Supplementary Materials for Article "Seminparametric Stochastic Modeling of the Rate Function in Longitudinal Studies"

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This document supplies the technical details related to the proof of Theorem 1, including (a) the derivation of the solution for Ornstein-Uhlenbeck stochastic differential equation (OU SDE) and (b) the derivation of the solution of integrated Ornstein-Uhlenbeck stochastic differential equation (IOU SDE).

A Solution for OU SDE:

Let $Z(t) = V(t) - \bar{\nu}$. We have

$$dZ(t) = dV(t) = -\rho Z(t)dt + \sigma_\xi dW(t).$$

Due the fact that for any two continuous functions $g(t)$ and $f(t)$,

$$\int_{t_1}^{t_2} g(s)df(s) = g(t_2)f(t_2) - g(t_1)f(t_1) - \int_{t_1}^{t_2} f(s)dg(s) \ ,$$

we obtain

$$g(t_2)f(t_2) = g(t_1)f(t_1) + \int_{t_1}^{t_2} g(s)df(s) + \int_{t_1}^{t_2} f(s)dg(s) \ .$$

Thus, for $t_1 = t_{j-1}$, $t_2 = t_j$, $g(s) = Z(s)$ and $f(s) = \exp(\rho s)$,

$$\begin{aligned} Z(t_j)\exp(\rho t_j) &= Z(t_{j-1})\exp(\rho t_{j-1}) + \int_{t_{j-1}}^{t_j} Z(s)d\exp(\rho s) + \int_{t_{j-1}}^{t_j} \exp(\rho s)dZ(s) \\ &= Z(t_{j-1})\exp(\rho t_{j-1}) + \int_{t_{j-1}}^{t_j} \exp(\rho s)\left\{\rho Z(s)ds + dZ(s)\right\} \\ &= Z(t_{j-1})\exp(\rho t_{j-1}) + \int_{t_{j-1}}^{t_j} \exp(\rho s)\ \sigma_\xi dW(s). \end{aligned}$$

It follows that for $\delta_j = t_j - t_{j-1}$, we get

$$Z(t_j) = Z(t_{j-1})\exp(-\rho\delta_j) + \sigma_\xi \zeta_j \ ,$$

where $\zeta_j = \exp(-\rho t_j)\int_{t_{j-1}}^{t_j}\exp(\rho s)dW(s)$. It is easy to show that $E(\zeta_j) = 0$, and

$$\begin{aligned} E(\zeta_j^2) &= \exp(-2\rho t_j)E[\int_{t_{j-1}}^{t_j}\exp(\rho s_1)dW(s_1)\int_{t_{j-1}}^{t_j}\exp(\rho s_2)dW(s_2)] \\ &= \exp(-2\rho t_j)\int_{t_{j-1}}^{t_j}\exp(2\rho s)ds \\ &= \frac{1}{2\rho}\{1-\exp(-2\rho\delta_j)\}, \end{aligned}$$

and

$$Var(\zeta_j) = E(\zeta_j^2) = \frac{1}{2\rho}\left\{1-\exp(-2\rho\delta_j)\right\}.$$

Thus,

$$V(t_j) = \bar{\nu} + \{V(t_{j-1}) - \bar{\nu}\}\exp(-\rho\delta_j) + \sigma_\xi\zeta_j, \tag{1}$$

with $E\left\{V(t_j) \mid U(t_{j-1}), V(t_{j-1}), \bar{\nu}, \rho, \sigma_\xi\right\}$ and $Var\left\{V(t_j) \mid U(t_{j-1}), V(t_{j-1}), \bar{\nu}, \rho, \sigma_\xi)\right\}$ equal to the expressions given in Theorem 1.

B Solution for IOU SDE

From equation (1), we have

$$\begin{aligned} U(t_j) &= U(t_{j-1}) + \int_{t_{j-1}}^{t_j} V(t)dt \\ &= U(t_{j-1}) + \bar{\nu}\delta_j + \int_{t_{j-1}}^{t_j}\{V(t_{j-1}) - \bar{\nu}\}\exp\{-\rho(t-t_{j-1})\}dt + \\ &\quad \sigma_\varepsilon\int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^{t}\exp\{-\rho(t-s)\}dW(s)dt \\ &= U(t_{j-1}) + \bar{\nu}\delta_j + \{V(t_{j-1}) - \bar{\nu}\}\left\{\frac{1-\exp(-\rho\delta_j)}{\rho}\right\} + \\ &\quad \sigma_\varepsilon\int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^{t}\exp\{-\rho(t-s)\}dW(s)dt. \end{aligned}$$

This implies that

$$U(t_j) = U(t_{j-1}) + \bar{\nu}\delta_j + \{V(t_{j-1}) - \bar{\nu}\}\left\{\frac{1-\exp(-\rho\delta_j)}{\rho}\right\} + \sigma_\varepsilon\xi_j,$$

where

$$\begin{aligned}\xi_j &= \int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^{t} \exp\{-\rho(t-s)\}dW(s)dt \\ &= \int_{t_{j-1}}^{t_j}\int_{s}^{t_j} \exp\{-\rho(t-s)\}dtdW(s) \\ &= \frac{1}{\rho}\int_{t_{j-1}}^{t_j} \left[1-\exp\{-\rho(t_j-s)\}\right] dW(s).\end{aligned}$$

Moreover, we can show that $E[\xi_j]=0$, and

$$\begin{aligned}Var(\xi_j)=E(\xi_j^2) &= \frac{1}{\rho^2}\int_{t_{j-1}}^{t_j}\left[\exp\{-2\rho(t_j-s)\}-2\exp\{-\rho(t_j-s)\}+1\right] ds \\ &= \frac{\delta_j}{\rho^2}+\frac{1}{2\rho^3}\left\{-3+4\exp(-\rho\delta_j)-\exp(-2\rho\delta_j)\right\},\end{aligned}$$

$$\begin{aligned}E(\xi_j\zeta_j) &= E\left[\frac{1}{\rho}\int_{t_{j-1}}^{t_j}[1-\exp\{-\rho(t_j-s_1)\}dW(s_1)\int_{t_{j-1}}^{t_j}\exp\{-\rho(t_j-s_2)\}dW(s_2)\right] \\ &= \frac{1}{\rho}\int_{t_{j-1}}^{t_j}\left[\exp\{-\rho(t_j-s)\}-\exp\{-2\rho(t_j-s)\}\right] ds \\ &= \frac{1}{2\rho^2}\{1-2\exp(-\rho\delta_j)+\exp(-2\rho\delta_j)\}.\end{aligned}$$

Thus,

$$Cov(\xi_j,\zeta_j)=\frac{1}{2\rho^2}\{1-2\exp(-\rho\delta_j)+\exp(-2\rho\delta_j)\}.$$

Finally, it is trivial to show that $E\{U(t_j)\mid U(t_{j-1}),V(t_{j-1}),\bar{\nu},\rho,\sigma_\xi\}$, $Var\{U(t_j)\mid U(t_{j-1}),V(t_{j-1}),\bar{\nu},\rho,\sigma_\xi\}$ and $Cov\{U(t_j),V(t_j)\mid U(t_{j-1}),V(t_{j-1}),\bar{\nu},\rho,\sigma_\xi\}$ are the expressions given in Theorem 1.