Appendix A. Supplementary Data

Gap Function

The gap function represents the distance between the contacting surfaces $\gamma^{(1)}$ and $\gamma^{(2)}$. Each surface $\gamma^{(i)}$ $(i = 1, 2)$ is described by the positition of points on the surface, $\mathbf{x}^{(i)}\left(\eta^1_{(i)}, \eta^2_{(i)}\right)$, as a function of the given parametric coordinates $\eta^\alpha_{(i)}$ $(\alpha = 1, 2)$. Then, the covariant basis vectors on each surface are

$$\mathbf{g}^{(i)}_\alpha = \frac{\partial \mathbf{x}^{(i)}}{\partial \eta^\alpha_{(i)}}, \quad \alpha = 1, 2 \,. \tag{A.1}$$

The unit outward normal on each surface is

$$\mathbf{n}^{(i)} = \frac{\mathbf{g}^{(i)}_1 \times \mathbf{g}^{(i)}_2}{\left|\mathbf{g}^{(i)}_1 \times \mathbf{g}^{(i)}_2\right|} \,. \tag{A.2}$$

The gap function g is defined from

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + g\mathbf{n}^{(1)}, \quad g = \left(\mathbf{x}^{(2)} - \mathbf{x}^{(1)}\right) \cdot \mathbf{n}^{(1)} \,. \tag{A.3}$$

Directional Derivatives

The directional derivatives of the position, effective pressure and effective concentration, as well as equivalent virtual increments, are given by

$$\begin{aligned}
D\mathbf{x}^{(1)} &= \Delta\mathbf{u}^{(1)}, & D\mathbf{x}^{(2)} &= \Delta\mathbf{u}^{(2)} + \mathbf{g}^{(2)}_\alpha \Delta\eta^\alpha_{(2)} \\
D\tilde{p}^{(1)} &= \Delta\tilde{p}^{(1)}, & D\tilde{p}^{(2)} &= \Delta\tilde{p}^{(2)} + \frac{\partial \tilde{p}^{(2)}}{\partial \eta^\alpha_{(2)}} \Delta\eta^\alpha_{(2)} \\
D\tilde{c}^{(1)} &= \Delta\tilde{c}^{(1)}, & D\tilde{c}^{(2)} &= \Delta\tilde{c}^{(2)} + \frac{\partial \tilde{c}^{(2)}}{\partial \eta^\alpha_{(2)}} \Delta\eta^\alpha_{(2)} \\
D\delta\mathbf{v}^{(1)} &= \mathbf{0}, & D\mathbf{v}^{(2)} &= \frac{\partial \delta\mathbf{v}^{(2)}}{\partial \eta^\alpha_{(2)}} \Delta\eta^\alpha_{(2)} \\
D\delta\tilde{p}^{(1)} &= 0, & D\delta\tilde{p}^{(2)} &= \frac{\partial \delta\tilde{p}^{(2)}}{\partial \eta^\alpha_{(2)}} \Delta\eta^\alpha_{(2)} \\
D\delta\tilde{c}^{(1)} &= 0, & D\delta\tilde{c}^{(2)} &= \frac{\partial \delta\tilde{c}^{(2)}}{\partial \eta^\alpha_{(2)}} \Delta\eta^\alpha_{(2)}
\end{aligned} \,. \tag{A.4}$$

In these expressions,

$$\Delta\eta^{\alpha}_{(2)} = \left(\Delta\mathbf{u}^{(1)} - \Delta\mathbf{u}^{(2)}\right)\cdot a^{\alpha\beta}\mathbf{g}^{(1)}_{\beta} - a^{\alpha\beta} g \mathbf{n}^{(1)}\cdot\frac{\partial\Delta\mathbf{u}^{(1)}}{\partial\eta^{\beta}_{(1)}}\,, \tag{A.5}$$

where $a^{\alpha\beta} = (A_{\alpha\beta})^{-1}$ and $A_{\alpha\beta} = \mathbf{g}^{(1)}_{\alpha}\cdot\mathbf{g}^{(2)}_{\beta}$. Similarly, it can be shown that

