## Web Appendix

**Lemma 1:** $c(z,e) = E[c^*(z,X) \mid e(X)=e]$ for $z \in \{0,1\}$

**Proof**: By definition, $c^*(z,X) = E[Y_z \mid Z=1,X] - E[Y_z \mid Z=0,X]$. For $z \in \{0,1\}$, let $f(X \mid Z=z, e(X)=e)$ denote the conditional distribution of the confounder vector *X* given the treatment group $Z=z$ and the PS $e(X)=e$, and $f(X \mid e(X)=e)$ denote the conditional distribution of *X* given $e(X)=e$ only. By the definition of the propensity score, $\Pr(Z=1 \mid X, e(X)=e) = e = \Pr(Z=1 \mid e(X)=e)$, i.e., $Z \amalg X \mid e(X)=e$. Then, $f(X \mid Z=z, e(X)=e) = f(X \mid e(X)=e)$. Moreover,

$$
\begin{aligned}
&E[E(Y_z \mid Z=1,X) \mid e(X)=e] \\
&= \int E(Y_z \mid Z=1,X) f(X \mid e(X)=e) dX \\
&= \int E(Y_z \mid Z=1,X,e(X)=e) f(X \mid Z=1, e(X)=e) dX \\
&= E[Y_z \mid Z=1, e(X)=e]
\end{aligned}
$$

Following similar argument, it can be shown that

$$E[E(Y_z \mid Z=0,X) \mid e(X)=e] = E[Y_z \mid Z=0, e(X)=e].$$

As a direct consequence, we have proved that $c(z,e) = E[c^*(z,X) \mid e(X)=e]$.

**Lemma 2**: Suppose the sensitivity function $c(z,e)$ is correctly specified, then the bias correction term in the SF-corrected estimator will remove the bias due to uncontrolled confounding.

**Proof**: By definition,

$$
\begin{aligned}
Y^{SF} &\equiv Y - \left(E[Y_Z \mid Z, e(X)] - E[Y_Z \mid e(X)]\right) \\
&= \begin{cases} Y - (1-e(X))c(1,e) & \text{if } Z=1 \\ Y + e(X)c(0,e) & \text{if } Z=0 \end{cases} \qquad (1)
\end{aligned}
$$

We first consider the daily users with $Z_i = 1$, then $Y_i = Y_{1,i}$,

$$Y_i^{SF} = Y_i - \left(E[Y_{1,i} \mid Z_i = 1, e(X_i)] - E[Y_{1,i} \mid e(X_i)]\right),$$

and

$$
\begin{aligned}
&E\left[\frac{Z_i}{e(X_i)} Y_i^{SF}\right] = E\left[\frac{Z_i}{e(X_i)} E\left(Y_i^{SF} \mid Z_i = 1, e(X_i)\right)\right] \\
&= E\left[\frac{Z_i}{e(X_i)} \begin{Bmatrix} E(Y_i \mid Z_i = 1, e(X_i)) \\ -\left(E[Y_{1,i} \mid Z_i = 1, e(X_i)] - E[Y_{1,i} \mid e(X_i)]\right) \end{Bmatrix}\right] . \\
&= E\left[\frac{Z_i}{e(X_i)} E[Y_{1,i} \mid e(X_i)]\right] = E\left[\frac{E[Z_i \mid e(X_i)]}{e(X_i)} E[Y_{1,i} \mid e(X_i)]\right] \\
&= E\left(E[Y_1 \mid e(X)]\right) = E[Y_1]
\end{aligned}
$$

Similarly, it can be proved that for the periodic users ( $Z_i = 0$ ), $Y_i = Y_{0,i}$ and $E\left[\frac{1-Z_i}{1-e(X_i)}Y_i^{SF}\right] = E[Y_0]$. Apparently, the bias correction term $(E[Y_Z \mid Z, e(X)] - E[Y_Z \mid e(X)])$ removes the bias due to uncontrolled confounding.

In addition, for $z \in \{0,1\}$, by the definition of conditional expectation,

$$\begin{aligned} & E[Y_z \mid e(X)] \\ &= E[Y_z \mid Z=1, e(X)]\Pr(Z=1 \mid e(X)) + E[Y_z \mid Z=0, e(X)]\Pr(Z=0 \mid e(X)) \\ &= E[Y_z \mid Z=1, e(X)]e(X) + E[Y_z \mid Z=0, e(X)](1-e(X)) \end{aligned}.$$

Then, after simple algebra, it can be shown that,

$$\begin{aligned} E[Y_1 \mid Z=1, e(X)] - E[Y_1 \mid e(X)] &= (1-e(X))c(1,e) \\ E[Y_0 \mid Z=0, e(X)] - E[Y_0 \mid e(X)] &= -e(X)c(0,e) \end{aligned}.$$

Furthermore, for any given sensitivity function $c(z,e)$, $\hat{\psi}^{SF}$ can be easily obtained by replacing the unknown PS using $\{\hat{e}(X_i), i = 1,\ldots,n\}$.