (3.5)	5.4 (7.2)	−0.8	-
39. Abdominal aorta IV	10.6	5.6 (7.3)	4.4 (5.9)	−0.1	0.0
40. Inferior mesenteric	5.0	1.6 (2.5)	6.2 (7.6)	−0.9	-
41. Abdominal aorta V	1.0	5.2 (6.8)	4.2 (5.7)	−0.1	0.1
42. R. common iliac	5.8	3.6 (5.1)	4.9 (6.3)	−0.5	0.2
43. L. common iliac	5.8	3.6 (5.1)	4.9 (6.3)	−0.5	0.2
44. L. external iliac	14.4	3.0 (4.8)	7.2 (8.7)	−0.4	0.0
45. L. internal iliac	5.0	2.0 (4.1)	10.7 (11.6)	−0.8	-
46. L. femoral	44.3	2.2 (3.6)	8.0 (9.7)	−0.5	0.2
47. L. deep femoral	12.6	2.2 (3.6)	7.8 (9.5)	−0.5	-
48. L. posterior tibial	32.1	1.9 (3.8)	11.5 (12.8)	−0.4	-
49. L. anterior tibial	34.3	1.3 (2.0)	13.1 (16.5)	−0.8	-
50. R. external iliac	14.4	3.0 (4.8)	7.2 (8.7)	−0.4	0.0
51. R. internal iliac	5.0	2.0 (4.1)	10.7 (11.6)	−0.8	-
52. R. femoral	44.3	2.2 (3.6)	8.0 (9.7)	−0.5	0.2
53. R. deep femoral	12.6	2.2 (3.6)	7.8 (9.5)	−0.5	-
54. R. posterior tibial	32.1	1.9 (3.8)	11.5 (12.8)	−0.4	-
55. R. anterior tibial	34.3	1.3 (2.0)	13.1 (16.4)	−0.8	-

Table 2: Peripheral resistances and compliances at the terminal segments of the 55-artery network in Fig. 1. They lead to a total resistance of 134.2 MPa s m^{-3} and a total compliance (as defined by Alastruey (2010)) of 12.0 m^3 GPa^{-1}. In all these segments, the pressure at which flow to the microcirculation ceases is zero. The compliances in brackets correspond to the network with well-matched junctions for forward-travelling waves.

Arterial segment	Resistance (10^{10} Pa s m^{-3})	Compliance (10^{-10} m^3 Pa^{-1})
6. R. vertebral	0.60	0.93 (0.84)
8. R. radial	0.53	1.06 (0.96)
10. R. interosseous	8.43	0.07 (0.06)
11. R. ulnar II	0.53	1.06 (0.96)
12. R. internal carotid	1.39	0.40 (0.36)
13. R. external carotid	1.39	0.40 (0.36)
16. L. internal carotid	1.39	0.40 (0.36)
17. L. external carotid	1.39	0.40 (0.36)
20. L. vertebral	0.60	0.93 (0.84)
22. L. radial	0.53	1.06 (0.96)
24. L. interosseous	8.43	0.07 (0.06)
25. L. ulnar II	0.53	1.06 (0.96)
26. Intercostals	0.14	4.02 (3.64)
31. Hepatic	0.36	1.54 (1.39)
32. Gastric	0.54	1.03 (0.93)
33. Splenic	0.23	2.41 (2.18)
34. Superior mesenteric	0.09	6.00 (5.43)
36. L. renal	0.11	4.94 (4.47)
38. R. renal	0.11	4.94 (4.47)
40. Inferior mesenteric	0.69	0.81 (0.73)
45. L. internal iliac	0.79	0.70 (0.64)
47. L. deep femoral	0.48	1.17 (1.06)
48. L. posterior tibial	0.48	1.17 (1.06)
49. L. anterior tibial	0.56	1.00 (0.90)
51. R. internal iliac	0.79	0.70 (0.64)
53. R. deep femoral	0.48	1.17 (1.06)
54. R. posterior tibial	0.48	1.17 (1.06)
55. R. anterior tibial	0.56	1.00 (0.90)

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