Supplementary Materials

Appendix A Proof of Relations (16)

A.1 The Subscripts ije Do Not Contain 2

Notice

$$\begin{aligned}
\frac{\partial f_{000}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{000}\log\frac{P_{000}}{P_{0..}P_{.0.}P_{..0}}\right] = \log\frac{P_{000}}{P_{0..}P_{.0.}P_{..0}}+\left[1-\frac{P_{000}}{P_{0..}}-\frac{P_{000}}{P_{.0.}}-\frac{P_{000}}{P_{..0}}\right]\log e,\\
\frac{\partial f_{001}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{001}\log\frac{P_{001}}{P_{0..}P_{.0.}P_{..1}}\right] = -\left[\frac{P_{001}}{P_{0..}}+\frac{P_{001}}{P_{.0.}}\right]\log e,\\
\frac{\partial f_{002}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{002}\log\frac{P_{002}}{P_{0..}P_{.0.}P_{..2}}\right] = \left[-\frac{P_{002}}{P_{0..}}-\frac{P_{002}}{P_{.0.}}+\frac{P_{002}}{P_{..2}}\right]\log e,\\
\frac{\partial f_{010}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{010}\log\frac{P_{010}}{P_{0..}P_{.1.}P_{..0}}\right] = -\left[\frac{P_{010}}{P_{0..}}+\frac{P_{010}}{P_{..0}}\right]\log e,\\
\frac{\partial f_{011}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{011}\log\frac{P_{011}}{P_{0..}P_{.1.}P_{..1}}\right] = -\frac{P_{011}}{P_{0..}}\log e,\\
\frac{\partial f_{012}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{012}\log\frac{P_{012}}{P_{0..}P_{.1.}P_{..2}}\right] = \left[-\frac{P_{012}}{P_{0..}}+\frac{P_{012}}{P_{..2}}\right]\log e,\\
\frac{\partial f_{020}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{020}\log\frac{P_{020}}{P_{0..}P_{.2.}P_{..0}}\right] = \left[-\frac{P_{020}}{P_{0..}}+\frac{P_{020}}{P_{.2.}}-\frac{P_{020}}{P_{..0}}\right]\log e,\\
\frac{\partial f_{021}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{021}\log\frac{P_{021}}{P_{0..}P_{.2.}P_{..1}}\right] = \left[-\frac{P_{021}}{P_{0..}}+\frac{P_{021}}{P_{.2.}}\right]\log e,\\
\frac{\partial f_{022}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{022}\log\frac{P_{022}}{P_{0..}P_{.2.}P_{..2}}\right] = \left[-\frac{P_{022}}{P_{0..}}+\frac{P_{022}}{P_{.2.}}+\frac{P_{022}}{P_{..2}}\right]\log e.
\end{aligned} \tag{A.1}$$

In addition, we have

$$\begin{aligned}
\frac{\partial f_{100}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{100}\log\frac{P_{100}}{P_{1..}P_{.0.}P_{..0}}\right] = -\left[\frac{P_{100}}{P_{.0.}}+\frac{P_{100}}{P_{..0}}\right]\log e,\\
\frac{\partial f_{101}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{101}\log\frac{P_{101}}{P_{1..}P_{.0.}P_{..1}}\right] = -\frac{P_{101}}{P_{.0.}}\log e,\\
\frac{\partial f_{102}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{102}\log\frac{P_{102}}{P_{1..}P_{.0.}P_{..2}}\right] = \left[-\frac{P_{102}}{P_{.0.}}+\frac{P_{102}}{P_{..2}}\right]\log e,\\
\frac{\partial f_{110}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{110}\log\frac{P_{110}}{P_{1..}P_{.1.}P_{..0}}\right] = -\frac{P_{110}}{P_{..0}}\log e,\\
\frac{\partial f_{111}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{111}\log\frac{P_{111}}{P_{1..}P_{.1.}P_{..1}}\right] = 0,\\
\frac{\partial f_{112}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{112}\log\frac{P_{112}}{P_{1..}P_{.1.}P_{..2}}\right] = \frac{P_{112}}{P_{..2}}\log e,\\
\frac{\partial f_{120}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{120}\log\frac{P_{120}}{P_{1..}P_{.2.}P_{..0}}\right] = \left[\frac{P_{120}}{P_{.2.}}-\frac{P_{120}}{P_{..0}}\right]\log e,
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\frac{\partial f_{121}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{121}\log\frac{P_{121}}{P_{1..}P_{.2.}P_{..1}}\right] = \frac{P_{121}}{P_{.2.}}\log e,\\
\frac{\partial f_{122}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{122}\log\frac{P_{122}}{P_{1..}P_{.2.}P_{..2}}\right] = \left[\frac{P_{122}}{P_{.2.}}+\frac{P_{122}}{P_{..2}}\right]\log e,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial f_{200}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{200}\log\frac{P_{200}}{P_{2..}P_{.0.}P_{..0}}\right] = \left[\frac{P_{200}}{P_{2..}}-\frac{P_{200}}{P_{.0.}}-\frac{P_{200}}{P_{..0}}\right]\log e,\\
\frac{\partial f_{201}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{201}\log\frac{P_{201}}{P_{2..}P_{.0.}P_{..1}}\right] = \left[\frac{P_{201}}{P_{2..}}-\frac{P_{201}}{P_{.0.}}\right]\log e,\\
\frac{\partial f_{202}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{202}\log\frac{P_{202}}{P_{2..}P_{.0.}P_{..2}}\right] = \left[\frac{P_{202}}{P_{2..}}-\frac{P_{202}}{P_{.0.}}+\frac{P_{202}}{P_{..2}}\right]\log e,\\
\frac{\partial f_{210}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{210}\log\frac{P_{210}}{P_{2..}P_{.1.}P_{..0}}\right] = \left[\frac{P_{210}}{P_{2..}}-\frac{P_{210}}{P_{..0}}\right]\log e,\\
\frac{\partial f_{211}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{211}\log\frac{P_{211}}{P_{2..}P_{.1.}P_{..1}}\right] = \frac{P_{211}}{P_{2..}}\log e,\\
\frac{\partial f_{212}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{212}\log\frac{P_{212}}{P_{2..}P_{.1.}P_{..2}}\right] = \left[\frac{P_{212}}{P_{2..}}+\frac{P_{212}}{P_{..2}}\right]\log e,\\
\frac{\partial f_{220}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{220}\log\frac{P_{220}}{P_{2..}P_{.2.}P_{..0}}\right] = \left[\frac{P_{220}}{P_{2..}}+\frac{P_{220}}{P_{.2.}}-\frac{P_{220}}{P_{..0}}\right]\log e,\\
\frac{\partial f_{221}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{221}\log\frac{P_{221}}{P_{2..}P_{.2.}P_{..1}}\right] = \left[\frac{P_{221}}{P_{2..}}+\frac{P_{221}}{P_{.2.}}\right]\log e,\\
\frac{\partial f_{222}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{222}\log\frac{P_{222}}{P_{2..}P_{.2.}P_{..2}}\right] = -\log\frac{P_{222}}{P_{2..}P_{.2.}P_{..2}}+\left[-1+\frac{P_{222}}{P_{2..}}+\frac{P_{222}}{P_{.2.}}+\frac{P_{222}}{P_{..2}}\right]\log e.
\end{aligned} \tag{A.3}$$

By relations (A.1), (A.2), and (A.3), we have

$$\begin{aligned}
\frac{\partial f}{\partial P_{000}} &= \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{e=0}^{2}\frac{\partial f_{ije}}{\partial P_{000}}\\
&= \log\frac{P_{000}}{P_{0..}P_{.0.}P_{..0}}-\log\frac{P_{222}}{P_{2..}P_{.2.}P_{..2}}.
\end{aligned}$$

Similarly, we have for $i,j,e=0,1$

$$\frac{\partial f}{\partial P_{ije}} = \log\frac{P_{ije}}{P_{i..}P_{.j.}P_{..e}}-\log\frac{P_{222}}{P_{2..}P_{.2.}P_{..2}}.$$

