Supporting Information

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SI Text

Michaelis–Menten Uptake Rate. Let a system be composed of a closed recipient and a liquid medium where nutrient ions, S, and (unoccupied) uptake sites, E_f, diffuse freely. Following the enzymatic analogy for the uptake (1), we can describe the uptake process by means of the following reaction:

$$S + E_f \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} ES \xrightarrow{k_2} S_i + E_f, \qquad [S1]$$

where ES represents the enzyme–substrate compound and S_i is the nutrient incorporated by the transporter into the cell. Expression **S1** describes a series of reactions in which one nutrient ion is trapped by an uptake site at a constant rate k_1; the site remains occupied until the enzyme–ion pair is dissociated at a rate k_2. An ion–transporter complex can be dissociated outside the cell at a rate k_{-1} (rebound process), although we consider here that this process occurs at a frequency too small to be taken into account.

The Law of Mass Action (2) allows us to write a system of dynamic equations for the concentration (represented by [..]) of the reacting agents:

$$\frac{d[ES]}{dt} = k_1[E_f][S] - k_2[ES] \qquad [S2]$$

$$\frac{d[S_i]}{dt} = k_2[ES] \qquad [S3]$$

$$\frac{d[S]}{dt} = \left(-k_1[E_f][S]\right)\nu_c^{-1} \qquad [S4]$$

$$\frac{d[E_f]}{dt} = -k_1[E_f][S] + k_2[ES]. \qquad [S5]$$

Note that we use different units for the concentrations outside and inside the cell: The external nutrient concentration $[S]$ is measured in units of mol/L, whereas the internal concentration of nutrient, compound, and enzyme ($[S_i]$, $[ES]$, and $[E_f]$, respectively) are measured in mol per cell (Table S1). With this choice of units, the dynamic equation for the concentration of internal nutrient, Eq. **S3**, can be directly identified with the uptake rate of the single cell, $V = d[S_i]/dt$. The inclusion of the cell volume, ν_c, is thus necessary for the dimensional balance of Eq. **S4**.

Assuming a rapid first encounter of the uptake sites with their first target ions (i.e., the time needed for the sites to absorb ions at the initial stage of the reactions is small), the number of occupied sites can be considered approximately constant (2), that is, $d[ES]/dt \simeq 0$; using that, by definition, the (constant) total number of uptake sites is given by $[E] = [E_f] + [ES]$, we can write

$$V = \frac{k_2[E][S]}{[S] + k_2/k_1} = \frac{V_{max}[S]}{[S] + K_S}, \qquad [S6]$$

that is, the classic Michaelis–Menten (MM) form for the uptake rate. The kinetic parameters are given by $V_{max} = k_2[E]$ (maximum uptake rate) and $K_S = k_2/k_1$ (half-saturation constant), the former remaining constant due to the constant character of the total number of sites, $[E]$.

Diffusion-Limitation Correction for the Uptake Rate. Using the reaction scheme of Fig. 1 (main text), we deduce here a generalized expression for the uptake rate that takes into account the effects of the boundary layer developed by the cell in the diffusion-limitation regime. Although similar to those that can be obtained using the approaches of refs. 3 and 4, the deduction of this expression and some particularities are specially devised for the general framework introduced in this paper.

Replacing the bulk nutrient by the local nutrient surrounding the cell, S_0, the application of the Law of Mass Action (2) to the reaction scheme of Fig. 1 (main text) leads to the following equations for the different concentrations:

$$\frac{d[ES]}{dt} = k_1[E_f][S_0] - k_2[ES] \qquad [S7]$$

$$\frac{d[S_i]}{dt} = k_2[ES] \qquad [S8]$$

$$\frac{d[S_0]}{dt} = \left(-k_1[E_f][S_0] + \Phi_D\right)\nu_c^{-1} \qquad [S9]$$

$$\frac{d[E_f]}{dt} = -k_1[E_f][S_0] + k_2[ES] + \Phi_E. \qquad [S10]$$

Note again that the external (local) nutrient concentration $[S_0]$ is measured in units of mol/L, whereas $[S_i]$, $[ES]$, and $[E_f]$ are measured in mol per cell (Table S1). This allows us to equal Eq. **S8** with the uptake rate of the single cell, $V = d[S_i]/dt$, needing once more the cell volume, ν_c, in Eq. **S9**, for dimensional purposes. The terms in blue in Fig. 1, Φ_D and Φ_E, are the inflow of nutrient ions from the bulk to the cell surroundings and the "flow" of new uptake sites synthesized by the cell (main text).