$$\begin{aligned} Dt_n &= \varepsilon_n\left(\Delta\mathbf{u}^{(2)} - \Delta\mathbf{u}^{(1)} + \mathbf{g}^{(2)}_{\alpha}\Delta\eta^{\alpha}_{(2)}\right)\cdot\mathbf{n}^{(1)} \\ Dw_n &= \varepsilon_p\left(\Delta\tilde{p}^{(1)} - \Delta\tilde{p}^{(2)} - \frac{\partial\tilde{p}^{(2)}}{\partial\eta^{\alpha}_{(2)}}\Delta\eta^{\alpha}_{(2)}\right) \\ Dj_n &= \varepsilon_c\left(\Delta\tilde{c}^{(1)} - \Delta\tilde{c}^{(2)} - \frac{\partial\tilde{c}^{(2)}}{\partial\eta^{\alpha}_{(2)}}\Delta\eta^{\alpha}_{(2)}\right) \end{aligned}. \tag{A.6}$$

Given these relations, the directional derivative of the various terms appearing in the integrand of δG_c are

$$\begin{aligned} &D\left(t_n\left(\delta\mathbf{v}^{(1)} - \delta\mathbf{v}^{(2)}\right)\cdot\mathbf{g}^{(1)}_1\times\mathbf{g}^{(1)}_2\right) \\ &= -J^{(1)}_{\eta}\varepsilon_n\left(\delta\mathbf{v}^{(1)} - \delta\mathbf{v}^{(2)}\right)\cdot\left(\mathbf{n}^{(1)}\otimes\mathbf{n}^{(1)}\right)\cdot\left(\Delta\mathbf{u}^{(1)} - \Delta\mathbf{u}^{(2)}\right) \\ &+ J^{(1)}_{\eta} t_n\frac{\partial\delta\mathbf{v}^{(2)}}{\partial\eta^{\alpha}_{(2)}}\cdot\left(\mathbf{n}^{(2)}\otimes\mathbf{g}^{\alpha}_{(2)}\right)\cdot\left(\Delta\mathbf{u}^{(1)} - \Delta\mathbf{u}^{(2)}\right) \\ &+ t_n\left(\delta\mathbf{v}^{(1)} - \delta\mathbf{v}^{(2)}\right)\cdot\left(\frac{\partial\Delta\mathbf{u}^{(1)}}{\partial\eta^{1}_{(1)}}\times\mathbf{g}^{(1)}_2 + \mathbf{g}^{(1)}_1\times\frac{\partial\Delta\mathbf{u}^{(1)}}{\partial\eta^{2}_{(1)}}\right) \end{aligned}, \tag{A.7}$$

$$\begin{aligned} &D\left(w_n\left(\delta\tilde{p}^{(1)} - \delta\tilde{p}^{(2)}\right)\left|\mathbf{g}^{(1)}_1\times\mathbf{g}^{(1)}_2\right|\right) \\ &= J^{(1)}_{\eta}\varepsilon_p\left(\delta\tilde{p}^{(1)} - \delta\tilde{p}^{(2)}\right)\left(\Delta\tilde{p}^{(1)} - \Delta\tilde{p}^{(2)}\right) \\ &- J^{(1)}_{\eta}\left[\varepsilon_p\left(\delta\tilde{p}^{(1)} - \delta\tilde{p}^{(2)}\right)\frac{\partial\tilde{p}^{(1)}}{\partial\eta^{\alpha}_{(1)}}\mathbf{g}^{\alpha}_{(1)} + w_n\frac{\partial\delta\tilde{p}^{(2)}}{\partial\eta^{\alpha}_{(2)}}\mathbf{g}^{\alpha}_{(2)}\right]\cdot\left(\Delta\mathbf{u}^{(1)} - \Delta\mathbf{u}^{(2)}\right), \\ &+ w_n\left(\delta\tilde{p}^{(1)} - \delta\tilde{p}^{(2)}\right)\mathbf{n}^{(1)}\cdot\left(\frac{\partial\Delta\mathbf{u}^{(1)}}{\partial\eta^{1}_{(1)}}\times\mathbf{g}^{(1)}_2 + \mathbf{g}^{(1)}_1\times\frac{\partial\Delta\mathbf{u}^{(1)}}{\partial\eta^{2}_{(1)}}\right) \end{aligned} \tag{A.8}$$