A.2 The Subscripts ije Contain One 2

Notice

$$\frac{\partial f_{000}}{\partial P_{002}} = \frac{\partial}{\partial P_{002}}\left[P_{000}\log\frac{P_{000}}{P_{0..}P_{.0.}P_{..0}}\right] = \left[-\frac{P_{000}}{P_{0..}}-\frac{P_{000}}{P_{.0.}}\right]\log e,$$

$$
\begin{aligned}
\frac{\partial f_{001}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{001}\log\frac{P_{001}}{P_{0..}P_{.0.}P_{..1}}\right] = \left[-\frac{P_{001}}{P_{0..}}-\frac{P_{001}}{P_{.0.}}\right]\log e,\\
\frac{\partial f_{002}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{002}\log\frac{P_{002}}{P_{0..}P_{.0.}P_{..2}}\right] = \log\frac{P_{002}}{P_{0..}P_{.0.}P_{..2}}+\left[1-\frac{P_{002}}{P_{0..}}-\frac{P_{002}}{P_{.0.}}\right]\log e,\\
\frac{\partial f_{010}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{010}\log\frac{P_{010}}{P_{0..}P_{.1.}P_{..0}}\right] = -\frac{P_{010}}{P_{0..}}\log e,\\
\frac{\partial f_{011}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{011}\log\frac{P_{011}}{P_{0..}P_{.1.}P_{..1}}\right] = -\frac{P_{011}}{P_{0..}}\log e,\\
\frac{\partial f_{012}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{012}\log\frac{P_{012}}{P_{0..}P_{.1.}P_{..2}}\right] = -\frac{P_{012}}{P_{0..}}\log e,\\
\frac{\partial f_{020}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{020}\log\frac{P_{020}}{P_{0..}P_{.2.}P_{..0}}\right] = \left[-\frac{P_{020}}{P_{0..}}+\frac{P_{020}}{P_{.2.}}\right]\log e,\\
\frac{\partial f_{021}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{021}\log\frac{P_{021}}{P_{0..}P_{.2.}P_{..1}}\right] = \left[-\frac{P_{021}}{P_{0..}}+\frac{P_{021}}{P_{.2.}}\right]\log e,\\
\frac{\partial f_{022}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{022}\log\frac{P_{022}}{P_{0..}P_{.2.}P_{..2}}\right] = \left[-\frac{P_{022}}{P_{0..}}+\frac{P_{022}}{P_{.2.}}\right]\log e.
\end{aligned}
\tag{A.4}
$$

In addition, we have

$$
\begin{aligned}
\frac{\partial f_{100}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{100}\log\frac{P_{100}}{P_{1..}P_{.0.}P_{..0}}\right] = -\frac{P_{100}}{P_{.0.}}\log e,\\
\frac{\partial f_{101}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{101}\log\frac{P_{101}}{P_{1..}P_{.0.}P_{..1}}\right] = -\frac{P_{101}}{P_{.0.}}\log e,\\
\frac{\partial f_{102}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{102}\log\frac{P_{102}}{P_{1..}P_{.0.}P_{..2}}\right] = -\frac{P_{102}}{P_{.0.}}\log e,\\
\frac{\partial f_{110}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{110}\log\frac{P_{110}}{P_{1..}P_{.1.}P_{..0}}\right] = 0,\\
\frac{\partial f_{111}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{111}\log\frac{P_{111}}{P_{1..}P_{.1.}P_{..1}}\right] = 0,\\
\frac{\partial f_{112}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{112}\log\frac{P_{112}}{P_{1..}P_{.1.}P_{..2}}\right] = 0,\\
\frac{\partial f_{120}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{120}\log\frac{P_{120}}{P_{1..}P_{.2.}P_{..0}}\right] = \frac{P_{120}}{P_{.2.}}\log e,\\
\frac{\partial f_{121}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{121}\log\frac{P_{121}}{P_{1..}P_{.2.}P_{..1}}\right] = \frac{P_{121}}{P_{.2.}}\log e,\\
\frac{\partial f_{122}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{122}\log\frac{P_{122}}{P_{1..}P_{.2.}P_{..2}}\right] = \frac{P_{122}}{P_{.2.}}\log e,
\end{aligned}
\tag{A.5}
$$

and

$$
\begin{aligned}
\frac{\partial f_{200}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{200}\log\frac{P_{200}}{P_{2..}P_{.0.}P_{..0}}\right] = \left[\frac{P_{200}}{P_{2..}}-\frac{P_{200}}{P_{.0.}}\right]\log e,\\
\frac{\partial f_{201}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{201}\log\frac{P_{201}}{P_{2..}P_{.0.}P_{..1}}\right] = \left[\frac{P_{201}}{P_{2..}}-\frac{P_{201}}{P_{.0.}}\right]\log e,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f_{202}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{202}\log\frac{P_{202}}{P_{2..}P_{\cdot 0\cdot}P_{..2}}\right] = \left[\frac{P_{202}}{P_{2..}}-\frac{P_{202}}{P_{\cdot 0\cdot}}\right]\log e,\\
\frac{\partial f_{210}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{210}\log\frac{P_{210}}{P_{2..}P_{\cdot 1\cdot}P_{..0}}\right] = \frac{P_{210}}{P_{2..}}\log e,\\
\frac{\partial f_{211}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{211}\log\frac{P_{211}}{P_{2..}P_{\cdot 1\cdot}P_{..1}}\right] = \frac{P_{211}}{P_{2..}}\log e,\\
\frac{\partial f_{212}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{212}\log\frac{P_{212}}{P_{2..}P_{\cdot 1\cdot}P_{..2}}\right] = \frac{P_{212}}{P_{2..}}\log e,\\
\frac{\partial f_{220}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{220}\log\frac{P_{220}}{P_{2..}P_{\cdot 2\cdot}P_{..0}}\right] = \left[\frac{P_{220}}{P_{2..}}+\frac{P_{220}}{P_{\cdot 2\cdot}}\right]\log e,\\
\frac{\partial f_{221}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{221}\log\frac{P_{221}}{P_{2..}P_{\cdot 2\cdot}P_{..1}}\right] = \left[\frac{P_{221}}{P_{2..}}+\frac{P_{221}}{P_{\cdot 2\cdot}}\right]\log e,\\
\frac{\partial f_{222}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{222}\log\frac{P_{222}}{P_{2..}P_{\cdot 2\cdot}P_{..2}}\right] = -\log\frac{P_{222}}{P_{2..}P_{\cdot 2\cdot}P_{..2}}+\left[-1+\frac{P_{222}}{P_{2..}}+\frac{P_{222}}{P_{\cdot 2\cdot}}\right]\log e.
\end{aligned}
\tag{A.6}
$$

By relations (A.4), (A.5), and (A.6), we have

$$
\begin{aligned}
\frac{\partial f}{\partial P_{002}} &= \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{e=0}^{2}\frac{\partial f_{ije}}{\partial P_{002}}\\
&= \log\frac{P_{002}}{P_{0..}P_{\cdot 0\cdot}P_{..2}}-\log\frac{P_{222}}{P_{2..}P_{\cdot 2\cdot}P_{..2}}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\frac{\partial f}{\partial P_{012}} &= \log\frac{P_{012}}{P_{0..}P_{\cdot 1\cdot}P_{..2}}-\log\frac{P_{222}}{P_{2..}P_{\cdot 2\cdot}P_{..2}},\\
\frac{\partial f}{\partial P_{020}} &= \log\frac{P_{020}}{P_{0..}P_{\cdot 2\cdot}P_{..0}}-\log\frac{P_{222}}{P_{2..}P_{\cdot 2\cdot}P_{..2}},\\
\frac{\partial f}{\partial P_{021}} &= \log\frac{P_{021}}{P_{0..}P_{\cdot 2\cdot}P_{..1}}-\log\frac{P_{222}}{P_{2..}P_{\cdot 2\cdot}P_{..2}},\\
\frac{\partial f}{\partial P_{102}} &= \log\frac{P_{102}}{P_{1..}P_{\cdot 0\cdot}P_{..2}}-\log\frac{P_{222}}{P_{2..}P_{\cdot 2\cdot}P_{..2}},\\
\frac{\partial f}{\partial P_{112}} &= \log\frac{P_{112}}{P_{1..}P_{\cdot 1\cdot}P_{..2}}-\log\frac{P_{222}}{P_{2..}P_{\cdot 2\cdot}P_{..2}},\\
\frac{\partial f}{\partial P_{120}} &= \log\frac{P_{120}}{P_{1..}P_{\cdot 2\cdot}P_{..0}}-\log\frac{P_{222}}{P_{2..}P_{\cdot 2\cdot}P_{..2}},\\
\frac{\partial f}{\partial P_{121}} &= \log\frac{P_{121}}{P_{1..}P_{\cdot 2\cdot}P_{..1}}-\log\frac{P_{222}}{P_{2..}P_{\cdot 2\cdot}P_{..2}},\\
\frac{\partial f}{\partial P_{200}} &= \log\frac{P_{200}}{P_{2..}P_{\cdot 0\cdot}P_{..0}}-\log\frac{P_{222}}{P_{2..}P_{\cdot 2\cdot}P_{..2}},\\
\frac{\partial f}{\partial P_{201}} &= \log\frac{P_{201}}{P_{2..}P_{\cdot 0\cdot}P_{..1}}-\log\frac{P_{222}}{P_{2..}P_{\cdot 2\cdot}P_{..2}},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial P_{210}} &= \log \frac{P_{210}}{P_{2..}P_{.1.}P_{..0}} - \log \frac{P_{222}}{P_{2..}P_{.2.}P_{..2}}, \\
\frac{\partial f}{\partial P_{211}} &= \log \frac{P_{211}}{P_{2..}P_{.1.}P_{..1}} - \log \frac{P_{222}}{P_{2..}P_{.2.}P_{..2}}.
\end{aligned}
$$