As discussed in the main text, there are two main limiting regimes for the cell, namely diffusion limitation ($V > \Phi_D$, if expressed in terms of fluxes) and porter limitation ($V < \Phi_D$). We now assume stationary conditions for the local nutrient concentration, that is, the uptake equals the diffusive flow, and subsequently explore those limits. Under stationary conditions, we can write $\Phi_D = 4\pi D r_c([S] - [S_0])$, with D being the diffusion constant of the nutrient in the medium under consideration and r_c being the cell radius (3, 4). We can, as well, set Eq. **S9** equal to zero, obtaining in this way a closed-form expression for the local nutrient concentration:

$$[S_0] = \frac{4\pi D r_c[S]}{k_1[E_f] + 4\pi D r_c}. \qquad [S11]$$

If we assume that the time needed for the sites to absorb the first nutrient ions at the initial stage of the reactions is small, the number of occupied sites can be considered approximately constant (2). From that condition, $d[ES]/dt \simeq 0$, and we arrive at the equation

$$[ES] = \frac{k_1[E][S_0]}{k_2 - k_1[S_0]}. \qquad [S12]$$

Combining Eqs. **S11** and **S12**, and using the definitions $[E_f] = [E] - [ES]$, $V = k_2[ES]$, $V_{max} = k_2[E]$, and $K_S = k_2/k_1$ (see above), we obtain after some algebra the dimensionless quadratic equation

$$\frac{V_{\max}}{4\pi D r_c K_S}\left(\frac{V}{V_{\max}}\right)^2-\left(1+\frac{V_{\max}}{4\pi D r_c K_S}+\frac{[S]}{K_S}\right)\frac{V}{V_{\max}}+\frac{[S]}{K_S}=0. \quad \textbf{[S13]}$$

This expression is similar to equation 9 in ref. 4, where an equation for the uptake rate per unit area was obtained following a different deduction based on the Pasciak and Gavis model (3). After solving Eq. **S13**, we can find an approximate solution valid for both the diffusion and porter limitation regimes (4) that leads to the generalized MM formulation:

$$V=\frac{V_{\max}[S]}{K_S\left(1+\frac{V_{\max}}{4\pi D r_c K_S}\right)+[S]}=\frac{V_{\max}[S]}{\tilde{K}_S+[S]}. \quad \textbf{[S14]}$$

This is the generalized expression for the uptake rate of a cell used in the individual-based model introduced in the main text.

Individual-Based Model. As mentioned in the main text, the individual-based model is composed of the equation for the number of uptake sites synthesized by a cell $n(t)$ (main text, Eq. **6**), plus the equations that keep track of the changes in the internal content of organic carbon and the different nutrients (in this case, only nitrogen). We refer to the main text for a detailed explanation of the dynamic equation for $n(t)$.

In accordance with Shuter (5) and Geider et al. (6), among others, the equation for the individual's internal amount of carbon, C, is a balance between the incoming organic-carbon flow, with a source term due to photosynthesis dC_P/dt, and two loss terms, the first due to the expenditure of carbon due to maintenance costs, dC_M/dt (cost of keeping the various apparatuses and biostructures operational), and the other representing the biosynthesis of new biomaterial, dC_B/dt:

$$\frac{dC}{dt}(t)=\frac{dC_P}{dt}(t)-\frac{dC_M}{dt}(t)-\frac{dC_B}{dt}(t). \quad \textbf{[S15]}$$

Similarly, in the balance equation of the internal nitrogen content of the cell, N, we take into account the uptake of nitrogen, $dN_U/dt = V$, together with the maintenance and biosynthetic costs, dN_M/dt and dN_B/dt, respectively:

$$\frac{dN}{dt}(t)=\frac{dN_U}{dt}(t)-\frac{dN_M}{dt}(t)-\frac{dN_B}{dt}(t). \quad \textbf{[S16]}$$

Now we explain in detail each of those terms.

