Supporting Information

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SI Text

In the following we endeavor to provide some technical details on the RG analysis discussed in the main text. A complete presentation will appear in a forthcoming technical paper.

The starting point of our approach is the replica mean field theory of structural glasses, in which one studies the distribution of putative metastable glassy states by introducing $m-1$ copies (or replicas) of the same liquid system coupled with a small attractive interaction whose amplitude is set to zero after taking the thermodynamic limit (1, 2). By keeping the leading terms in the local order parameter, which is the similarity or "overlap" between different states, one obtains the following Ginzburg-Landau functional (3, 4)

$$S[\mathbf{q}] = \int \frac{d^d x}{a_0^d}\left\{\frac{a_0^2}{2}\sum_{a,b=1}^{m}(\partial q_{ab}(x))^2 + V(\mathbf{q}(x))\right\} \qquad \textbf{[S1]}$$

with

$$V = \sum_{a,b=1}^{m}\left(\frac{t}{2}q_{ab}^2 - \frac{u+w}{3}q_{ab}^3 + \frac{y}{4}q_{ab}^4\right) - \frac{u}{3}\sum_{a,b,c=1}^{m} q_{ab}q_{bc}q_{ca}, \qquad \textbf{[S2]}$$

where $\mathbf{q}$ denotes the set of elements $\{q_{ab}\}$ (by construction, $q_{aa}=0$) and the overlap $q_{ab}(x)$ is physically associated with a local Debye-Waller factor characterizing molecular motion in the glass-forming liquid (3, 4). a_0 is the microscopic length scale corresponding to the first peak in the radial distribution function (henceforth we shall put $a_0=1$ and measure lengths in unit of a_0). For simplicity, the only temperature dependence is taken in $t=\frac{T-T_0}{T_0}$, with T_0 setting the temperature scale, while $u, w, y > 0$ are considered as independent of temperature. This "real replica" method allows one to obtain the properties of the metastable states from the knowledge of the replica partition function,

$$\mathcal{Z}(m) = \int \prod_{ab}\mathcal{D}q_{ab}(x)\exp(-S[\mathbf{q}]). \qquad \textbf{[S3]}$$

The mean free energy of a typical equilibrium state and the corresponding configurational entropy respectively read $\beta F = -\partial \log \mathcal{Z}(m)/\partial m$ and $S_c = -m^2\partial(\log \mathcal{Z}(m)/m)/\partial m$. The number m of replicas should be analytically continued to 1 in the equilibrium liquid phase and to a value less than one in the ideal glass phase (2). At the mean field level; i.e., by looking for the uniform saddle-points of Eq. **S1**, one finds that the order parameter q_{ab} is zero above a temperature T_d, such that $t_d=\frac{w^2}{4y}$, and that below T_d appears another uniform solution with a replica symmetric (RS) structure $q_{ab}=q_{EA}>0$ for $a\neq b$. By explicitly using the RS structure of q_{ab} one finds that q_{EA} is the secondary local minimum of $\tilde{V}(q)=\frac{V(q)}{m-1}|_{m=1}$ and that the configurational entropy per unit volume is $\tilde{V}(q_{EA})$. Finally, at a temperature T_K such that $t_K=\frac{2w^2}{9y}$, there is a random first order transition with a coexistence between a zero-overlap phase and a high-overlap one, transition with zero latent heat and vanishing configurational entropy density. Below T_K, the system is in an ideal glass phase characterized by a nonzero overlap matrix and a value of m less than 1.

The effect of freezing a fraction f of particles is thoroughly studied within mean field theory for the REM case in the main text. A more refined analysis is obtained focusing on the p-spin spherical model and will be discussed in detail elsewhere in a forthcoming paper. In the following, in order to go beyond the mean field analysis, we focus on the replica field theory. Pinning particles at random can be schematically represented in the field theory by forcing the overlap between replica to be equal to q_{EA} at a set of points associated with frozen particles. In reality, the effect is more complicated than that but this is irrelevant as far as the large length scale properties are concerned. Thus, we impose to the measure in **[S3]** the constraints $q_{ab}(x)=q_{EA}$ in a random set of Poisson-distributed points characterized by a density $f\rho$, where ρ is the particle density.

In order to go beyond the mean field analysis discussed in the previous section and study the critical properties of the glass transitions obtained approaching the $T_K(f)$ line, we follow ref. 5). We consider a real-space Migdal Kadanoff (MK) renormalization group (RG) approach, which becomes exact on hierarchical diamond-like lattices, and apply it to a lattice version of the effective Hamiltonian in Eq. **S1**. Such lattices are built iteratively by replacing each bond between sites by a fixed number of new bonds which, to mimic Euclidean d-dimensional lattices, is taken equal to 2^d. After n iterations, the volume of the system, which is equal to the total number of original bonds, is equal to 2^{nd} whereas the "distance" between the boundary sites is equal to 2^n bonds: this naturally fixes the length scale after n iterations as $\ell_n=2^n$. The procedure is illustrated in Fig. 1. The main advantage of this RG procedure is that the renormalized effective pair interaction on a link between two sites at the n-th step of renormalization, $W_n^i(\mathbf{q}^1,\mathbf{q}^2)$, satisfies a closed equation written in terms of the pair interactions $W_{n-1}^{j_L}(\mathbf{q}^1,\mathbf{q}^2)$ and $W_{n-1}^{j_R}(\mathbf{q}^1,\mathbf{q}^2)$ of the links connecting those sites:

$$W_n^i(\mathbf{q}^1,\mathbf{q}^2) = \sum_{j_{L,R}=1}^{2^{d-1}} \log\int\prod_{a,b}dq_{ab}\exp\{W_{n-1}^{j_L}(\mathbf{q}^1,\mathbf{q}) + V(\mathbf{q}) + W_{n-1}^{j_R}(\mathbf{q},\mathbf{q}^2)\}, \qquad \textbf{[S4]}$$

where the labels 1 and 2 denote the value of two renormalized sites from which emanate $2^{n(d-1)}$ original bonds, and the labels L and R indicate the jth left and right link respectively, see Fig. S1. At the nth iteration, the original lattice is replaced by a renormalized one where the unit length is ℓ_n and the pair interaction between sites is $W_n^i(\mathbf{q}^1,\mathbf{q}^2)$.

In this framework, freezing particles at random leads to the requirement that in the first iterative equations corresponding to $n=1$, q_{ab} is fixed equal to q_{EA} with probability f for each given intermediate site. In particular we only fix the constraint for the sites in the center of the hierarchical lattice, corresponding to $n=1$. The reason is that the other sites of the lattice, entering in the RG equations for $n>1$, actually correspond to renormalized regions and not microscopic ones. As discussed in the following, the presence of these random constraints considerably complicates the analysis because the W_n's become random variables. In order to obtain a tractable problem we use the insight gained in the analysis of ref. 5. Without frozen particles we have found that the nature of the RG flow is similar to that found for first order discontinuity fixed points. There are only two essential couplings: the field h favoring one phase with respect to the other and the coupling J opposing spatial variations of the order parameter. In our context, h and J respectively correspond to the configurational entropy s_c favoring the zero-overlap phase and the interface free-energy loss Y between a high-overlap and low-overlap phase. Guided by this result, we repeat in the following the analysis performed in ref. 5, "projecting" the functional MKRG equation on simpler ones, where the matrix q_{ab} is a two-state variable, $q_{ab} = 0$ or $q_{ab} = q_{EA}$ for all pairs a, b. As in ref. 5 we focus on $\tilde{W}^i_n(q_1, q_2) = \frac{W^i_n(q_1,q_2)}{m-1}|_{m=1}$ and $\tilde{V}(q)$, and we approximate the full integral by a steepest-descent calculation (see ref. 5 for a discussion of this approximation). The iteration equation simplifies to

$$\tilde{W}^i_n(q_1, q_2) = \sum_{j_{L,R}=1}^{2^{d-1}} \min_{q=0,q_{EA}} \{\tilde{W}^{j_L}_{n-1}(q_1, q) + \tilde{W}^{j_R}_{n-1}(q, q_2) + \tilde{V}(q)\}, \quad \textbf{[S5]}$$

for q_1 and q_2 that take the values 0 and q_{EA} only. We have chosen, without much loss of generality, a simpler form of $V(q)$ (3, 4): by imposing that $q_{EA} = 1$ at T_K, one finds that

$$\tilde{V}(q) = \frac{q^2}{2}\left(\frac{T - T_K}{T_K} + 8\frac{T_d - T_K}{T_K}(1-q)^2\right).$$

The initial condition is $\tilde{W}^{j_{L,R}}_0(q_1, q_2) = (q_1 - q_2)^2/2$. We have checked that the results of ref. 5 continue to hold within this approximation, as required. Because of the frozen particles, the above equation is modified at the first stage of iteration, $n = 1$. For a fraction f of randomly chosen initial sites, instead of minimizing over q, one just evaluates the expression within brackets in **[S5]** for $q = q_{EA}$. In consequence, even though the RG equations do not contain quenched disorder, their initial conditions do. Thus, the $W^i_n(q_1, q_2)$s become random variables and Eq. **S5** defines the flow equation for their probability distributions.

We have numerically solved this flow equation and characterized its solution. The corresponding results are discussed in the main text. A detailed technical presentation will appear in a forthcoming paper.

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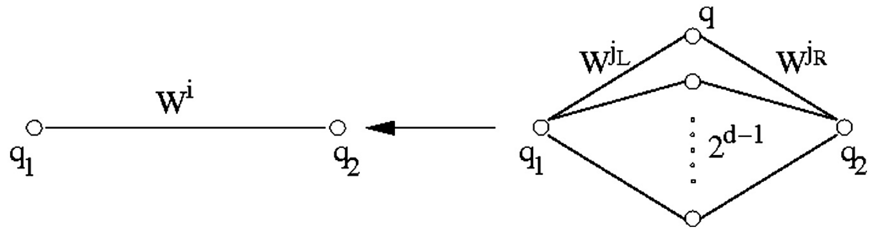

Fig. S1. Elementary step illustrating the RG procedure on a hierachical lattice.