and

$$\begin{aligned}
&D\left(j_n\left(\delta\tilde{c}^{(1)}-\delta\tilde{c}^{(2)}\right)\left|\mathbf{g}_1^{(1)}\times\mathbf{g}_2^{(1)}\right|\right)\\
&=J_\eta^{(1)}\varepsilon_c\left(\delta\tilde{c}^{(1)}-\delta\tilde{c}^{(2)}\right)\left(\Delta\tilde{c}^{(1)}-\Delta\tilde{c}^{(2)}\right)\\
&-J_\eta^{(1)}\left[\varepsilon_c\left(\delta\tilde{c}^{(1)}-\delta\tilde{c}^{(2)}\right)\frac{\partial\tilde{c}^{(1)}}{\partial\eta_{(1)}^{\alpha}}\mathbf{g}_{(1)}^{\alpha}+j_n\frac{\partial\delta\tilde{c}^{(2)}}{\partial\eta_{(2)}^{\alpha}}\mathbf{g}_{(2)}^{\alpha}\right]\cdot\left(\Delta\mathbf{u}^{(1)}-\Delta\mathbf{u}^{(2)}\right),\\
&+j_n\left(\delta\tilde{c}^{(1)}-\delta\tilde{c}^{(2)}\right)\mathbf{n}^{(1)}\cdot\left(\frac{\partial\Delta\mathbf{u}^{(1)}}{\partial\eta_{(1)}^{1}}\times\mathbf{g}_2^{(1)}+\mathbf{g}_1^{(1)}\times\frac{\partial\Delta\mathbf{u}^{(1)}}{\partial\eta_{(1)}^{2}}\right)
\end{aligned}\tag{A.9}$$

where

$$J_\eta^{(1)}=\left|\mathbf{g}_1^{(1)}\times\mathbf{g}_2^{(1)}\right|\,. \tag{A.10}$$

Discretization

The contact integral may be discretized as

$$\begin{aligned}\delta G_c=\sum_{e=1}^{n_e^{(1)}}\sum_{k=1}^{n_{\text{int}}^{(e)}}W_kJ_\eta^{(1)}\Big[t_n\left(\delta\mathbf{v}^{(1)}-\delta\mathbf{v}^{(2)}\right)\cdot\mathbf{n}^{(1)}&\,,\\
+w_n\left(\delta\tilde{p}^{(1)}-\delta\tilde{p}^{(2)}\right)+j_n\left(\delta\tilde{c}^{(1)}-\delta\tilde{c}^{(2)}\right)\Big]&\end{aligned}\tag{A.11}$$

where $n_e^{(1)}$ is the number of element faces on $\gamma^{(1)}$, $n_{\text{int}}^{(e)}$ is the number of integration points on the $e-$th element face of $\gamma^{(1)}$, W_k is the weight associated with the $k-$th integration point. In this expression it should be understood that terms associated with $\gamma^{(1)}$ (such as $J_\eta^{(1)}$,$\delta\mathbf{v}^{(1)}$, t_n, etc.) are evaluated at the parametric coordinates $\left(\eta_{(1)}^1,\eta_{(1)}^2\right)$ of the $k-$th integration point $\mathbf{x}^{(1)}$ on $\gamma^{(1)}$, whereas terms associated with $\gamma^{(2)}$ (such as $\delta\mathbf{v}^{(2)}$, $\delta\tilde{p}^{(2)}$, etc.) are evaluated at the parametric coordinates $\left(\eta_{(2)}^1,\eta_{(2)}^2\right)$ of the point $\mathbf{x}^{(2)}$ on $\gamma^{(2)}$ closest to that integration point on $\gamma^{(1)}$, in accordance with (A.3).