Photosynthesis term. Photosynthesis is the process by which phytoplankton obtain C from inorganic carbon using sunlight as a source of energy. Following a standard formulation, we consider that the contribution of photosynthesis to the change of organic carbon is proportional to the total amount present (7), where the proportionality constant is known as the photosynthetic efficiency, P_C:

$$\frac{dC_P}{dt}(t)=P_C(t)C(t). \quad \textbf{[S17]}$$

This efficiency depends mainly on light and temperature, but must also receive feedback from the internal state of the organism (see, for instance, ref. 8). We study here only cases with optimal T and I values, so that there is no need to write an explicit dependence of P_C on those variables. Thus, the efficiency can be written as

$$P_C(t)=P_{\max}P_{\text{int}}(t), \quad \textbf{[S18]}$$

where $P_{\max}$ is known as the maximum photosynthetic rate, and P_{int} keeps track of the internal state of the cell by using a function of the quota (this dependence is well-discussed in the literature; see, for instance, refs. 6 and 9. We use here, for simplicity, a linear function, but more complex functions can be also used):

$$P_{int}(t)=1-\frac{Q_{\max}-Q(t)}{Q_{\max}-Q_{\min}}=\frac{Q(t)-Q_{\min}}{Q_{\max}-Q_{\min}}. \quad \textbf{[S19]}$$

In this way, in cases in which Q is close to $Q_{\min}$ (i.e., very high levels of C for the current concentration of internal nutrient), it would be unnecessary for the production of more C, and photosynthesis becomes very inefficient. The opposite situation is found when Q is close to $Q_{\max}$.

Nutrient-uptake term. To account for the nutrient incorporated into the cell, we use the diffusion limitation-corrected uptake rate described in the main text, Eq. **3**, with the appropriate units [grams of nitrogen per cell (gN/cell), in this case; Table S1]:

$$\frac{dN_U}{dt}(t)=V=\frac{V_{\max}[S]}{K_S\left(1+\frac{V_{\max}}{4\pi D r_c K_S}\right)+[S]}=\frac{V_{\max}[S]}{\tilde{K}_S+[S]}, \quad \textbf{[S20]}$$

with $V_{\max}=k_2[E]$ and $K_S=k_2/k_1$. Thus, if we couple this with Eq. **6** (where we transform the units from sites per cell into mol per cell, i.e., we transform $n(t)$ into $[E]$), we obtain the dynamic, flexible description for the uptake rate introduced in the main text.

On the other hand, by fixing $V_{\max}$ to a given value, we obtain the static representation for MM that is used in the main text to benchmark the performance of the dynamic approach.

Maintenance term. This terminology refers to the expenditure, in terms of C and N, not directly related to growth processes but to maintaining the integrity of the cell (e.g., cell wall) and running cellular functions (e.g., uptake process, photosynthesis, repair of structures) (5).

Maintenance metabolic rates depend on the efficiency of the processes that are taken into account (6). For the sake of simplicity, we assume an explicit dependence only on the nutrient-harvesting apparatus, whereas the rest of contributions (including that of photosynthesis) collapse into a single constant. (We assume that this apparatus uses ATP molecules as an energy source. Therefore, its maintenance cost is supported only by the C budget.) Then:

$$\frac{dC_M}{dt}(t)=M_C C(t)+M_n^C Q^{-1}\frac{dN_U}{dt} \quad \textbf{[S21]}$$

$$\frac{dN_M}{dt}=M_N N(t). \quad \textbf{[S22]}$$

Note that the cost of maintenance of the uptake apparatus depends mainly on the ratio of uptake performance to internal nutrient (second term of Eq. **S21**). In this way, we can use a single, dynamic term for the high expense resulting from the synthesis of a high number of uptake sites for low $[S]$, together with the cost of the large number of assimilating enzymes for high $[S]$ (10).

Biosynthesis term. The only biosynthetic process considered here is the production of new uptake sites. Temperature should influence the cost of the synthesis of new sites, but as T does not play a relevant role in this paper (see above), the simplest way to consider the cost of new components for the nutrient-harvesting apparatus is

$$\frac{dC_B}{dt}(t)=B_C\Theta\left(\frac{dn}{dt}(t)\right) \quad \textbf{[S23]}$$

$$\frac{dN_B}{dt}(t)=B_N\Theta\left(\frac{dn}{dt}(t)\right), \quad \textbf{[S24]}$$

where the coefficients B_X reflect the cost in terms of the currency X (either nutrient or carbon) of the manufacture of a new uptake site. The $\Theta(x)$ function is defined as $\Theta(x) = x$ when $x > 0$ and $\Theta(x) = 0$ otherwise. (We use the Θ function because, following

ref. 11, in our model biomass coming from the cell's own biomaterial should not be regained.)