A.3 The Subscripts ije Contain Two 2

Notice

$$
\begin{aligned}
\frac{\partial f_{000}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{000}\log\frac{P_{000}}{P_{0..}P_{.0.}P_{..0}}\right] = 0, \\
\frac{\partial f_{001}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{001}\log\frac{P_{001}}{P_{0..}P_{.0.}P_{..1}}\right] = 0, \\
\frac{\partial f_{002}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{002}\log\frac{P_{002}}{P_{0..}P_{.0.}P_{..2}}\right] = 0, \\
\frac{\partial f_{010}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{010}\log\frac{P_{010}}{P_{0..}P_{.1.}P_{..0}}\right] = 0, \\
\frac{\partial f_{011}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{011}\log\frac{P_{011}}{P_{0..}P_{.1.}P_{..1}}\right] = 0, \\
\frac{\partial f_{012}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{012}\log\frac{P_{012}}{P_{0..}P_{.1.}P_{..2}}\right] = 0, \\
\frac{\partial f_{020}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{020}\log\frac{P_{020}}{P_{0..}P_{.2.}P_{..0}}\right] = 0, \\
\frac{\partial f_{021}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{021}\log\frac{P_{021}}{P_{0..}P_{.2.}P_{..1}}\right] = 0, \\
\frac{\partial f_{022}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{022}\log\frac{P_{022}}{P_{0..}P_{.2.}P_{..2}}\right] = 0.
\end{aligned}
\tag{A.7}
$$

In addition, we have

$$
\begin{aligned}
\frac{\partial f_{100}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{100}\log\frac{P_{100}}{P_{1..}P_{.0.}P_{..0}}\right] = -\frac{P_{100}}{P_{1..}}\log e, \\
\frac{\partial f_{101}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{101}\log\frac{P_{101}}{P_{1..}P_{.0.}P_{..1}}\right] = -\frac{P_{101}}{P_{1..}}\log e, \\
\frac{\partial f_{102}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{102}\log\frac{P_{102}}{P_{1..}P_{.0.}P_{..2}}\right] = -\frac{P_{102}}{P_{1..}}\log e, \\
\frac{\partial f_{110}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{110}\log\frac{P_{110}}{P_{1..}P_{.1.}P_{..0}}\right] = -\frac{P_{110}}{P_{1..}}\log e, \\
\frac{\partial f_{111}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{111}\log\frac{P_{111}}{P_{1..}P_{.1.}P_{..1}}\right] = -\frac{P_{111}}{P_{1..}}\log e, \\
\frac{\partial f_{112}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{112}\log\frac{P_{112}}{P_{1..}P_{.1.}P_{..2}}\right] = -\frac{P_{112}}{P_{1..}}\log e,
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
\frac{\partial f_{120}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{120}\log\frac{P_{120}}{P_{1..}P_{.2.}P_{..0}}\right] = -\frac{P_{120}}{P_{1..}}\log e,\\
\frac{\partial f_{121}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{121}\log\frac{P_{121}}{P_{1..}P_{.2.}P_{..1}}\right] = -\frac{P_{121}}{P_{1..}}\log e,\\
\frac{\partial f_{122}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{122}\log\frac{P_{122}}{P_{1..}P_{.2.}P_{..2}}\right] = \log\frac{P_{122}}{P_{1..}P_{.2.}P_{..2}} + \left[1-\frac{P_{122}}{P_{1..}}\right]\log e,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial f_{200}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{200}\log\frac{P_{200}}{P_{2..}P_{.0.}P_{..0}}\right] = \frac{P_{200}}{P_{2..}}\log e,\\
\frac{\partial f_{201}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{201}\log\frac{P_{201}}{P_{2..}P_{.0.}P_{..1}}\right] = \frac{P_{201}}{P_{2..}}\log e,\\
\frac{\partial f_{202}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{202}\log\frac{P_{202}}{P_{2..}P_{.0.}P_{..2}}\right] = \frac{P_{202}}{P_{2..}}\log e,\\
\frac{\partial f_{210}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{210}\log\frac{P_{210}}{P_{2..}P_{.1.}P_{..0}}\right] = \frac{P_{210}}{P_{2..}}\log e,\\
\frac{\partial f_{211}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{211}\log\frac{P_{211}}{P_{2..}P_{.1.}P_{..1}}\right] = \frac{P_{211}}{P_{2..}}\log e,\\
\frac{\partial f_{212}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{212}\log\frac{P_{212}}{P_{2..}P_{.1.}P_{..2}}\right] = \frac{P_{212}}{P_{2..}}\log e,\\
\frac{\partial f_{220}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{220}\log\frac{P_{220}}{P_{2..}P_{.2.}P_{..0}}\right] = \frac{P_{220}}{P_{2..}}\log e,\\
\frac{\partial f_{221}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{221}\log\frac{P_{221}}{P_{2..}P_{.2.}P_{..1}}\right] = \frac{P_{221}}{P_{2..}}\log e,\\
\frac{\partial f_{222}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{222}\log\frac{P_{222}}{P_{2..}P_{.2.}P_{..2}}\right] = -\log\frac{P_{222}}{P_{2..}P_{.2.}P_{..2}} + \left[-1+\frac{P_{222}}{P_{2..}}\right]\log e.
\end{aligned}
\tag{A.9}
$$

By relations (A.7), (A.8), and (A.9), we have

$$
\begin{aligned}
\frac{\partial f}{\partial P_{122}} &= \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{e=0}^{2}\frac{\partial f_{ije}}{\partial P_{122}}\\
&= \log\frac{P_{122}}{P_{1..}P_{.2.}P_{..2}} - \log\frac{P_{222}}{P_{2..}P_{.2.}P_{..2}}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\frac{\partial f}{\partial P_{022}} &= \log\frac{P_{022}}{P_{0..}P_{.2.}P_{..2}} - \log\frac{P_{222}}{P_{2..}P_{.2.}P_{..2}},\\
\frac{\partial f}{\partial P_{202}} &= \log\frac{P_{202}}{P_{2..}P_{.0.}P_{..2}} - \log\frac{P_{222}}{P_{2..}P_{.2.}P_{..2}},\\
\frac{\partial f}{\partial P_{212}} &= \log\frac{P_{212}}{P_{2..}P_{.1.}P_{..2}} - \log\frac{P_{222}}{P_{2..}P_{.2.}P_{..2}},
\end{aligned}
$$

$$\begin{aligned}\frac{\partial f}{\partial P_{220}} &= \log\frac{P_{220}}{P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 0}} - \log\frac{P_{222}}{P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}},\\ \frac{\partial f}{\partial P_{221}} &= \log\frac{P_{221}}{P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 1}} - \log\frac{P_{222}}{P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}.\end{aligned}$$

Appendix B Proof of Relations (17)

