Supplementary Data

Preliminaries

We consider the following $\beta \to \infty$ limit for the reaction diffusion equation with the source function (15)

$$\frac{\partial\phi}{\partial t} = D\frac{\partial^2\phi}{\partial x^2} - b\phi + \frac{1}{2}\Gamma_I k_b h(x)^2 \frac{n_\infty}{1+K_e^0}\theta(\frac{A_I}{K_m}\phi - \frac{1}{2}k_b h^2 + \frac{1}{\beta}\ln K_e^0) \tag{1}$$

where $\theta(u) = 1$ if $u \geq 1$ and 0 otherwise. To keep the notation as simple as possible, it will be useful to renormalize ϕ and Γ_I as

$$\psi(x) = A_I\phi(x)/K_m \tag{2}$$
$$\Gamma = A_I\Gamma_I/K_m \tag{3}$$

The source function (15) possesses three zeros, $\psi^\star_{1,2,3}$, with $\psi^\star_1 = 0 < \frac{1}{2}k_b h^2 - 1/\beta \ln K_e^0 < \psi^\star_3$ if

$$\frac{\Gamma k_b n_\infty}{b\,(1+K_e^0)} > 1 - \frac{2}{\beta k_b h(x)^2}\ln K_e^0 \tag{4}$$

at some value of x.

Type 1 and 2 solutions

We derive in this Appendix the condition under which Eq. (1) possesses two families of solutions $\phi_{1,2}(x)$ with the following properties:

1. Case 1 : $\psi_1(x) > \frac{1}{2}k_b h(x)^2 - 1/\beta \ln K_e^0$ in some interval $[-x_0, x_0]$ and $\psi_1(x) < \frac{1}{2}k_b h(x)^2 - 1/\beta \ln K_e^0$ otherwise;
2. Case 2 : $\psi_2(x) < \frac{1}{2}k_b h(x)^2 - 1/\beta \ln K_e^0$ in some interval $[-x_0, x_0]$ and $\psi_2(x) > \frac{1}{2}k_b h(x)^2 - 1/\beta \ln K_e^0$ otherwise.

Type 1 solutions are maximum at $x = 0$ but type 2 solutions have two symmetric maxima with a depletion hole centered at $x = 0$. The condition of existence of type 2 solution is found by matching the concentration $\psi(x)$ and the current $\psi'(x)$ at $x = \pm x_0$.

Finding the solution for case 1 is straightforward. First, let us define the error function (1) :

$$\text{Erf}(z_0, z_1) = \frac{2}{\sqrt{\pi}}\int_{z_0}^{z_1} e^{-t^2}dt \tag{5}$$

with $\text{Erf}(x) = \text{Erf}(0, x)$. We find in case 1 for $x \in [-x_0, x_0]$:

$$\psi_1(x) = u(x)e^{x/\lambda} + v(x)e^{-x/\lambda} + c\cosh(x/\lambda) \tag{6}$$

with

$$u(x) = C_A\text{Erf}(x/w + w/2\lambda, +\infty) \tag{7}$$
$$v(x) = C_A\text{Erf}(-\infty, x/w - w/2\lambda) \tag{8}$$

where

$$C_A = \frac{\sqrt{\pi}}{8\lambda}n_\infty k_b h_0^2 \frac{w}{b}\frac{\Gamma}{1+K_e^0}e^{w^2/4\lambda^2} \tag{9}$$

The constant c in (6) is determined by matching Eq. (6) with $(\frac{1}{2}k_b h(x)^2 - 1/\beta \ln K_e^0) \exp[-(x-x_0)/\lambda]$ at the boundary point $x = x_0$. In practice, we can take $c = 0$ and $x_0 \to \infty$ as shown in Fig. 4-a (dashed curve compared to plain curve).

In the limit of small diffusion length, $\lambda/w \ll 1$, type 1 solutions have the following asymptotic behavior ($1 - \mathrm{Erf}(x) \simeq 1/(\sqrt{\pi}x)\exp(-x^2)$ when x goes to infinity, see (1))

$$\psi_1(x) \simeq \frac{k_b}{2b(1+K_e^0)} \frac{\Gamma}{w} n_\infty h(x)^2 \tag{10}$$

Type 2 solution

For type 2 solutions, we have :

$$\begin{aligned}
\psi_2^<(x) &= \left[\frac{1}{2}k_b h(x)^2 - 1/\beta \ln K_e^0\right] \frac{\cosh(x/\lambda)}{\cosh(x_0/\lambda)} \text{ if } -x_0 < x < x_0 && (11)\\
\psi_2^>(x) &= C_A e^{x/\lambda} \mathrm{Erf}(x/w + w/2\lambda, \infty) && (12)\\
&\quad + C_A e^{-x/\lambda} \mathrm{Erf}(B, x/w - w/2\lambda) \text{ if } |x| > x_0 && (13)
\end{aligned}$$

where x_0 and B are found by matching both solutions at $x = \pm x_0$. Instead of these conditions, we will use the equivalent conditions at $x = \pm x_0$:

$$\begin{aligned}
\psi_2^>(x_0) &= \frac{1}{2}k_b h(x)^2 - 1/\beta \ln K_e^0 && (14)\\
\psi_2^<(x_0) + \lambda \psi_2^{<\prime}(x_0) &= \psi_2^>(x_0) + \lambda \psi_2^{>\prime}(x_0) && (15)
\end{aligned}$$

where the second equation gives x_0. Once done, B can be found by using the first equation.

Condition for the existence of type 2 solution

From Eq. (15), one finds that x_0 is solution of the following equation :

$$\begin{aligned}
&\frac{\lambda}{w} \frac{e^{-(x_0/w + w/2\lambda)^2}}{\mathrm{Erf}(x_0/w + w/2\lambda, \infty)} (1 + \tanh(x_0/\lambda)) \\
&\qquad = \frac{\sqrt{\pi}}{2} n_\infty k_b \frac{\Gamma}{b(1+K_e^0)} \left(1 - 2\frac{\ln K_e^0}{\beta k_b h(x_0)^2}\right)
\end{aligned} \tag{16}$$

In order for (16) to have a solution, we have two conditions :

1. Condition (4) must hold.
2. There exists a minimum value of w_0, i.e. w_c, such that for $w > w_c$, Eq. (16) has one unique solution. This condition corresponds to $x_0 = 0$ in (16).

The argument goes as follows for $\beta \to \infty$. For a given w_0, the lefthand side of (16) is a strictly increasing function of x_0. Thus, its is minimum for $x_0 = 0$, since x_0 is positive. However,

$$\frac{\lambda}{w} \frac{e^{-(w/2\lambda)^2}}{\mathrm{Erf}(w/2\lambda)} \tag{17}$$

decreases with increasing w/λ. Thus, for $w < w_c$ defined by

$$\frac{\lambda}{w_c} \frac{e^{-(w_c/2\lambda)^2}}{\mathrm{Erf}(w_c/2\lambda)} = \frac{\sqrt{\pi}}{2} n_\infty k_b h_0^2 \frac{\Gamma}{b(1+K_e^0)} \tag{18}$$

(16) cannot have a solution, since the lefthand side of (16) is always larger than the righthand side. Moreover, $\lambda/w e^{-(w_c/2\lambda)^2}/\mathrm{Erf}(w_c/2\lambda) > \sqrt{\pi}/2$. The last inequality implies condition (4) for the source function to have three zeros.

Bibliography

[1] Abramowitz, M., and I. Stegun. 1972. Handbook of Mathematical Functions. National Bureau of Standards.