# eAppendix to On the Nondifferential Misclassification of a Binary Confounder

**Result 2** For $A$, and $C$ binary, the partial control result holds for the effect of treatment on the treated on the RD, RR, and OR scales.

**Proof:** We will show that $E_{C'|A=1}[Y|A=1]{=}E[Y|A=1]{=}E[Y_1|A=1]$ and that $E_{C'|A=1}[Y|A=0]$ is between $E[Y|A=0]$ and $E[Y_0|A=1]$. It then follows immediately that the observed adjusted effect is between the crude and true effects on the RD, RR, and OR scales. First, note that $E[Y_1|A=1]=E[Y|A=1]$ by the consistency assumption, and

$$\begin{aligned} E_{C'|A=1}[Y|A=1] &= E[Y|A=1,C'=1]P[C'=1|A=1]+E[Y|A=1,C'=0]P[C'=0|A=1] \\ &= E[Y|A=1] \end{aligned}$$

We now prove that $E_{C'|A=1}[Y|A=0]$ is between $E[Y|A=0]$ and $E[Y_0|A=1]$. It is enough to show that $E[Y_0|A=1]-E_{C'|A=1}[Y|A=0]$ and $E_{C'|A=1}[Y|A=0]-E[Y|A=0]$ have the same sign, where

$$\begin{aligned} & E[Y_0|A=1]-E_{C'|A=1}[Y|A=0] \\ = & \left(E[Y|A=0,C=1]-E[Y|A=0,C=0]\right) \\ & \times\left(\{E[C|A=1,C'=1]-E[C|A=0,C'=1]\}\,E[C'|A=1]\right. \\ & \left.+\,\{E[C|A=1,C'=0]-E[C|A=0,C'=0]\}\,(1-E[C'|A=1])\right) \end{aligned}$$

and

$$\begin{aligned}
& E_{C'|A=1}[Y|A=0] - E[Y|A=0] \\
= \; & (E[Y_0|A=1] - E[Y|A=0]) - \big(E[Y_0|A=1] - E_{C'|A=1}[Y|A=0]\big) \\
= \; & (E[Y|A=0,C=1] - E[Y|A=0,C=0])\,(E[C|A=1] - E[C|A=0]) \\
& - (E[Y|A=0,C=1] - E[Y|A=0,C=0]) \\
& \times (\{E[C|A=1,C'=1] - E[C|A=0,C'=1]\}\,E[C'|A=1] \\
& + \{E[C|A=1,C'=0] - E[C|A=0,C'=0]\}\,(1 - E[C'|A=1])) \\
= \; & (E[Y|A=0,C=1] - E[Y|A=0,C=0]) \\
& \times (E[C|A=0,C'=1]\,\{E[C'|A=1] - E[C'|A=0]\} \\
& - E[C|A=0,C'=0]\,\{E[C'|A=1] - E[C'|A=0]\})
\end{aligned}$$

Then, it is enough to show that

$$\begin{aligned}
& \big\{E[C|A=1,C'=1] - E[C|A=0,C'=1]\big\}\,E[C'|A=1] \\
& + \big\{E[C|A=1,C'=0] - E[C|A=0,C'=0]\big\}\,\big(1 - E[C'|A=1]\big)
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
& E[C|A=0,C'=1]\,\big\{E[C'|A=1] - E[C'|A=0]\big\} \\
& - E[C|A=0,C'=0]\,\big\{E[C'|A=1] - E[C'|A=0]\big\}
\end{aligned} \tag{2}$$

have the same sign. We will prove this by showing that (1) and (2) are simultaneously either maximized or minimized at 0.

We consider two cases: $e \geq 1 - f$ and $e \leq 1 - f$, where $e$ is sensitivity and $f$ is specificity. In the first case $E[C|A = a, C']$ is increasing in $C'$ for all $a$, while in the second case it is decreasing in $C'$ for all $a$, with equality in both cases when $e = f = 0.5$.

We now prove that $E[C|A = 1] \leq E[C|A = 0] \iff E[C'|A = 1] \leq E[C'|A = 0]$ when $e \geq 1 - f$. That $E[C|A = 1] \leq E[C|A = 0] \iff E[C'|A = 1] \geq E[C'|A = 0]$ when $e \leq 1 - f$ follows by a similar argument. Note that

$$E[C'|A = 1] = E[C|A = 1]e + (1 - E[C|A = 1])\,(1 - f)$$

and

$$E[C'|A = 0] = E[C|A = 0]e + (1 - E[C|A = 0])\,(1 - f)$$

are both convex combinations of $e$ and $(1 - f)$. If $E[C|A = 1] > E[C|A = 0]$, then $E[C|A = 1] : (1 - E[C|A = 1]) > E[C|A = 0] : (1 - E[C|A = 0])$ and $E[C'|A = 1]$ will be closer to $e$ and therefore greater than $E[C'|A = 0]$. If $E[C|A = 1] < E[C|A = 0]$, the reverse relationship holds and $E[C'|A = 1] < E[C'|A = 0]$.