B.1 The Subscripts ije Do Not Contain 2

Notice

$$\begin{aligned}\frac{\partial h_{000}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{000}\log\frac{P_{000}P_{0\cdot\cdot}P_{\cdot 0\cdot}P_{\cdot\cdot 0}}{P_{00\cdot}P_{0\cdot 0}P_{\cdot 00}}\right]\\ &= \log\frac{P_{000}P_{0\cdot\cdot}P_{\cdot 0\cdot}P_{\cdot\cdot 0}}{P_{00\cdot}P_{0\cdot 0}P_{\cdot 00}} + \left[1+\frac{P_{000}}{P_{0\cdot\cdot}}+\frac{P_{000}}{P_{\cdot 0\cdot}}+\frac{P_{000}}{P_{\cdot\cdot 0}}-\frac{P_{000}}{P_{00\cdot}}-\frac{P_{000}}{P_{0\cdot 0}}-\frac{P_{000}}{P_{\cdot 00}}\right]\log e,\\ \frac{\partial h_{001}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{001}\log\frac{P_{001}P_{0\cdot\cdot}P_{\cdot 0\cdot}P_{\cdot\cdot 1}}{P_{00\cdot}P_{0\cdot 1}P_{\cdot 01}}\right] = \left[\frac{P_{001}}{P_{0\cdot\cdot}}+\frac{P_{001}}{P_{\cdot 0\cdot}}-\frac{P_{001}}{P_{00\cdot}}\right]\log e,\\ \frac{\partial h_{002}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{002}\log\frac{P_{002}P_{0\cdot\cdot}P_{\cdot 0\cdot}P_{\cdot\cdot 2}}{P_{00\cdot}P_{0\cdot 2}P_{\cdot 02}}\right] = \left[\frac{P_{002}}{P_{0\cdot\cdot}}+\frac{P_{002}}{P_{\cdot 0\cdot}}-\frac{P_{002}}{P_{\cdot\cdot 2}}-\frac{P_{002}}{P_{00\cdot}}\right]\log e,\\ \frac{\partial h_{010}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{010}\log\frac{P_{010}P_{0\cdot\cdot}P_{\cdot 1\cdot}P_{\cdot\cdot 0}}{P_{01\cdot}P_{0\cdot 0}P_{\cdot 10}}\right] = \left[\frac{P_{010}}{P_{0\cdot\cdot}}+\frac{P_{010}}{P_{\cdot\cdot 0}}-\frac{P_{010}}{P_{0\cdot 0}}\right]\log e,\\ \frac{\partial h_{011}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{011}\log\frac{P_{011}P_{0\cdot\cdot}P_{\cdot 1\cdot}P_{\cdot\cdot 1}}{P_{01\cdot}P_{0\cdot 1}P_{\cdot 11}}\right] = \frac{P_{011}}{P_{0\cdot\cdot}}\log e,\\ \frac{\partial h_{012}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{012}\log\frac{P_{012}P_{0\cdot\cdot}P_{\cdot 1\cdot}P_{\cdot\cdot 2}}{P_{01\cdot}P_{0\cdot 2}P_{\cdot 12}}\right] = \left[\frac{P_{012}}{P_{0\cdot\cdot}}-\frac{P_{012}}{P_{\cdot\cdot 2}}\right]\log e,\\ \frac{\partial h_{020}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{020}\log\frac{P_{020}P_{0\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 0}}{P_{02\cdot}P_{0\cdot 0}P_{\cdot 20}}\right] = \left[\frac{P_{020}}{P_{0\cdot\cdot}}-\frac{P_{020}}{P_{\cdot 2\cdot}}+\frac{P_{020}}{P_{\cdot\cdot 0}}-\frac{P_{020}}{P_{0\cdot 0}}\right]\log e,\\ \frac{\partial h_{021}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{021}\log\frac{P_{021}P_{0\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 1}}{P_{02\cdot}P_{0\cdot 1}P_{\cdot 21}}\right] = \left[\frac{P_{021}}{P_{0\cdot\cdot}}-\frac{P_{021}}{P_{\cdot 2\cdot}}\right]\log e,\\ \frac{\partial h_{022}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{022}\log\frac{P_{022}P_{0\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}{P_{02\cdot}P_{0\cdot 2}P_{\cdot 22}}\right] = \left[\frac{P_{022}}{P_{0\cdot\cdot}}-\frac{P_{022}}{P_{\cdot 2\cdot}}-\frac{P_{022}}{P_{\cdot\cdot 2}}+\frac{P_{022}}{P_{\cdot 22}}\right]\log e.\end{aligned} \tag{B.1}$$

In addition, we have

$$\begin{aligned}\frac{\partial h_{100}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{100}\log\frac{P_{100}P_{1\cdot\cdot}P_{\cdot 0\cdot}P_{\cdot\cdot 0}}{P_{10\cdot}P_{1\cdot 0}P_{\cdot 00}}\right] = \left[\frac{P_{100}}{P_{\cdot 0\cdot}}+\frac{P_{100}}{P_{\cdot\cdot 0}}-\frac{P_{100}}{P_{\cdot 00}}\right]\log e,\\ \frac{\partial h_{101}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{101}\log\frac{P_{101}P_{1\cdot\cdot}P_{\cdot 0\cdot}P_{\cdot\cdot 1}}{P_{10\cdot}P_{1\cdot 1}P_{\cdot 01}}\right] = \frac{P_{101}}{P_{\cdot 0\cdot}}\log e,\\ \frac{\partial h_{102}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{102}\log\frac{P_{102}P_{1\cdot\cdot}P_{\cdot 0\cdot}P_{\cdot\cdot 2}}{P_{10\cdot}P_{1\cdot 2}P_{\cdot 02}}\right] = \left[\frac{P_{102}}{P_{\cdot 0\cdot}}-\frac{P_{102}}{P_{\cdot\cdot 2}}\right]\log e,\\ \frac{\partial h_{110}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{110}\log\frac{P_{110}P_{1\cdot\cdot}P_{\cdot 1\cdot}P_{\cdot\cdot 0}}{P_{11\cdot}P_{1\cdot 0}P_{\cdot 10}}\right] = \frac{P_{110}}{P_{\cdot\cdot 0}}\log e,\end{aligned}$$

$$
\begin{aligned}
\frac{\partial h_{111}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{111}\log\frac{P_{111}P_{1\cdot\cdot}P_{\cdot1\cdot}P_{\cdot\cdot1}}{P_{11\cdot}P_{1\cdot1}P_{\cdot11}}\right] = 0, \\
\frac{\partial h_{112}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{112}\log\frac{P_{112}P_{1\cdot\cdot}P_{\cdot1\cdot}P_{\cdot\cdot2}}{P_{11\cdot}P_{1\cdot2}P_{\cdot12}}\right] = -\frac{P_{112}}{P_{\cdot\cdot2}}\log e, \\
\frac{\partial h_{120}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{120}\log\frac{P_{120}P_{1\cdot\cdot}P_{\cdot2\cdot}P_{\cdot\cdot0}}{P_{12\cdot}P_{1\cdot0}P_{\cdot20}}\right] = \left[-\frac{P_{120}}{P_{\cdot2\cdot}}+\frac{P_{120}}{P_{\cdot\cdot0}}\right]\log e, \\
\frac{\partial h_{121}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{121}\log\frac{P_{121}P_{1\cdot\cdot}P_{\cdot2\cdot}P_{\cdot\cdot1}}{P_{12\cdot}P_{1\cdot1}P_{\cdot21}}\right] = -\frac{P_{121}}{P_{\cdot2\cdot}}\log e, \\
\frac{\partial h_{122}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{122}\log\frac{P_{122}P_{1\cdot\cdot}P_{\cdot2\cdot}P_{\cdot\cdot2}}{P_{12\cdot}P_{1\cdot2}P_{\cdot22}}\right] = -\left[\frac{P_{122}}{P_{\cdot2\cdot}}+\frac{P_{122}}{P_{\cdot\cdot2}}+\frac{P_{122}}{P_{\cdot22}}\right]\log e,
\end{aligned}
\tag{B.2}
$$

