Supplementary Information for "Dynamical robustness in complex networks: the crucial role of low-degree nodes"

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TABLE OF CONTENTS

I. CRITICAL RATIO FOR RANDOM INACTIVATION IN HETEROGENEOUS NETWORKS

We derive the critical ratio p_c for random inactivation in heterogeneous networks. As described in the Methods section of the main text, the degree-weighted mean field approximation yields the approximated form of the local field as follows:

$$h_j \equiv \sum_{k=1}^{N} A_{jk} z_k \simeq (1-p) k_j H_A(t) + p k_j H_I(t), \tag{1}$$

where the mean fields for active and inactive subpopulations are given by

$$H_A(t) \equiv \frac{\sum_{j \in S_A} k_j z_j(t)}{\sum_{j \in S_A} k_j} \text{ and } H_I(t) \equiv \frac{\sum_{j \in S_I} k_j z_j(t)}{\sum_{j \in S_I} k_j}. \tag{2}$$

Figure S1 shows that the local fields of individual oscillators are well approximated by equation (1). Therefore, the original system equation (3) in the main text is approximated as follows:

$$\dot{z}_j = (\alpha_j + i\Omega - |z_j|^2) z_j + \frac{K k_j}{N} \left((1-p) H_A(t) + p H_I(t) - z_j \right). \tag{3}$$

From numerical observation, we have confirmed that all the oscillators exhibit phase synchronization with frequency Ω. Thus, we suppose that the state variables can be written as $z_j(t) = r_j(t) \exp(i(\Omega t + \theta))$, where r_j is the amplitude and θ, the phase shift. The mean fields are represented as follows:

$$H_A(t) = R_A(t) e^{i(\Omega t + \theta)} \text{ and } H_I(t) = R_I(t) e^{i(\Omega t + \theta)}, \tag{4}$$

where

$$R_A(t) = \frac{\sum_{j \in S_A} k_j r_j(t)}{\sum_{j \in S_A} k_j} \text{ and } R_I(t) = \frac{\sum_{j \in S_I} k_j r_j(t)}{\sum_{j \in S_I} k_j}. \tag{5}$$

By substituting equation (4) into equation (3), we obtain the following equation for the oscillation amplitude r_j:

$$\dot{r}_j = (\alpha_j - \frac{K k_j}{N} - r_j^2) r_j + \frac{K k_j}{N} ((1-p) R_A(t) + p R_I(t)). \tag{6}$$

We can assume that $R_A(t)$ and $R_I(t)$ are time-independent in the stationary oscillatory behavior. Once R_A and R_I are given, the stationary amplitude is obtained as $r_j^*(R_A, R_I)$ from
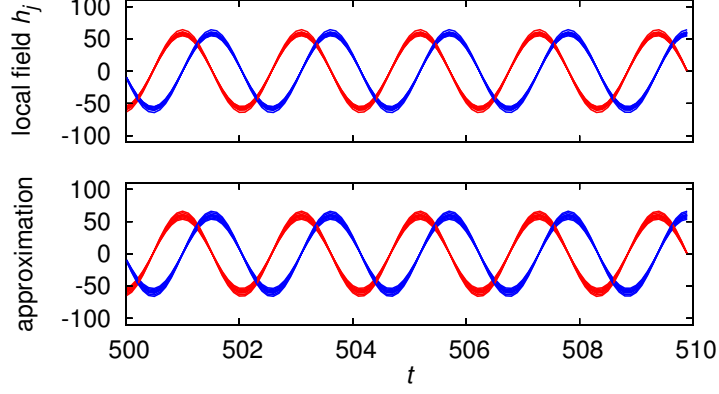
FIG. S1: **Validation of the mean field approximation.** The real (red curves) and imaginary (blue curves) parts of the original local fields $h_j = \sum_{k=1}^{N} A_{jk} z_k$ (upper panel) and the approximated local fields $h_j \simeq (1-p)k_j H_A(t) + pk_j H_I(t)$ (lower panel) of several oscillators in a heterogeneous network with $d \sim 0.08$ ($N = 1000$, $\langle k \rangle = 80$), $K = 30$, and $p = 0.5$.