The variables may be interpolated over each element face according to

$$
\begin{aligned}
\delta\mathbf{v}^{(1)} &= \sum_{a=1}^{m^{(1)}} N_a^{(1)}\delta\mathbf{v}_a^{(1)} & \delta\mathbf{v}^{(2)} &= \sum_{b=1}^{m^{(2)}} N_b^{(2)}\delta\mathbf{v}_b^{(2)} \\
\Delta\mathbf{u}^{(1)} &= \sum_{c=1}^{m^{(1)}} N_c^{(1)}\Delta\mathbf{u}_c^{(1)} & \Delta\mathbf{u}^{(2)} &= \sum_{d=1}^{m^{(2)}} N_d^{(2)}\Delta\mathbf{u}_d^{(2)} \\
\delta\tilde{p}^{(1)} &= \sum_{a=1}^{m^{(1)}} N_a^{(1)}\delta\tilde{p}_a^{(1)} & \delta\tilde{p}^{(2)} &= \sum_{b=1}^{m^{(2)}} N_b^{(2)}\delta\tilde{p}_b^{(2)} \\
\Delta\tilde{p}^{(1)} &= \sum_{c=1}^{m^{(1)}} N_c^{(1)}\Delta\tilde{p}_c^{(1)} & \Delta\tilde{p}^{(2)} &= \sum_{d=1}^{m^{(2)}} N_d^{(2)}\Delta\tilde{p}_d^{(2)} \\
\delta\tilde{c}^{(1)} &= \sum_{a=1}^{m^{(1)}} N_a^{(1)}\delta\tilde{c}_a^{(1)} & \delta\tilde{c}^{(2)} &= \sum_{b=1}^{m^{(2)}} N_b^{(2)}\delta\tilde{c}_b^{(2)} \\
\Delta\tilde{c}^{(1)} &= \sum_{c=1}^{m^{(1)}} N_c^{(1)}\Delta\tilde{c}_c^{(1)} & \Delta\tilde{c}^{(2)} &= \sum_{d=1}^{m^{(2)}} N_d^{(2)}\Delta\tilde{c}_d^{(2)}
\end{aligned}
\tag{A.12}
$$

where $N_a^{(i)}\left(\eta^1_{(i)}, \eta^2_{(i)}\right)$ are the shape functions of element faces on $\gamma^{(i)}$ and $m^{(i)}$ is the number of nodes on an element face. Then

$$
\begin{aligned}
\delta G_c = \sum_{e=1}^{n_e^{(1)}} \sum_{k=1}^{n_{\text{int}}^{(e)}} W_k J_\eta^{(1)} \Bigg(& \sum_{a=1}^{m^{(1)}} \begin{bmatrix} \delta\mathbf{v}_a^{(1)} & \delta\tilde{p}_a^{(1)} & \delta\tilde{c}_a^{(1)} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{f}_a^{(1)} \\ w_a^{(1)} \\ j_a^{(1)} \end{bmatrix} \\
& + \sum_{b=1}^{m_k^{(2)}} \begin{bmatrix} \delta\mathbf{v}_{b,k}^{(1)} & \delta\tilde{p}_{b,k}^{(1)} & \delta\tilde{c}_{b,k}^{(1)} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{f}_{b,k}^{(1)} \\ w_{b,k}^{(1)} \\ j_{b,k}^{(1)} \end{bmatrix} \Bigg)
\end{aligned}
\tag{A.13}
$$

where

$$
\begin{aligned}
\mathbf{f}_a^{(1)} &= N_a^{(1)} t_n \mathbf{n}^{(1)} & \mathbf{f}_{b,k}^{(2)} &= -N_b^{(2)} t_n \mathbf{n}^{(1)} \\
w_a^{(1)} &= N_a^{(1)} w_n & w_{b,k}^{(2)} &= -N_b^{(2)} w_n \\
j_a^{(1)} &= N_a^{(1)} j_n & j_{b,k}^{(2)} &= -N_b^{(2)} j_n
\end{aligned}
\tag{A.14}
$$