and

$$
\begin{aligned}
\frac{\partial h_{200}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{200}\log\frac{P_{200}P_{2\cdot\cdot}P_{\cdot0\cdot}P_{\cdot\cdot0}}{P_{20\cdot}P_{2\cdot0}P_{\cdot00}}\right] = \left[-\frac{P_{200}}{P_{2\cdot\cdot}}+\frac{P_{200}}{P_{\cdot0\cdot}}+\frac{P_{200}}{P_{\cdot\cdot0}}-\frac{P_{200}}{P_{\cdot00}}\right]\log e, \\
\frac{\partial h_{201}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{201}\log\frac{P_{201}P_{2\cdot\cdot}P_{\cdot0\cdot}P_{\cdot\cdot1}}{P_{20\cdot}P_{2\cdot1}P_{\cdot01}}\right] = \left[-\frac{P_{201}}{P_{2\cdot\cdot}}+\frac{P_{201}}{P_{\cdot0\cdot}}\right]\log e, \\
\frac{\partial h_{202}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{202}\log\frac{P_{202}P_{2\cdot\cdot}P_{\cdot0\cdot}P_{\cdot\cdot2}}{P_{20\cdot}P_{2\cdot2}P_{\cdot02}}\right] = \left[-\frac{P_{202}}{P_{2\cdot\cdot}}+\frac{P_{202}}{P_{\cdot0\cdot}}-\frac{P_{202}}{P_{\cdot\cdot2}}+\frac{P_{202}}{P_{2\cdot2}}\right]\log e, \\
\frac{\partial h_{210}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{210}\log\frac{P_{210}P_{2\cdot\cdot}P_{\cdot1\cdot}P_{\cdot\cdot0}}{P_{21\cdot}P_{2\cdot0}P_{\cdot10}}\right] = \left[-\frac{P_{210}}{P_{2\cdot\cdot}}+\frac{P_{210}}{P_{\cdot\cdot0}}\right]\log e, \\
\frac{\partial h_{211}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{211}\log\frac{P_{211}P_{2\cdot\cdot}P_{\cdot1\cdot}P_{\cdot\cdot1}}{P_{21\cdot}P_{2\cdot1}P_{\cdot11}}\right] = -\frac{P_{211}}{P_{2\cdot\cdot}}\log e, \\
\frac{\partial h_{212}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{212}\log\frac{P_{212}P_{2\cdot\cdot}P_{\cdot1\cdot}P_{\cdot\cdot2}}{P_{21\cdot}P_{2\cdot2}P_{\cdot12}}\right] = \left[-\frac{P_{212}}{P_{2\cdot\cdot}}-\frac{P_{212}}{P_{\cdot\cdot2}}+\frac{P_{212}}{P_{2\cdot2}}\right]\log e, \\
\frac{\partial h_{220}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{220}\log\frac{P_{220}P_{2\cdot\cdot}P_{\cdot2\cdot}P_{\cdot\cdot0}}{P_{22\cdot}P_{2\cdot0}P_{\cdot20}}\right] = \left[-\frac{P_{220}}{P_{2\cdot\cdot}}-\frac{P_{220}}{P_{\cdot2\cdot}}+\frac{P_{220}}{P_{\cdot\cdot0}}+\frac{P_{220}}{P_{22\cdot}}\right]\log e, \\
\frac{\partial h_{221}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{221}\log\frac{P_{221}P_{2\cdot\cdot}P_{\cdot2\cdot}P_{\cdot\cdot1}}{P_{22\cdot}P_{2\cdot1}P_{\cdot21}}\right] = \left[-\frac{P_{221}}{P_{2\cdot\cdot}}-\frac{P_{221}}{P_{\cdot2\cdot}}+\frac{P_{221}}{P_{22\cdot}}\right]\log e, \\
\frac{\partial h_{222}}{\partial P_{000}} &= \frac{\partial}{\partial P_{000}}\left[P_{222}\log\frac{P_{222}P_{2\cdot\cdot}P_{\cdot2\cdot}P_{\cdot\cdot2}}{P_{22\cdot}P_{2\cdot2}P_{\cdot22}}\right] \\
&= -\log\frac{P_{222}P_{2\cdot\cdot}P_{\cdot2\cdot}P_{\cdot\cdot2}}{P_{22\cdot}P_{2\cdot2}P_{\cdot22}}+\left[-1-\frac{P_{222}}{P_{2\cdot\cdot}}-\frac{P_{222}}{P_{\cdot2\cdot}}-\frac{P_{222}}{P_{\cdot\cdot2}}+\frac{P_{222}}{P_{22\cdot}}+\frac{P_{222}}{P_{2\cdot2}}+\frac{P_{222}}{P_{\cdot22}}\right]\log e.
\end{aligned}
\tag{B.3}
$$

By relations (B.1), (B.2), and (B.3), we have

$$
\begin{aligned}
\frac{\partial h}{\partial P_{000}} &= \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{e=0}^{2}\frac{\partial h_{ije}}{\partial P_{000}} \\
&= \log\frac{P_{000}P_{0\cdot\cdot}P_{\cdot0\cdot}P_{\cdot\cdot0}}{P_{00\cdot}P_{0\cdot0}P_{\cdot00}}-\log\frac{P_{222}P_{2\cdot\cdot}P_{\cdot2\cdot}P_{\cdot\cdot2}}{P_{22\cdot}P_{2\cdot2}P_{\cdot22}}.
\end{aligned}
$$

Similarly, we have for $i, j, e = 0, 1$

$$\frac{\partial h}{\partial P_{ije}} = \log \frac{P_{ije} P_{i\cdot\cdot} P_{\cdot j\cdot} P_{\cdot\cdot e}}{P_{ij\cdot} P_{i\cdot e} P_{\cdot je}} - \log \frac{P_{222} P_{2\cdot\cdot} P_{\cdot 2\cdot} P_{\cdot\cdot 2}}{P_{22\cdot} P_{2\cdot 2} P_{\cdot 22}}.$$

B.2 The Subscripts ije Contain One 2

Notice

$$\begin{aligned}
\frac{\partial h_{000}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}} \left[P_{000} \log \frac{P_{000} P_{0\cdot\cdot} P_{\cdot 0\cdot} P_{\cdot\cdot 0}}{P_{00\cdot} P_{0\cdot 0} P_{\cdot 00}} \right] = \left[\frac{P_{000}}{P_{0\cdot\cdot}} + \frac{P_{000}}{P_{\cdot 0\cdot}} - \frac{P_{000}}{P_{00\cdot}} \right] \log e, \\
\frac{\partial h_{001}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}} \left[P_{001} \log \frac{P_{001} P_{0\cdot\cdot} P_{\cdot 0\cdot} P_{\cdot\cdot 1}}{P_{00\cdot} P_{0\cdot 1} P_{\cdot 01}} \right] = \left[\frac{P_{001}}{P_{0\cdot\cdot}} + \frac{P_{001}}{P_{\cdot 0\cdot}} - \frac{P_{001}}{P_{00\cdot}} \right] \log e, \\
\frac{\partial h_{002}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}} \left[P_{002} \log \frac{P_{002} P_{0\cdot\cdot} P_{\cdot 0\cdot} P_{\cdot\cdot 2}}{P_{00\cdot} P_{0\cdot 2} P_{\cdot 02}} \right] \\
&= \log \frac{P_{002} P_{0\cdot\cdot} P_{\cdot 0\cdot} P_{\cdot\cdot 2}}{P_{00\cdot} P_{0\cdot 2} P_{\cdot 02}} + \left[1 + \frac{P_{002}}{P_{0\cdot\cdot}} + \frac{P_{002}}{P_{\cdot 0\cdot}} - \frac{P_{002}}{P_{00\cdot}} - \frac{P_{002}}{P_{0\cdot 2}} - \frac{P_{002}}{P_{\cdot 02}} \right] \log e, \\
\frac{\partial h_{010}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}} \left[P_{010} \log \frac{P_{010} P_{0\cdot\cdot} P_{\cdot 1\cdot} P_{\cdot\cdot 0}}{P_{01\cdot} P_{0\cdot 0} P_{\cdot 10}} \right] = \frac{P_{010}}{P_{0\cdot\cdot}} \log e, \\
\frac{\partial h_{011}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}} \left[P_{011} \log \frac{P_{011} P_{0\cdot\cdot} P_{\cdot 1\cdot} P_{\cdot\cdot 1}}{P_{01\cdot} P_{0\cdot 1} P_{\cdot 11}} \right] = \frac{P_{011}}{P_{0\cdot\cdot}} \log e, \\
\frac{\partial h_{012}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}} \left[P_{012} \log \frac{P_{012} P_{0\cdot\cdot} P_{\cdot 1\cdot} P_{\cdot\cdot 2}}{P_{01\cdot} P_{0\cdot 2} P_{\cdot 12}} \right] = \left[\frac{P_{012}}{P_{0\cdot\cdot}} - \frac{P_{012}}{P_{0\cdot 2}} \right] \log e, \\
\frac{\partial h_{020}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}} \left[P_{020} \log \frac{P_{020} P_{0\cdot\cdot} P_{\cdot 2\cdot} P_{\cdot\cdot 0}}{P_{02\cdot} P_{0\cdot 0} P_{\cdot 20}} \right] = \left[\frac{P_{020}}{P_{0\cdot\cdot}} - \frac{P_{020}}{P_{\cdot 2\cdot}} \right] \log e, \\
\frac{\partial h_{021}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}} \left[P_{021} \log \frac{P_{021} P_{0\cdot\cdot} P_{\cdot 2\cdot} P_{\cdot\cdot 1}}{P_{02\cdot} P_{0\cdot 1} P_{\cdot 21}} \right] = \left[\frac{P_{021}}{P_{0\cdot\cdot}} - \frac{P_{021}}{P_{\cdot 2\cdot}} \right] \log e, \\
\frac{\partial h_{022}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}} \left[P_{022} \log \frac{P_{022} P_{0\cdot\cdot} P_{\cdot 2\cdot} P_{\cdot\cdot 2}}{P_{02\cdot} P_{0\cdot 2} P_{\cdot 22}} \right] = \left[\frac{P_{022}}{P_{0\cdot\cdot}} - \frac{P_{022}}{P_{\cdot 2\cdot}} - \frac{P_{022}}{P_{0\cdot 2}} + \frac{P_{022}}{P_{\cdot 22}} \right] \log e.
\end{aligned} \tag{B.4}$$