Now we turn our attention to the minima and maxima of expressions (1) and (2); we will find the extreme values with respect to $E[C|A, C']$. The derivative of (2) with respect to $E[C|A = 0, C' = 1]$ is $E[C'|A = 1] - E[C'|A = 0]$ and the derivative with respect to $E[C|A = 0, C' = 0]$ is $-\{E[C'|A = 1] - E[C'|A = 0]\}$. Therefore (2) is monotonic in both $E[C|A = 0, C' = 1]$ and $E[C|A = 0, C' = 0]$, and furthermore it is non-decreasing in one and non-increasing in the other. Therefore, the unconstrained extrema occur at $E[C|A = 0, C' = 1] = 1$ and $E[C|A = 0, C' = 0] = 0$, and at $E[C|A = 0, C' = 1] = 0$ and $E[C|A = 0, C' = 0] = 1$. But we know that when $e \geq 1 - f$ these conditional expectations are constrained to be nondecreasing in $C'$; therefore the constrained extrema occur at $E[C|A = 0, C' = 1] = 1$ and $E[C|A = 0, C' = 0] = 0$ and at $E[C|A = 0, C' = 1] = E[C|A = 0, C' = 0] = 1$. On the other hand, when $e \leq 1 - f$

these conditional expectations are constrained to be nonincreasing in $C'$ and therefore the constrained extrema occur at $E[C|A=0,C'=1]=0$ and $E[C|A=0,C'=0]=1$ and at $E[C|A=0,C'=1]=E[C|A=0,C'=0]=1$.

When $E[C|A=0,C'=1]=E[C|A=0,C'=0]$, (2) equals 0. If $E[C'|A=1]<E[C'|A=0]$, then this is the maximum of (2) if $e\geq 1-f$ and the minimum if $e\leq 1-f$. If $E[C'|A=1]>E[C'|A=0]$ then (2) is maximized at 0 if $e\leq 1-f$ and minimized at 0 if $e\geq 1-f$.

To find the extrema of (1), we restrict our attention to the case where $E[C'|A=0]<E[C'|A=1]$ and $e\geq 1-f$. We will show that (1) is also minimized at 0. Because (1) is equal to

$$E[C|A=1]-E[C|A=0,C'=1]P[C'=1|A=1]-E[C=1|A=0,C'=0]P[C'=0|A=1]$$

proving that it is minimized at 0 is equivalent to proving that

$$E[C|A=0,C'=1]P[C'=1|A=1]+E[C=1|A=0,C'=0]P[C'=0|A=1] \tag{3}$$

is maximized at $E[C|A=1]$. We will maximize (3) with respect to $e$ and $f$.

Let $D=E[C|A=0]$ and $B=E[C|A=1]$.

$$\begin{aligned} & E[C|A=0,C'=1]P[C'=1|A=1]-E[C=1|A=0,C'=0]P[C'=0|A=1] \\ = \; & D\left(e\frac{Be+(1-B)(1-f)}{De+(1-D)(1-f)}+(1-e)\,\frac{1-\{Be+(1-B)(1-f)\}}{1-\{De+(1-D)(1-f)\}}\right) \end{aligned} \tag{4}$$

If $e\geq 1-f$, then $D\leq B$ and (4) is increasing in $e$ (the derivative with respect to $e$ is positive), so in order to maximize it we set $e = 1$. Now (4) reduces to $D\frac{B+(1-B)(1-f)}{D+(1-D)(1-f)}$, which is increasing in $f$ (the derivative with respect to $f$ is positive) and is therefore maximized at $f = 1$. At $e = f = 1$, $\sum_{C'} E[C|A = 0, C']P[C'|A = 1] = E[C|A = 1]$. Therefore, (1) is minimized at 0.

That (1) is minimized at 0 when $E[C'|A = 0] < E[C'|A = 1]$ and $e \leq 1 - f$, and that it is maximized at 0 when $E[C'|A = 0] < E[C'|A = 1]$ and $e \geq 1 - f$ or when $E[C'|A = 0] > E[C'|A = 1]$ and $e \leq 1 - f$, follow by analogous arguments.

**Result 3** For $A$ and $C$ binary, if sensitivity+specificity $\geq 1$ then the bias of the observed adjusted effect of treatment on the treated decreases with increasing sensitivity and specificity. If in addition $E[Y|A, C]$ is monotonic in $C$, then the bias of the observed adjusted average treatment effect decreases with increasing sensitivity and specificity.

**Proof:** Under monotonicity, $E_{C'}[Y|A = a]$ lies between $E[Y_a]$ and $E[Y|A = a]$ for $a = 0, 1$. We will show that $E_{C'}[Y|A = 1]$ and $E_{C'}[Y|A = 0]$ move further from $E[Y|A = a]$ and closer to $E[Y_a]$ as $e$ and $f$ increase. The result follows immediately.

When $e + f \geq 1$, the signs of $\frac{\partial}{\partial e}E_{C'}[Y|A = 1]$ and $\frac{\partial}{\partial f}E_{C'}[Y|A = 1]$ depend only on the sign of $\{E[Y|A = 1, C = 1] - E[Y|A = 1, C = 0]\}\{E[A|C = 0] - E[A|C = 1]\}$.When $E[Y|A, C]$ and $E[A|C]$ are both either non-increasing or non-decreasing in $C$ this product is negative (or 0) and $E_{C'}[Y|A = 1]$ is non-increasing in $e$ and $f$, i.e. as $e$ and $f$ increase $E_{C'}[Y|A = 1]$ moves closer to $E[Y_1]$ and farther from $E[Y|A = 1]$. (Recall that $E[Y_1] \leq E[Y|A = 1]$ when both conditional expectations are either non-increasing or non-decreasing in $C$.) When one of $E[Y|A, C]$ and $E[A|C]$ is non-increasing and the other non-decreasing then the product is non-negative and $E_{C'}[Y|A = 1]$ is non-increasing in $e$ and $f$, which again entails that as $e$ and $f$ increase $E_{C'}[Y|A = 1]$ moves closer to $E[Y_1]$ and further from $E[Y|A = 1]$.

The proof that $E_{C'}[Y|A = 0]$ approaches $E[Y_0]$ as $e$ and $f$ increase is similar.