Similarly,

$$
\begin{aligned}
-D\delta G_c = &\sum_{e=1}^{n_e^{(1)}} \sum_{k=1}^{n_{\text{int}}^{(e)}} W_k J_\eta^{(1)} \\
&\times \left(\sum_{a=1}^{m^{(1)}} \begin{bmatrix} \delta \mathbf{v}_a^{(1)} & \delta \tilde{p}_a^{(1)} & \delta \tilde{c}_a^{(1)} \end{bmatrix} \cdot \right. \\
&\left(\sum_{c=1}^{m^{(1)}} \begin{bmatrix} \mathbf{K}_{ac}^{(1,1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{g}_{ac}^{(1,1)} & g_{ac}^{(1,1)} & 0 \\ \mathbf{h}_{ac}^{(1,1)} & 0 & h_{ac}^{(1,1)} \end{bmatrix} \cdot \begin{bmatrix} \Delta \mathbf{u}_c^{(1)} \\ \Delta \tilde{p}_c^{(1)} \\ \Delta \tilde{c}_c^{(1)} \end{bmatrix} \right. \\
&\left. + \sum_{d=1}^{m_k^{(2)}} \begin{bmatrix} \mathbf{K}_{ad,k}^{(1,2)} & \mathbf{0} & \mathbf{0} \\ \mathbf{g}_{ad,k}^{(1,2)} & g_{ad,k}^{(1,2)} & 0 \\ \mathbf{h}_{ad,k}^{(1,2)} & 0 & h_{ad,k}^{(1,2)} \end{bmatrix} \cdot \begin{bmatrix} \Delta \mathbf{u}_d^{(2)} \\ \Delta \tilde{p}_d^{(2)} \\ \Delta \tilde{c}_d^{(2)} \end{bmatrix} \right) \\
&+ \sum_{b=1}^{m_k^{(2)}} \begin{bmatrix} \delta \mathbf{v}_{b,k}^{(2)} & \delta \tilde{p}_{b,k}^{(2)} & \delta \tilde{c}_{b,k}^{(2)} \end{bmatrix} \cdot \\
&\left(\sum_{c=1}^{m^{(1)}} \begin{bmatrix} \mathbf{K}_{bc,k}^{(2,1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{g}_{bc,k}^{(2,1)} & g_{bc,k}^{(2,1)} & 0 \\ \mathbf{h}_{bc,k}^{(2,1)} & 0 & h_{bc,k}^{(2,1)} \end{bmatrix} \cdot \begin{bmatrix} \Delta \mathbf{u}_c^{(1)} \\ \Delta \tilde{p}_c^{(1)} \\ \Delta \tilde{c}_c^{(1)} \end{bmatrix} \right. \\
&\left.\left. + \sum_{d=1}^{m_k^{(2)}} \begin{bmatrix} \mathbf{K}_{bd,k}^{(2,2)} & \mathbf{0} & \mathbf{0} \\ \mathbf{g}_{bd,k}^{(2,2)} & g_{bd,k}^{(2,2)} & 0 \\ \mathbf{h}_{bd,k}^{(2,2)} & 0 & h_{bd,k}^{(2,2)} \end{bmatrix} \cdot \begin{bmatrix} \Delta \mathbf{u}_d^{(2)} \\ \Delta \tilde{p}_d^{(2)} \\ \Delta \tilde{c}_d^{(2)} \end{bmatrix} \right) \right), \qquad \text{(A.15)}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{K}_{ac}^{(1,1)} &= N_a^{(1)} \left(\varepsilon_n N_c^{(1)} \mathbf{N}^{(1)} + t_n \mathbf{A}_c^{(1)} \right) \\
\mathbf{K}_{ad,k}^{(1,2)} &= -\varepsilon_n N_a^{(1)} N_d^{(2)} \mathbf{N}^{(1)} \\
\mathbf{K}_{bc,k}^{(2,1)} &= -N_c^{(1)} \left(\varepsilon_n N_b^{(2)} \mathbf{N}^{(1)} + t_n \mathbf{M}_b^{(2)} \right) - t_n N_b^{(2)} \mathbf{A}_c^{(1)}, \\
\mathbf{K}_{bd,k}^{(2,2)} &= N_d^{(2)} \left(\varepsilon_n N_b^{(2)} \mathbf{N}^{(1)} + t_n \mathbf{M}_b^{(2)} \right)
\end{aligned} \qquad \text{(A.16)}
$$