In addition, we have

$$\begin{aligned}
\frac{\partial h_{100}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}} \left[P_{100} \log \frac{P_{100} P_{1\cdot\cdot} P_{\cdot 0\cdot} P_{\cdot\cdot 0}}{P_{10\cdot} P_{1\cdot 0} P_{\cdot 00}} \right] = \frac{P_{100}}{P_{\cdot 0\cdot}} \log e, \\
\frac{\partial h_{101}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}} \left[P_{101} \log \frac{P_{101} P_{1\cdot\cdot} P_{\cdot 0\cdot} P_{\cdot\cdot 1}}{P_{10\cdot} P_{1\cdot 1} P_{10\cdot}} \right] = \frac{P_{101}}{P_{\cdot 0\cdot}} \log e, \\
\frac{\partial h_{102}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}} \left[P_{102} \log \frac{P_{102} P_{1\cdot\cdot} P_{\cdot 0\cdot} P_{\cdot\cdot 2}}{P_{10\cdot} P_{1\cdot 2} P_{\cdot 02}} \right] = \left[\frac{P_{102}}{P_{\cdot 0\cdot}} - \frac{P_{102}}{P_{\cdot 02}} \right] \log e, \\
\frac{\partial h_{110}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}} \left[P_{110} \log \frac{P_{110} P_{1\cdot\cdot} P_{\cdot 1\cdot} P_{\cdot\cdot 0}}{P_{11\cdot} P_{1\cdot 0} P_{\cdot 10}} \right] = 0, \\
\frac{\partial h_{111}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}} \left[P_{111} \log \frac{P_{111} P_{1\cdot\cdot} P_{\cdot 1\cdot} P_{\cdot\cdot 1}}{P_{11\cdot} P_{1\cdot 1} P_{\cdot 11}} \right] = 0, \\
\frac{\partial h_{112}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}} \left[P_{112} \log \frac{P_{112} P_{1\cdot\cdot} P_{\cdot 1\cdot} P_{\cdot\cdot 2}}{P_{11\cdot} P_{1\cdot 2} P_{\cdot 12}} \right] = 0,
\end{aligned} \tag{B.5}$$

$$
\begin{aligned}
\frac{\partial h_{120}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{120}\log\frac{P_{120}P_{1\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 0}}{P_{12\cdot}P_{1\cdot 0}P_{\cdot 20}}\right] = -\frac{P_{120}}{P_{\cdot 2\cdot}}\log e,\\
\frac{\partial h_{121}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{121}\log\frac{P_{121}P_{1\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 1}}{P_{12\cdot}P_{1\cdot 1}P_{\cdot 21}}\right] = -\frac{P_{121}}{P_{\cdot 2\cdot}}\log e,\\
\frac{\partial h_{122}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{122}\log\frac{P_{122}P_{1\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}{P_{12\cdot}P_{1\cdot 2}P_{\cdot 22}}\right] = \left[-\frac{P_{122}}{P_{\cdot 2\cdot}}+\frac{P_{122}}{P_{\cdot 22}}\right]\log e,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial h_{200}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{200}\log\frac{P_{200}P_{2\cdot\cdot}P_{\cdot 0\cdot}P_{\cdot\cdot 0}}{P_{20\cdot}P_{2\cdot 0}P_{\cdot 00}}\right] = \left[-\frac{P_{200}}{P_{2\cdot\cdot}}+\frac{P_{200}}{P_{\cdot 0\cdot}}\right]\log e,\\
\frac{\partial h_{201}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{201}\log\frac{P_{201}P_{2\cdot\cdot}P_{\cdot 0\cdot}P_{\cdot\cdot 1}}{P_{20\cdot}P_{2\cdot 1}P_{\cdot 01}}\right] = \left[-\frac{P_{201}}{P_{2\cdot\cdot}}+\frac{P_{201}}{P_{\cdot 0\cdot}}\right]\log e,\\
\frac{\partial h_{202}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{202}\log\frac{P_{202}P_{2\cdot\cdot}P_{\cdot 0\cdot}P_{\cdot\cdot 2}}{P_{20\cdot}P_{2\cdot 2}P_{\cdot 02}}\right] = \left[-\frac{P_{202}}{P_{2\cdot\cdot}}+\frac{P_{202}}{P_{\cdot 0\cdot}}-\frac{P_{202}}{P_{\cdot 02}}+\frac{P_{202}}{P_{2\cdot 2}}\right]\log e,\\
\frac{\partial h_{210}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{210}\log\frac{P_{210}P_{2\cdot\cdot}P_{\cdot 1\cdot}P_{\cdot\cdot 0}}{P_{21\cdot}P_{2\cdot 0}P_{\cdot 10}}\right] = -\frac{P_{210}}{P_{2\cdot\cdot}}\log e,\\
\frac{\partial h_{211}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{211}\log\frac{P_{211}P_{2\cdot\cdot}P_{\cdot 1\cdot}P_{\cdot\cdot 1}}{P_{21\cdot}P_{2\cdot 1}P_{\cdot 11}}\right] = -\frac{P_{211}}{P_{2\cdot\cdot}}\log e,\\
\frac{\partial h_{212}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{212}\log\frac{P_{212}P_{2\cdot\cdot}P_{\cdot 1\cdot}P_{\cdot\cdot 2}}{P_{21\cdot}P_{2\cdot 2}P_{\cdot 12}}\right] = \left[-\frac{P_{212}}{P_{2\cdot\cdot}}+\frac{P_{212}}{P_{2\cdot 2}}\right]\log e,\\
\frac{\partial h_{220}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{220}\log\frac{P_{220}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 0}}{P_{22\cdot}P_{2\cdot 0}P_{\cdot 20}}\right] = \left[-\frac{P_{220}}{P_{2\cdot\cdot}}-\frac{P_{220}}{P_{\cdot 2\cdot}}+\frac{P_{220}}{P_{22\cdot}}\right]\log e,\\
\frac{\partial h_{221}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{221}\log\frac{P_{221}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 1}}{P_{22\cdot}P_{2\cdot 1}P_{\cdot 21}}\right] = \left[-\frac{P_{221}}{P_{2\cdot\cdot}}-\frac{P_{221}}{P_{\cdot 2\cdot}}+\frac{P_{221}}{P_{22\cdot}}\right]\log e,\\
\frac{\partial h_{222}}{\partial P_{002}} &= \frac{\partial}{\partial P_{002}}\left[P_{222}\log\frac{P_{222}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}{P_{22\cdot}P_{2\cdot 2}P_{\cdot 22}}\right]\\
&= -\log\frac{P_{222}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}{P_{22\cdot}P_{2\cdot 2}P_{\cdot 22}}+\left[-1-\frac{P_{222}}{P_{2\cdot\cdot}}-\frac{P_{222}}{P_{\cdot 2\cdot}}+\frac{P_{222}}{P_{22\cdot}}+\frac{P_{222}}{P_{2\cdot 2}}+\frac{P_{222}}{P_{\cdot 22}}\right]\log e.
\end{aligned}
\tag{B.6}
$$

By relations (B.4), (B.5), and (B.6), we have

$$
\begin{aligned}
\frac{\partial h}{\partial P_{002}} &= \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{e=0}^{2}\frac{\partial h_{ije}}{\partial P_{002}}\\
&= \log\frac{P_{002}P_{0\cdot\cdot}P_{\cdot 0\cdot}P_{\cdot\cdot 2}}{P_{00\cdot}P_{0\cdot 2}P_{\cdot 02}}-\log\frac{P_{222}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}{P_{22\cdot}P_{2\cdot 2}P_{\cdot 22}}.
\end{aligned}
$$

Similarly, we can show relations (17) when the subscripts ije contain one 2.