Observables. The chemostat conditions allow us to control the stationary nutrient concentration present in the system. We can, thus, study the behavior of various population-level observables as that concentration is changed.

We measure here the uptake rate and uptake parameters normalized by unit of biomass. For instance, the normalized population-level uptake rate is defined as

$$V = \left(\sum_{i=1}^{L(t)} \frac{dN_{U_i}}{dt}(t) \right) \left(\sum_{i=1}^{L(t)} C_i(t) \right)^{-1}, \qquad \textbf{[S25]}$$

where $dN_{U_i}/dt(t)$ represents each individual's uptake rate, and C_i is its internal amount of carbon. In the same fashion, we can define and monitor the population-level maximum uptake rate, $V_{\max}$, and the affinity, a_{ff}.

In addition, we measure the average number of uptake sites per organism, n, at the stationary state:

$$n = \frac{1}{L(t)} \sum_{i=1}^{L(t)} n_i(t). \qquad \textbf{[S26]}$$

Parameterization. Although our approach can be applied to any phytoplankton species, for the sake of concreteness we focus our attention here on a generic diatom strain. For the parameterization of the IBM, we use ranges of values for which we have found consensus in the literature, using both experimental and theoretical works (6, 12–16) (see Table S1 for the actual values). For instance, we have consistently found the values $Q_{\max}$ = 0.16–0.2 gN/gC, K_S = 1–3 μmol/L, and $V([S] \to \infty) = V_{\max}^{\text{lo}} = 0.1 - 0.6 gN/gC/d$, for various strains of diatoms. Concretely, in ref. 12, data from many different experimental sources are collected; importantly, the kinetic parameters for diatoms form a clear cluster around the values K_S = 1 μmol/L and $V_{\max}{}^{\text{lo}}$ = 0.2–0.5 gN·gC^{-1}·d^{-1}. Because, in our model, the value of $V_{\max}{}^{\text{lo}}$ emerges from the dynamics, a "correct" parameterization will give as a result an emergent $V([S]\to\infty) = V_{\max}^{\text{lo}}$ within the proposed range.

The value for the radius of the cell is deduced from the following allometric expression, valid for round-shaped diatoms (17):

$$C = \begin{cases} 0.1167 \nu_c^{0.881} \text{ if } \nu_c > 3000\ \mu m^3 \\ 0.2877 \nu_c^{0.811} \text{ otherwise} \end{cases}. \qquad \textbf{[S27]}$$

For parameters specifically related to the uptake sites, we use r_s = 10–40 A (16, 18), $\nu = 10^4 - 5{\cdot}10^5$, $k_1 = 4Dr_s$ (19), and $k_2 = K_S k_1$ (see above). [There is no specific experimental source for those numbers; this is the order of magnitude of the number of uptake sites for a diatom (see, for instance, ref. 16), and we are only assuming that the cell is able to develop those sites during its lifetime.]

Finally, to parameterize $V_{\max}$ in the case of the static MM model, we used the $V_{\max}^{lo}$ value obtained with the dynamic description (in gN per cell per d) to compare the results obtained.

We obtained a qualitatively similar behavior for all of the set of parameters used in our simulations (Table S1).