equation (6). The self-consistency of the mean field approximation requires that the mean fields calculated from the stationary amplitudes are equivalent to the originally given mean fields. Hence, there should exist a solution $(R_A, R_I) = (R'_A, R'_I)$ satisfying $R'_A = G_A(R'_A, R'_I)$ and $R'_I = G_I(R'_A, R'_I)$, where

$$G_A(R_A, R_I) \equiv \frac{\sum_{j \in S_A} k_j r_j^*(R_A, R_I)}{\sum_{j \in S_A} k_j}, \tag{7}$$

$$G_I(R_A, R_I) \equiv \frac{\sum_{j \in S_I} k_j r_j^*(R_A, R_I)}{\sum_{j \in S_I} k_j}. \tag{8}$$

The solution with $R'_A > 0$ and $R'_I > 0$ is stable before the phase transition to the non-oscillatory regime, whereas the fixed point at $R'_A = R'_I = 0$ should be stable subsequently. Therefore, the phase transition takes place when the stability of the fixed point at the origin changes. The stability is determined by the following linearized matrix:

$$J_0 = \begin{bmatrix} \frac{\partial G_A(R_A, R_I)}{\partial R_A} & \frac{\partial G_A(R_A, R_I)}{\partial R_I} \\ \frac{\partial G_I(R_A, R_I)}{\partial R_A} & \frac{\partial G_I(R_A, R_I)}{\partial R_I} \end{bmatrix} \Bigg|_{R_A = R_I = 0} . \tag{9}$$

We evaluate the entries of J_0. From equation (6), the stationary amplitude r_j^* before the phase transition is given by a positive real solution of the following cubic equation:

$$r_j^3 - (\alpha_j - \frac{K k_j}{N}) r_j - \frac{K k_j}{N}((1-p)R_A + pR_I) = 0. \tag{10}$$

3

Here we assume $\alpha_j - Kk_j/N < 0$ for all $j \in S_A$ so that equation (10) has only one positive real root and the mean field approximation works well. This assumption holds independently of the choice of inactive oscillators if the minimum degree, denoted by $k_{min} \equiv \min_{1 \leq j \leq N} k_j$, is larger than aN/K. By differentiating the right-hand side of equation (7) with respect to R_A, the (1,1)th entry of J_0 is obtained as follows:

$$\left. \frac{\partial G_A}{\partial R_A} \right|_{R_A = R_I = 0} = \frac{(1-p)K}{\sum_{j \in S_A} k_j} \left(\frac{1}{N} \sum_{j \in S_A} \frac{k_j^2}{Kk_j/N - \alpha_j} \right)$$

$$\simeq \frac{1}{d} \left(\frac{1}{N} \sum_{j \in S_A} \frac{d_j^2}{d_j - \alpha_j/K} \right), \tag{11}$$

where $d_j = k_j/N$ is the degree of the jth oscillator, normalized by the system size. The above approximation comes from $\sum_{j \in S_A} k_j \simeq (1-p)dN^2$. Note that the average of the normalized degrees is equal to the link density $d = \langle k \rangle/(N-1)$ in the limit of $N \to \infty$. Similarly, we can approximate all the other entries of J_0. Moreover, the following approximations also hold:

$$\frac{1}{N} \sum_{j \in S_A} \frac{d_j^2}{d_j - \alpha_j/K} \simeq (1-p)F(K, a), \tag{12}$$

$$\frac{1}{N} \sum_{j \in S_I} \frac{d_j^2}{d_j - \alpha_j/K} \simeq pF(K, -b), \tag{13}$$

where

$$F(K, \alpha) \equiv \frac{1}{N} \sum_{j=1}^{N} \frac{d_j^2}{d_j - \alpha/K}. \tag{14}$$

Therefore, we obtain

$$J_0 = \begin{bmatrix} (1-p)F(K, a)/d & pF(K, a)/d \\ (1-p)F(K, -b)/d & pF(K, -b)/d \end{bmatrix}. \tag{15}$$