B.3 The Subscripts ije Contain Two 2

Notice

$$\begin{aligned}
\frac{\partial h_{000}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{000}\log\frac{P_{000}P_{0..}P_{.0.}P_{..0}}{P_{00.}P_{0\cdot0}P_{\cdot00}}\right] = 0,\\
\frac{\partial h_{001}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{001}\log\frac{P_{001}P_{0..}P_{.0.}P_{..1}}{P_{00.}P_{0\cdot1}P_{\cdot01}}\right] = 0,\\
\frac{\partial h_{002}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{002}\log\frac{P_{002}P_{0..}P_{.0.}P_{..2}}{P_{00.}P_{0\cdot2}P_{\cdot02}}\right] = 0,\\
\frac{\partial h_{010}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{010}\log\frac{P_{010}P_{0..}P_{.1.}P_{..0}}{P_{01.}P_{0\cdot0}P_{\cdot10}}\right] = 0,\\
\frac{\partial h_{011}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{011}\log\frac{P_{011}P_{0..}P_{.1.}P_{..1}}{P_{01.}P_{0\cdot1}P_{\cdot11}}\right] = 0,\\
\frac{\partial h_{012}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{012}\log\frac{P_{012}P_{0..}P_{.1.}P_{..2}}{P_{01.}P_{0\cdot2}P_{\cdot12}}\right] = 0,\\
\frac{\partial h_{020}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{020}\log\frac{P_{020}P_{0..}P_{.2.}P_{..0}}{P_{02.}P_{0\cdot0}P_{\cdot20}}\right] = 0,\\
\frac{\partial h_{021}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{021}\log\frac{P_{021}P_{0..}P_{.2.}P_{..1}}{P_{02.}P_{0\cdot1}P_{\cdot21}}\right] = 0,\\
\frac{\partial h_{022}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{022}\log\frac{P_{022}P_{0..}P_{.2.}P_{..2}}{P_{02.}P_{0\cdot2}P_{\cdot22}}\right] = 0.
\end{aligned} \tag{B.7}$$

In addition, we have

$$\begin{aligned}
\frac{\partial h_{100}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{100}\log\frac{P_{100}P_{1..}P_{.0.}P_{..0}}{P_{10.}P_{1\cdot0}P_{\cdot00}}\right] = \frac{P_{100}}{P_{1..}}\log e,\\
\frac{\partial h_{101}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{101}\log\frac{P_{101}P_{1..}P_{.0.}P_{..1}}{P_{10.}P_{1\cdot1}P_{\cdot01}}\right] = \frac{P_{101}}{P_{1..}}\log e,\\
\frac{\partial h_{102}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{102}\log\frac{P_{102}P_{1..}P_{.0.}P_{..2}}{P_{10.}P_{1\cdot2}P_{\cdot02}}\right] = \left[\frac{P_{102}}{P_{1..}}-\frac{P_{102}}{P_{1\cdot2}}\right]\log e,\\
\frac{\partial h_{110}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{110}\log\frac{P_{110}P_{1..}P_{.1.}P_{..0}}{P_{11.}P_{1\cdot0}P_{\cdot10}}\right] = \frac{P_{110}}{P_{1..}}\log e,\\
\frac{\partial h_{111}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{111}\log\frac{P_{111}P_{1..}P_{.1.}P_{..1}}{P_{11.}P_{1\cdot1}P_{\cdot11}}\right] = \frac{P_{111}}{P_{1..}}\log e,\\
\frac{\partial h_{112}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{112}\log\frac{P_{112}P_{1..}P_{.1.}P_{..2}}{P_{11.}P_{1\cdot2}P_{\cdot12}}\right] = \left[\frac{P_{112}}{P_{1..}}-\frac{P_{112}}{P_{1\cdot2}}\right]\log e,\\
\frac{\partial h_{120}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{120}\log\frac{P_{120}P_{1..}P_{.2.}P_{..0}}{P_{12.}P_{1\cdot0}P_{\cdot20}}\right] = \left[\frac{P_{120}}{P_{1..}}-\frac{P_{120}}{P_{12.}}\right]\log e,\\
\frac{\partial h_{121}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{121}\log\frac{P_{121}P_{1..}P_{.2.}P_{..1}}{P_{12.}P_{1\cdot1}P_{\cdot21}}\right] = \left[\frac{P_{121}}{P_{1..}}-\frac{P_{121}}{P_{12.}}\right]\log e,\\
\frac{\partial h_{122}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{122}\log\frac{P_{122}P_{1..}P_{.2.}P_{..2}}{P_{12.}P_{1\cdot2}P_{\cdot22}}\right]\\
&= \log\frac{P_{122}P_{1..}P_{.2.}P_{..2}}{P_{12.}P_{1\cdot2}P_{\cdot22}}+\left[1+\frac{P_{122}}{P_{1..}}-\frac{P_{122}}{P_{12.}}-\frac{P_{122}}{P_{1\cdot2}}\right]\log e,
\end{aligned} \tag{B.8}$$

and

$$
\begin{aligned}
\frac{\partial h_{200}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{200}\log\frac{P_{200}P_{2\cdot\cdot}P_{\cdot 0\cdot}P_{\cdot\cdot 0}}{P_{20\cdot}P_{2\cdot 0}P_{\cdot 00}}\right] = -\frac{P_{200}}{P_{2\cdot\cdot}}\log e,\\
\frac{\partial h_{201}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{201}\log\frac{P_{201}P_{2\cdot\cdot}P_{\cdot 0\cdot}P_{\cdot\cdot 1}}{P_{20\cdot}P_{2\cdot 1}P_{\cdot 01}}\right] = -\frac{P_{201}}{P_{2\cdot\cdot}}\log e,\\
\frac{\partial h_{202}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{202}\log\frac{P_{202}P_{2\cdot\cdot}P_{\cdot 0\cdot}P_{\cdot\cdot 2}}{P_{20\cdot}P_{2\cdot 2}P_{\cdot 02}}\right] = \left[-\frac{P_{202}}{P_{2\cdot\cdot}}+\frac{P_{202}}{P_{2\cdot 2}}\right]\log e,\\
\frac{\partial h_{210}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{210}\log\frac{P_{210}P_{2\cdot\cdot}P_{\cdot 1\cdot}P_{\cdot\cdot 0}}{P_{21\cdot}P_{2\cdot 0}P_{\cdot 10}}\right] = -\frac{P_{210}}{P_{2\cdot\cdot}}\log e,\\
\frac{\partial h_{211}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{211}\log\frac{P_{211}P_{2\cdot\cdot}P_{\cdot 1\cdot}P_{\cdot\cdot 1}}{P_{21\cdot}P_{2\cdot 1}P_{\cdot 11}}\right] = -\frac{P_{211}}{P_{2\cdot\cdot}}\log e,\\
\frac{\partial h_{212}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{212}\log\frac{P_{212}P_{2\cdot\cdot}P_{\cdot 1\cdot}P_{\cdot\cdot 2}}{P_{21\cdot}P_{2\cdot 2}P_{\cdot 12}}\right] = \left[-\frac{P_{212}}{P_{2\cdot\cdot}}+\frac{P_{212}}{P_{2\cdot 2}}\right]\log e,\\
\frac{\partial h_{220}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{220}\log\frac{P_{220}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 0}}{P_{22\cdot}P_{2\cdot 0}P_{\cdot 20}}\right] = \left[-\frac{P_{220}}{P_{2\cdot\cdot}}+\frac{P_{220}}{P_{22\cdot}}\right]\log e,\\
\frac{\partial h_{221}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{221}\log\frac{P_{221}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 1}}{P_{22\cdot}P_{2\cdot 1}P_{\cdot 21}}\right] = \left[-\frac{P_{221}}{P_{2\cdot\cdot}}+\frac{P_{221}}{P_{22\cdot}}\right]\log e,\\
\frac{\partial h_{222}}{\partial P_{122}} &= \frac{\partial}{\partial P_{122}}\left[P_{222}\log\frac{P_{222}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}{P_{22\cdot}P_{2\cdot 2}P_{\cdot 22}}\right]\\
&= -\log\frac{P_{222}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}{P_{22\cdot}P_{2\cdot 2}P_{\cdot 22}}+\left[-1-\frac{P_{222}}{P_{2\cdot\cdot}}+\frac{P_{222}}{P_{22\cdot}}+\frac{P_{222}}{P_{2\cdot 2}}\right]\log e.
\end{aligned}
\tag{B.9}
$$

By relations (B.7), (B.8), and (B.9), we have

$$
\begin{aligned}
\frac{\partial h}{\partial P_{122}} &= \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{e=0}^{2}\frac{\partial h_{ije}}{\partial P_{122}}\\
&= \log\frac{P_{122}P_{1\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}{P_{12\cdot}P_{1\cdot 2}P_{\cdot 22}}-\log\frac{P_{222}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}{P_{22\cdot}P_{2\cdot 2}P_{\cdot 22}}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\frac{\partial h}{\partial P_{022}} &= \log\frac{P_{022}P_{0\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}{P_{02\cdot}P_{0\cdot 2}P_{\cdot 22}}-\log\frac{P_{222}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}{P_{22\cdot}P_{2\cdot 2}P_{\cdot 22}},\\
\frac{\partial h}{\partial P_{202}} &= \log\frac{P_{202}P_{2\cdot\cdot}P_{\cdot 0\cdot}P_{\cdot\cdot 2}}{P_{20\cdot}P_{2\cdot 2}P_{\cdot 02}}-\log\frac{P_{222}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}{P_{22\cdot}P_{2\cdot 2}P_{\cdot 22}},\\
\frac{\partial h}{\partial P_{212}} &= \log\frac{P_{212}P_{2\cdot\cdot}P_{\cdot 1\cdot}P_{\cdot\cdot 2}}{P_{21\cdot}P_{2\cdot 2}P_{\cdot 12}}-\log\frac{P_{222}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}{P_{22\cdot}P_{2\cdot 2}P_{\cdot 22}},\\
\frac{\partial h}{\partial P_{220}} &= \log\frac{P_{220}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 0}}{P_{22\cdot}P_{2\cdot 0}P_{\cdot 20}}-\log\frac{P_{222}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}{P_{22\cdot}P_{2\cdot 2}P_{\cdot 22}},\\
\frac{\partial h}{\partial P_{221}} &= \log\frac{P_{221}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 1}}{P_{22\cdot}P_{2\cdot 1}P_{\cdot 21}}-\log\frac{P_{222}P_{2\cdot\cdot}P_{\cdot 2\cdot}P_{\cdot\cdot 2}}{P_{22\cdot}P_{2\cdot 2}P_{\cdot 22}}.
\end{aligned}
$$