1. Caperon J (1967) Population growth in micro-organisms limited by food supply. *Ecology* 48:715–721.
2. Murray JD (2002) *Mathematical Biology I: An Introduction* (Springer, Berlin), 3rd Ed.
3. Pasciak WJ, Gavis J (1974) Transport limitation of nutrient uptake in phytoplankton. *Limnol Oceanogr* 19:881–888.
4. Armstrong RA (2008) Nutrient uptake rate as a function of cell size and surface transporter density: A Michaelis-like approximation to the model of Pasciak and Gavis. *Deep Sea Res Part I Oceanogr Res Pap* 55:1311–1317.
5. Shuter B (1979) A model of physiological adaptation in unicellular algae. *J Theor Biol* 78:519–552.
6. Geider RJ, MacIntyre HL, Kana TM (1998) A dynamic regulatory model of phytoplanktonic acclimation to light, nutrients, and temperature. *Limnol Oceanogr* 43:679–694.
7. Armstrong RA (2006) Optimality-based modeling of nitrogen allocation and photoacclimation in photosynthesis. *Deep Sea Res Part II Top Stud Oceanogr* 53:513–531.
8. Litchman E, Klausmeier CA (2008) Trait-based community ecology of plankton. *Annu Rev Ecol Evol Syst* 39:615–639.
9. Flynn KJ (2008) Use, abuse, misconceptions and insights from quota models: The Droop cell quota model 40 years on. *Oceanogr Mar Biol Ann Rev* 46:1–23.
10. Smith SL, Yamanaka Y, Pahlow M, Oschlies A (2009) Optimal uptake kinetics: Physiological acclimation explains the pattern of nitrate uptake by phytoplankton in the ocean. *Mar Ecol Prog Ser* 384:1–12.
11. Bruggeman J, Kooijman SALM (2007) A biodiversity-inspired approach to aquatic ecosystem modeling. *Limnol Oceanogr* 52:1533–1544.
12. Litchman EL, Klausmeier CA, Schofield OM, Falkowski PG (2007) The role of functional traits and trade-offs in structuring phytoplankton communities: Scaling from cellular to ecosystem level. *Ecol Lett* 10:1170–1181.
13. Flynn K (2001) A mechanistic model for describing dynamic multi-nutrient, light, temperature interactions in phytoplankton. *J Plankton Res* 23:977–997.
14. Woods JD (2005) The Lagrangian ensemble metamodel for simulating plankton ecosystems. *Prog Oceanogr* 67:84–159.
15. Hellweger FL, Kianirad E (2007) Individual-based modeling of phytoplankton: Evaluating approaches for applying the cell quota model. *J Theor Biol* 249: 554–565.
16. Aksnes DL, Egge JK (1991) A theoretical model for nutrient uptake in phytoplankton. *Mar Ecol Prog Ser* 70:65–72.
17. Menden-Deuer S, Lessard EJ (2000) Carbon to volume relationships for dinoflagellates, diatoms, and other protist plankton. *Limnol Oceanogr* 45:569–579.
18. Button DK (1998) Nutrient uptake by microorganisms according to kinetic parameters from theory as related to cytoarchitecture. *Microbiol Mol Biol Rev* 62:636–645.
19. Berg HC, Purcell EM (1977) Physics of chemoreception. *Biophys J* 20:193–219.

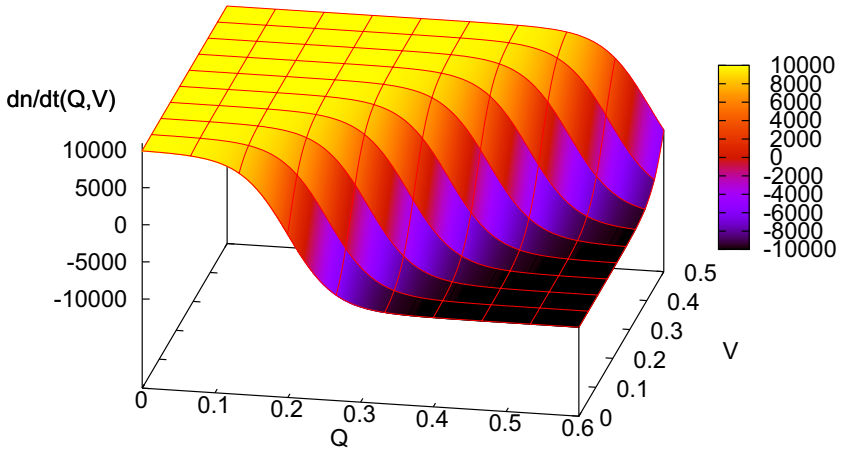

Fig. S1. Functional dependence of $\partial_t n$ on the main variables Q and V (Eq. **6**, main text). Note that the change in the number of sites is limited by the maximum number of sites that the cell can store at its surface (main text).

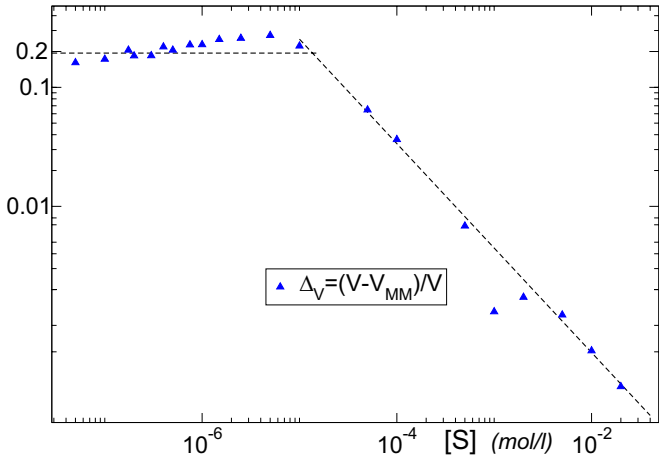

Fig. S2. Relative error between the uptake rate obtained with the dynamic approach proposed here, V, and a static version of the IBM where $V = V_{MM} = V^{lo}$. With this specific parameterization, the error reaches a stationary state around 21 ± 2%.

