Text S3 Average fixation probability

We wish to find the rate of change of the average probability of fixation of a focal allele,

$$\partial_t \bar{P} = \sum_{\underline{X}} g(\underline{X})\, \partial_t P(\underline{X}) + \sum_{\underline{X}} \partial_t g(\underline{X})\, P(\underline{X}) . \tag{21}$$

The first sum is given by the average of Eq. (2), weighted by genotype frequencies:

$$-\sum_{\underline{X}} g(\underline{X})\, \partial_t P(\underline{X}) = s\bar{P} + \sum_{\underline{X}} g(\underline{X})\, S(\underline{X})\, P(\underline{X}) + \sum_{\underline{X},\underline{Y}} g(\underline{X})\, r(\underline{X},\underline{Y})\, (P(\underline{Y}) - P(\underline{X})) - \frac{1}{2} \sum_{\underline{X}} g(\underline{X})\, P(\underline{X})^2 \tag{22}$$

To calculate the second sum in Eq. (21), we require the rate of change of background frequencies:

$$\partial_t g(\underline{X}) = S(\underline{X})\, g(\underline{X}) + \frac{1}{2} \sum_{\underline{Y}} \left(g(\underline{Y})\, R(\underline{Y},\underline{X}) - g(\underline{X})\, R(\underline{X},\underline{Y}) \right), \tag{23}$$

where $R(\underline{X},\underline{Y})$ is the rate at which individuals with genotype $\underline{X}$ recombine to form individuals with genotype $\underline{Y}$. (The factor of 1/2 in Eq. (23) is necessary because each recombination event involves two parents recombining to form two offspring.) Note that unlike $r(\underline{X},\underline{Y})$ (the rate at which recombination moves the focal allele from background $\underline{X}$ to background $\underline{Y}$), the definition of $R(\underline{X},\underline{Y})$ does not involve the focal allele, and in general $r(\underline{X},\underline{Y}) \neq \frac{1}{2} R(\underline{X},\underline{Y})$. However, it is true that $\sum_{\underline{X}} g(\underline{X})\, r(\underline{X},\underline{Y}) = \frac{1}{2} \sum_{\underline{X}} g(\underline{X})\, R(\underline{X},\underline{Y})$ for all $\underline{Y}$, since both sides are expressions for the total rate of recombination events producing offspring with genotype $\underline{Y}$. Thus, when we substitute Eqs. (22) and (23) into Eq. (21), we find that the terms involving S and r cancel, leaving

$$-\frac{\partial \bar{P}}{\partial t} = s\bar{P} - \frac{1}{2} \sum_{\underline{X}} g(\underline{X})\, P(\underline{X})^2 . \tag{24}$$

Rewriting the second term in terms of the mean and variance of $P(\underline{X})$, we obtain Eq. (3).