From the condition that the fixed point at $R_A = R_I = 0$ changes its stability at the phase transition point, we obtain the following critical ratio:

$$p_c^{het} = \frac{F(K, a) - d}{F(K, a) - F(K, -b)} \quad \text{for} \quad K > K_c^{het}. \tag{16}$$

This result is valid if $K > a/d_{min}$, where $d_{min} = k_{min}/N$. This condition guarantees our assumption that $a - Kk_j/N < 0$ for all $j \in S_A$. However, for a given network, the critical

4

coupling strength depends on the order according to which the randomly chosen oscillators are inactivated. In heterogeneous networks, an oscillator with a lower degree close to the minimum degree is more likely to be chosen. Hence, the critical coupling strength is given by $K_c^{het} \sim a/d_{min}$ ($\geq a/d$), below which $p_c^{het} = 1$. From equation (16), it is obvious that $p_c^{het} \to 1$ with $K \to K_c^{het} + 0$ and $p_c^{het} \to a/(a+b)$ with $K \to \infty$. Equation (16) is a general formula in the sense that it includes equation (19) for homogeneous networks in the next section II as a special case where $d_j = d$ for $j = 1, \ldots, N$.

II. CRITICAL RATIO FOR RANDOM INACTIVATION IN HOMOGENEOUS NETWORKS

We derive the critical ratio p_c for random inactivation in homogeneous networks. From the numerical observations that the oscillation amplitudes are almost the same within each subpopulation of active and inactive oscillators (see Fig. 2c in the main text), we consider that there is little effect of the difference in their degrees. Here, we assume that the degrees of all the oscillators are approximated by the mean degree, i.e. $k_j = \langle k \rangle$ for $j = 1, \ldots, N$. Then, the number of active oscillators in the neighborhood of each oscillator is expected to be $(1-p)\langle k \rangle$ and that of inactive oscillators, $p\langle k \rangle$. By setting $z_j = A$ for $j \in S_A$ and $z_j = I$ for $j \in S_I$ in the original equation (3) of the main text, we obtain the following reduced forms:

$$\dot{A} = (a - Kpd + i\Omega - |A|^2)A + KpdI, \tag{17}$$

$$\dot{I} = (-b - K(1-p)d + i\Omega - |I|^2)I + K(1-p)dA. \tag{18}$$

A linear stability analysis of the equilibrium point $A = I = 0$, which should be stable after the phase transition, yields

$$p_c^{hom} = \frac{a(Kd+b)}{(a+b)Kd} \quad \text{for} \quad K \geq K_c^{hom}. \tag{19}$$

When the link density d is fixed, the critical coupling strength is given by $K_c^{hom} = a/d$, below which $p_c^{hom} = 1$. It is obvious that $p_c^{hom} \to a/(a+b)$ with $K \to \infty$. Equation (19) shows that the critical ratio is invariant if the product of the link density d and the coupling strength K is kept constant. The critical ratio for the globally coupled (all-to-all) network[1], given by $p_c = a(K+b)/\{(a+b)K\}$, is included in equation (19) as a special case with $d = 1$.
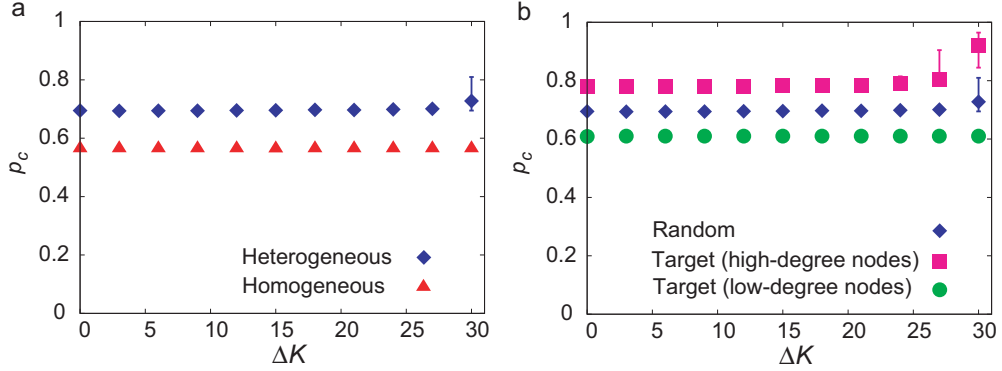
FIG. S2: **Effects of the uniform distribution of coupling strengths**. The crtical ratio p_c is plotted against ΔK in networks with $N = 3000$, where the coupling strengths K_{jk} are uniformly distributed in the range of $[\langle K \rangle - \Delta K, \langle K \rangle + \Delta K]$ with $\langle K \rangle = 30$. (a) Comparison between heterogeneous and homogeneous networks for random inactivation. (b) Comparison between random and targeted inactivation in heterogeneous networks.