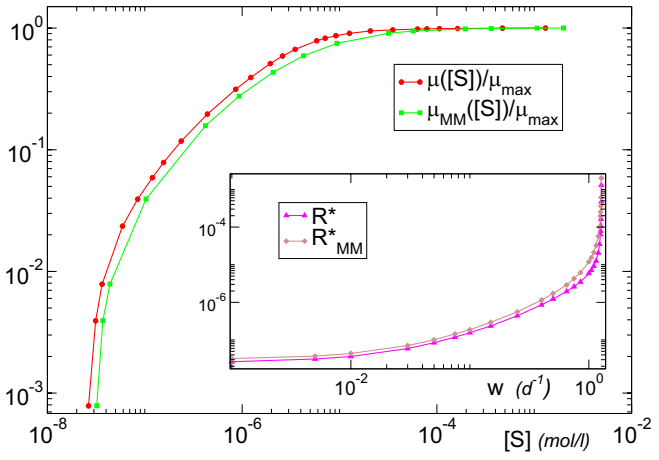

Fig. S3. Normalized growth rate obtained with our model (μ, red) and the diffusion limitation-corrected static MM model (μ_{MM}, green). (*Inset*) Break-even concentration of the only considered external resource plotted against the dilution rate of the chemostat w, for our approach (R^*, purple) and the static MM formulation (R^*_{MM}, orange).

Table S1. Symbols

Symbol	Also known as	Units	Value/range
Φ_E	Transporter flow	sites/d	–
Φ_D	Nutrient diffusive flow	mol/d	–
$[S]$	Bulk nutrient concentration	mol/L	–
$[S_0]$	Local nutrient concentration	mol/L	–
$[S_i]$	Internal nutrient concentration	mol/cell	–
$[E_f]$	Free enzyme/porter concentration	mol/cell	–
$[ES]$	Occupied enzyme/porter concentration	mol/cell	–
$[E]$	Total enzyme/porter concentration	mol/cell	–
C	Organic carbon per cell	gC/cell	–
N	Nitrogen per cell	gN/cell	–
n	Porters per cell	sites/cell	–
Q	C-based quota	gN/gC	–
A_{rel}	Ratio absorbing:total area	–	–
k_1	Encounter rate	$(mol/L)^{-1}\cdot d^{-1}$	–
k_2	Handling/processing rate	d^{-1}	–
V	Population uptake rate	$gN\cdot gC^{-1}\cdot d^{-1}$	–
V_{max}	Maximum uptake rate	$gN\cdot gC^{-1}\cdot d^{-1}$	–
K_S	Half-saturation constant	mol/L	10^{-6}
$\tilde{K}_S$	Effective half-saturation constant	mol/L	–
a_{ff}	Affinity	$l\cdot gC^{-1}\cdot d^{-1}$	–
w	Dilution rate	d^{-1}	0.001–2
D	Nutrient diffusivity	$m^2\cdot s^{-1}$	$1.5\cdot10^{-9}$
r_{site}	Uptake-site radius	m	$10\cdot10^{-10}$ to $40\cdot10^{-10}$
k	Sigmoid slope parameter	–	5;10
ν	Site production rate	sites/d	$10^4 - 5\cdot10^5$
Q_{max}	Maximum C-based quota	gN/gC	0.167;0.2
Q_{min}	Minimum C-based quota	gN/gC	0.04
C_{min}	Minimum species biomass	gC/cell	$2.66\cdot10^{-10}$;$9\cdot10^{-9}$
P_{max}	Maximum photosynthetic rate	d^{-1}	3.00
M_n^C	Uptake maintenance cost	–	0.10–3.00
M_C	Respiration cost	d^{-1}	$25\cdot10^{-3}$
B_C	Cost of biosynthesis	gC/site	0.00
M_n^N	Site maintenance cost	–	0.00
M_N	Respiration cost	d^{-1}	$25\cdot10^{-3}$
B_N	Cost of biosynthesis	gN/site	0.00