$$
\begin{aligned}
\mathbf{g}_{ac}^{(1,1)} &= N_a^{(1)}\left(\varepsilon_p N_c^{(1)}\mathbf{p}^{(1)} - w_n \mathbf{A}_c^{(1)} \cdot \mathbf{n}^{(1)}\right)\\
\mathbf{g}_{ad,k}^{(1,2)} &= -\varepsilon_p N_a^{(1)} N_d^{(2)} \mathbf{p}^{(1)}\\
\mathbf{g}_{bc,k}^{(2,1)} &= N_c^{(1)}\left(-\varepsilon_p N_b^{(2)}\mathbf{p}^{(1)} + w_n \mathbf{m}_b^{(2)}\right) + w_n N_b^{(2)} \mathbf{A}_c^{(1)} \cdot \mathbf{n}^{(1)}\,,\\
\mathbf{g}_{bd,k}^{(2,2)} &= N_d^{(2)}\left(\varepsilon_p N_b^{(2)}\mathbf{p}^{(1)} - w_n \mathbf{m}_b^{(2)}\right)
\end{aligned}
\tag{A.17}
$$

$$
\left.\begin{aligned}
g_{ac}^{(1,1)} &= -\varepsilon_p N_a^{(1)} N_c^{(1)}\\
g_{ad,k}^{(1,2)} &= \varepsilon_p N_a^{(1)} N_d^{(2)}\\
g_{bc,k}^{(2,1)} &= \varepsilon_p N_b^{(2)} N_c^{(1)}\\
g_{bd,k}^{(2,2)} &= -\varepsilon_p N_b^{(2)} N_d^{(2)}
\end{aligned}\right.\,,
\tag{A.18}
$$

$$
\begin{aligned}
\mathbf{h}_{ac}^{(1,1)} &= N_a^{(1)}\left(\varepsilon_c N_c^{(1)}\mathbf{q}^{(1)} - j_n \mathbf{A}_c^{(1)} \cdot \mathbf{n}^{(1)}\right)\\
\mathbf{h}_{ad,k}^{(1,2)} &= -\varepsilon_c N_a^{(1)} N_d^{(2)} \mathbf{q}^{(1)}\\
\mathbf{h}_{bc,k}^{(2,1)} &= N_c^{(1)}\left(-\varepsilon_c N_b^{(2)}\mathbf{q}^{(1)} + j_n \mathbf{m}_b^{(2)}\right) + j_n N_b^{(2)} \mathbf{A}_c^{(1)} \cdot \mathbf{n}^{(1)}\,,\\
\mathbf{h}_{bd,k}^{(2,2)} &= N_d^{(2)}\left(\varepsilon_c N_b^{(2)}\mathbf{q}^{(1)} - j_n \mathbf{m}_b^{(2)}\right)
\end{aligned}
\tag{A.19}
$$

$$
\left.\begin{aligned}
h_{ac}^{(1,1)} &= -\varepsilon_c N_a^{(1)} N_c^{(1)}\\
h_{ad,k}^{(1,2)} &= \varepsilon_c N_a^{(1)} N_d^{(2)}\\
h_{bc,k}^{(2,1)} &= \varepsilon_c N_b^{(2)} N_c^{(1)}\\
h_{bd,k}^{(2,2)} &= -\varepsilon_c N_b^{(2)} N_d^{(2)}
\end{aligned}\right.\,,
\tag{A.20}
$$