III. EFFECTS OF HETEROGENEITY IN COUPLING STRENGTH

In the coupled oscillator model (3) described in the Methods section of the main text, the coupling strength K is a constant common to all the links. However, the intensity of the interaction between nodes in a networked system is not necessarily identical. Here, we focus on heterogeneity in the coupling strength. Namely, we assume a distribution for the coupling strengths assigned to the links. The model equation is described as follows:

$$\dot{z}_j = (\alpha_j + i\Omega - |z_j|^2)z_j + \frac{1}{N}\sum_{k=1}^{N}K_{jk}A_{jk}(z_k - z_j) \quad \text{for } j = 1, \ldots, N, \qquad (20)$$

where K_{jk} represents the coupling strength for the link between the jth and the kth nodes. We assume $K_{jk} = K_{kj}$ for simplicity.

First, we examine the case in which the coupling strengths are uniformly distributed. We assume that K_{jk} is randomly chosen from a uniform distribution in the range of $[\langle K \rangle - \Delta K, \langle K \rangle + \Delta K]$, where $\langle K \rangle$ is the mean coupling strength and ΔK is the half of the width of the range. Figure S2 shows plots of the critical ratio p_c against ΔK in networks with $N = 3000$ and $\langle K \rangle = 30$. Although the value of p_c varies slightly when the width of the range of the uniformly distributed coupling strengths is very large, the comparison of p_c values show that our main results on the dynamical robustness are unchanged. Figure S2a shows
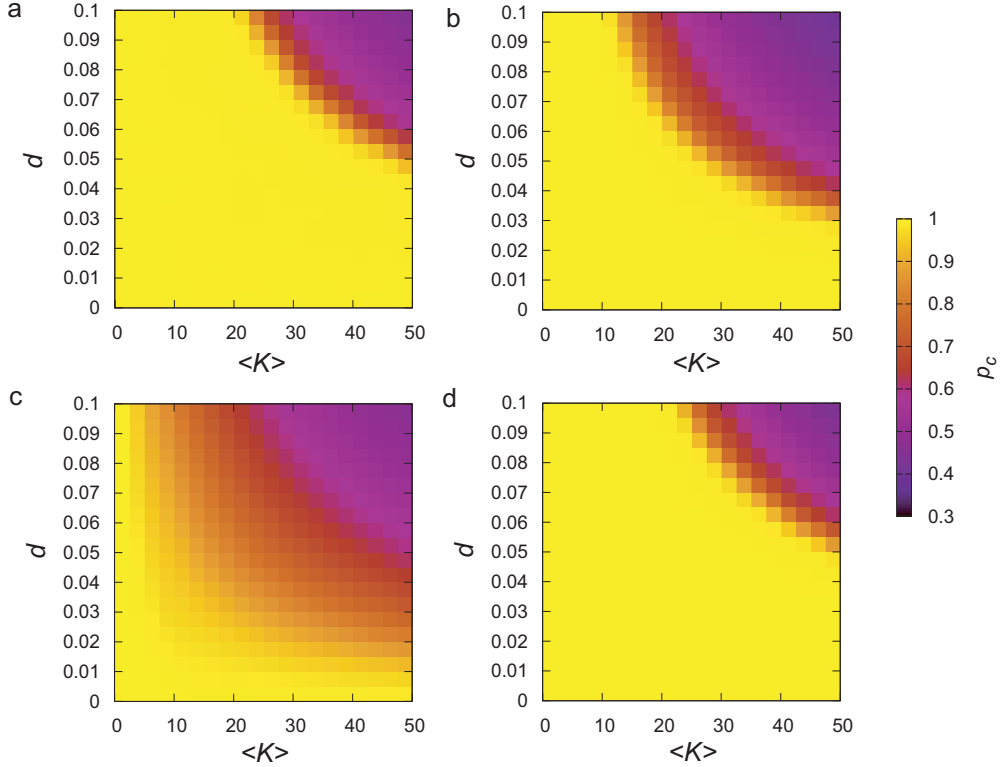
FIG. S3: **Dependence of the critical ratio on the mean coupling strength $\langle K \rangle$ and the link density d in the networks with heterogeneity in coupling strengths**. The color indicates the value of the crtical ratio p_c in networks with $N = 3000$ and $\Delta K = \langle K \rangle$. (a) Heterogeneous networks for random inactivation. (b) Homogeneous networks for random inactivation. (c) Heterogeneous networks for targeted inactivation of low-degree nodes. (d) Heterogeneous networks for targeted inactivation of high-degree nodes.

that heterogeneous networks are more tolerant to random inactivation than homogeneous networks, as also confirmed by the comparison between Figs. S3a and S3b. Figure S2b shows that heterogeneous networks are vulnerable to the targeted inactivation of low-degree nodes, as also seen by the comparison between Fig. S3a and S3c. The targeted inactivation of high-degree nodes has little effect on the critical ratio for random inactivation, as shown in Fig. S3a and S3d.

Second, we consider the case in which the coupling strengths are normally distributed. We assume that K_{jk} is randomly chosen from a truncated Gaussian distribution in the range of $[0, 2\langle K \rangle]$, where $\langle K \rangle$ is the mean coupling strength. The variance of the Gaussian
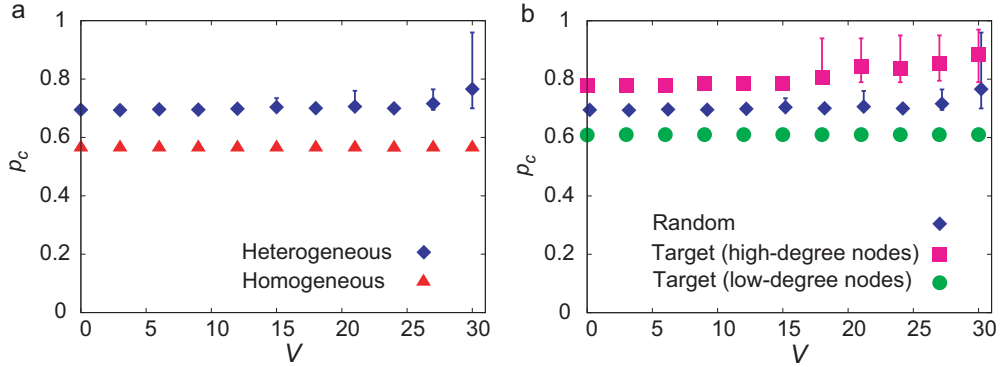
FIG. S4: **Effects of the Gaussian distribution of coupling strengths.** The crtical ratio p_c is plotted against V in networks with $N = 3000$ and $\langle K \rangle = 30$. (a) Comparison between heterogeneous and homogeneous networks for random inactivation. (b) Comparison between random and targeted inactivation in heterogeneous networks.

distribution is denoted by V. Figure S4 shows the effects of the variance on the critical ratio p_c in networks with $N = 3000$ and $K = 30$. The results indicate that our main results on the dynamical robustness also hold for networks with normally distributed coupling strengths, based on the average value of the critical ratio.

In summary, numerical results have verified that our main conclusions for the coupled oscillator model with identical coupling strengths are valid for the model with heterogeneous coupling strengths.

[1] Daido, H. & Nakanishi, K. Aging transition and universal scaling in oscillator networks. *Phys. Rev. Lett.*, **93**, 104101 (2004).