and

$$
\begin{aligned}
\mathbf{N}^{(1)} &= \mathbf{n}^{(1)} \otimes \mathbf{n}^{(1)} \\
\mathbf{A}_c^{(1)} &= \frac{1}{J_\eta^{(1)}} \mathcal{A} \left\{ \frac{\partial N_c^{(1)}}{\partial \eta_{(1)}^1} \mathbf{g}_2^{(1)} - \frac{\partial N_c^{(1)}}{\partial \eta_{(1)}^2} \mathbf{g}_1^{(1)} \right\} \\
\mathbf{M}_b^{(2)} &= \mathbf{n}^{(2)} \otimes \mathbf{m}_b^{(2)} \\
\mathbf{m}_b^{(2)} &= \frac{\partial N_b^{(2)}}{\partial \eta_{(2)}^\alpha} \mathbf{g}_{(2)}^\alpha \\
\mathbf{p}^{(1)} &= \frac{\partial \tilde{p}^{(1)}}{\partial \eta_{(1)}^\alpha} \mathbf{g}_{(1)}^\alpha \\
\mathbf{q}^{(1)} &= \frac{\partial \tilde{c}^{(1)}}{\partial \eta_{(1)}^\alpha} \mathbf{g}_{(1)}^\alpha
\end{aligned}
\quad . \tag{A.21}
$$

Note that the operator $\mathcal{A}\{\mathbf{v}\}$ represents the skew-symmetric tensor whose dual vector is $\mathbf{v}$. These relations provide the contributions to the global finite element stiffness matrix and load vector resulting from the contact interface.

Flux Relations

The solvent volumetric flux relative to the solid, $\mathbf{w}$, and the solute molar flux relative to the solid, $\mathbf{j}$, are related to the gradients of the effective fluid pressure and solute concentration via

$$
\begin{aligned}
\mathbf{w} &= -\tilde{\mathbf{k}} \cdot \left(\operatorname{grad} \tilde{p} + R\theta \frac{\tilde{\kappa}}{d_0} \mathbf{d} \cdot \operatorname{grad} \tilde{c} \right) \\
\mathbf{j} &= \tilde{\kappa} \mathbf{d} \cdot \left(-\varphi^w \operatorname{grad} \tilde{c} + \frac{\tilde{c}}{d_0} \mathbf{w} \right)
\end{aligned}
\quad , \tag{A.22}
$$

where

$$
\begin{aligned}
\tilde{\mathbf{k}} &= \left[\mathbf{k}^{-1} + \frac{R\theta}{\varphi^w} \frac{\tilde{\kappa}\tilde{c}}{d_0} \left(\mathbf{I} - \frac{\mathbf{d}}{d_0} \right) \right]^{-1} \\
\varphi^w &= 1 - \frac{\varphi_r^s}{J}
\end{aligned}
\quad , \tag{A.23}
$$

and φ_r^s is the solid volume fraction in the reference configuration. In these expressions $\mathbf{d}$ is the solute diffusivity tensor in the mixture, d_0 is the solute diffusivity in free solution (no solid matrix), and $\mathbf{k}$ is the hydraulic permeability tensor for the flow of pure solvent through the porous solid matrix; $\tilde{\mathbf{k}}$ is the hydraulic permeability of the solution (solvent+solute) through the porous matrix.

Reduction to Classical Diffusion-Convection Relations

The biphasic-solute equations summarized in (2.1)-(2.3) and (A.22) may be reduced to the classical equations of diffusion-convection by assuming that the solid deformation is static, $\mathbf{v}^s = \mathbf{0}$; that the physico-chemical behavior of dilute solutions is ideal and there is no volume exclusion of solute from the pore space of the solid matrix, $\tilde{\kappa} = 1$, $\Phi = 1$; and that the solute diffusivity in the porous medium is the same as in free solution, $\mathbf{d} = d_0\mathbf{I}$. In that case, $\tilde{c} = c$, $\tilde{p} = p - R\theta c$, and the flux relations of (A.22) become

$$\begin{aligned} \mathbf{w} &= -\mathbf{k} \cdot \operatorname{grad} p \\ \mathbf{j} &= -\varphi^w d_0 \operatorname{grad} c + c\mathbf{w} \end{aligned}$$

Furthermore, (2.2)-(2.3) reduce to

$$\begin{aligned} \operatorname{div} \mathbf{w} &= 0 \\ \varphi^w \frac{\partial c}{\partial t} - \varphi^w d_0 \operatorname{lap} c + \mathbf{w} \cdot \operatorname{grad} c &= 0 \end{aligned} \tag{A.24}$$

where 'lap' is the laplacian operator. For pure diffusion, let $\mathbf{w} = \mathbf{0}$; for steady-state convection, let $\partial c/\partial t = 0$.