Appendix C Proof of Relations (19) and (20)

Notice

$$\begin{aligned}
\frac{\partial h}{\partial P_{0\cdots 0}} &= \sum_{a_1=0}^{2}\cdots\sum_{a_K=0}^{2}\frac{\partial h_{a_1\cdots a_K}}{\partial P_{0\cdots 0}} \\
&= \sum_{a_1=0}^{2}\cdots\sum_{a_K=0}^{2}\frac{\partial}{\partial P_{0\cdots 0}}\left[P_{a_1\cdots a_K}\log\frac{P_{a_1\cdots a_K}\prod_{\substack{\vec{s}\subset(a_1,\cdots,a_K)\\ |(a_1,\cdots,a_K)\setminus\vec{s}|_{mod(2)}=0}}P_{\vec{s}}}{\prod_{\substack{\vec{s}\subset(a_1,\cdots,a_K)\\ |(a_1,\cdots,a_K)\setminus\vec{s}|_{mod(2)}=1}}P_{\vec{s}}}\right] \\
&= \log\frac{P_{0\cdots 0}\prod_{\substack{\vec{s}\subset(0,\cdots,0)\\ |(0,\cdots,0)\setminus\vec{s}|_{mod(2)}=0}}P_{\vec{s}}}{\prod_{\substack{\vec{s}\subset(0,\cdots,0)\\ |(0,\cdots,0)\setminus\vec{s}|_{mod(2)}=1}}P_{\vec{s}}}-\log\frac{P_{2\cdots 2}\prod_{\substack{\vec{s}\subset(2,\cdots,2)\\ |(2,\cdots,2)\setminus\vec{s}|_{mod(2)}=0}}P_{\vec{s}}}{\prod_{\substack{\vec{s}\subset(2,\cdots,2)\\ |(2,\cdots,2)\setminus\vec{s}|_{mod(2)}=1}}P_{\vec{s}}} \\
&\quad+\sum_{a_1=0}^{2}\cdots\sum_{a_K=0}^{2}P_{a_1\cdots a_K}\frac{\partial}{\partial P_{0\cdots 0}}\left[\log\frac{\prod_{\substack{\vec{s}\subset(a_1,\cdots,a_K)\\ |(a_1,\cdots,a_K)\setminus\vec{s}|_{mod(2)}=0}}P_{\vec{s}}}{\prod_{\substack{\vec{s}\subset(a_1,\cdots,a_K)\\ |(a_1,\cdots,a_K)\setminus\vec{s}|_{mod(2)}=1}}P_{\vec{s}}}\right]. \qquad \text{(C.1)}
\end{aligned}$$

It can be showed that

$$\sum_{a_1=0}^{2}\cdots\sum_{a_K=0}^{2}P_{a_1\cdots a_K}\frac{\partial}{\partial P_{0\cdots 0}}\left[\log\prod_{\substack{\vec{s}\subset(a_1,\cdots,a_K)\\ |(a_1,\cdots,a_K)\setminus\vec{s}|=i}}P_{\vec{s}}\right] = 0,\quad i=1,2,\cdots,K-1.$$

Therefore, we have

$$\begin{aligned}
&\sum_{a_1=0}^{2}\cdots\sum_{a_K=0}^{2}P_{a_1\cdots a_K}\frac{\partial}{\partial P_{0\cdots 0}}\left[\log\frac{\prod_{\substack{\vec{s}\subset(a_1,\cdots,a_K)\\ |(a_1,\cdots,a_K)\setminus\vec{s}|_{mod(2)}=0}}P_{\vec{s}}}{\prod_{\substack{\vec{s}\subset(a_1,\cdots,a_K)\\ |(a_1,\cdots,a_K)\setminus\vec{s}|_{mod(2)}=1}}P_{\vec{s}}}\right] \\
&= \sum_{a_1=0}^{2}\cdots\sum_{a_K=0}^{2}P_{a_1\cdots a_K}\frac{\partial}{\partial P_{0\cdots 0}}\left[\log\prod_{\substack{\vec{s}\subset(a_1,\cdots,a_K)\\ |(a_1,\cdots,a_K)\setminus\vec{s}|_{mod(2)}=0}}P_{\vec{s}}\right] \\
&\quad-\sum_{a_1=0}^{2}\cdots\sum_{a_K=0}^{2}P_{a_1\cdots a_K}\frac{\partial}{\partial P_{0\cdots 0}}\left[\log\prod_{\substack{\vec{s}\subset(a_1,\cdots,a_K)\\ |(a_1,\cdots,a_K)\setminus\vec{s}|_{mod(2)}=1}}P_{\vec{s}}\right] \\
&= 0.
\end{aligned}$$

Thus, the equation (C.1) implies

$$\frac{\partial h}{\partial P_{0\cdots 0}} = \log\frac{P_{0\cdots 0}\prod_{\substack{\vec{s}\subset(0,\cdots,0)\\ |(0,\cdots,0)\setminus\vec{s}|_{mod(2)}=0}}P_{\vec{s}}}{\prod_{\substack{\vec{s}\subset(0,\cdots,0)\\ |(0,\cdots,0)\setminus\vec{s}|_{mod(2)}=1}}P_{\vec{s}}}-\log\frac{P_{2\cdots 2}\prod_{\substack{\vec{s}\subset(2,\cdots,2)\\ |(2,\cdots,2)\setminus\vec{s}|_{mod(2)}=0}}P_{\vec{s}}}{\prod_{\substack{\vec{s}\subset(2,\cdots,2)\\ |(2,\cdots,2)\setminus\vec{s}|_{mod(2)}=1}}P_{\vec{s}}}.$$

Similarly, we may show the relations (19) and (20).

Appendix D SNP Combinations Used to Calculate the Empirical Type I Error Rates

In Table D.1, we present the SNP combinations and their joint genotype frequencies used in the calculations of empirical type I error rates of T_{IG}, T_{IIG}, and T_{TCIG}.

In Table D.2, we present the three SNP combinations used to calculate the empirical type I error rates of test statistic T_{IIG}. The three SNPs in each combination are strongly correlated to each other. Consider a combination of three SNPs A, B, and C. By significantly correlated to each other for the three SNPs, we mean all the four null hypotheses,

$$\begin{aligned} H_1 : P(G_A, G_B, G_C) &= P(G_A, G_B)P(G_C), \\ H_2 : P(G_A, G_B, G_C) &= P(G_A)P(G_B, G_C), \\ H_3 : P(G_A, G_B, G_C) &= P(G_B)P(G_A, G_C), \\ H_4 : P(G_A, G_B, G_C) &= P(G_A)P(G_B)P(G_C), \end{aligned}$$

all unlikely to be true. We use the Pearson χ^2 test to screen the three SNP combinations. The empirical type I error rates are reported in Table 2.

Table D.1: Joint genotype frequencies of bladder cancer data used to calculate the type I error rates of 2-way test statistic T_{IG} and 3-way test statistics T_{IIG} and T_{TCIG} in Table 2.

Test	**SNPs used to generate Joint Genotype Frequency**	**Joint Genotype Frequencies**								
T_{IG}	Xpd_751, Xpd_312	0.27907	0.07513	0.02683	0.10733	0.34526	0.03399	0.00894	0.05903	0.06440
T_{TCIG}	Xpd_751, Xpd_312, APE1 148	0.08229	0.01789	0.01252	0.03578	0.08766	0.00537	0.00358	0.01431	0.01252
		0.13775	0.03936	0.01252	0.06082	0.18068	0.02147	0.00179	0.02683	0.02862
		0.05903	0.01789	0.00179	0.01073	0.07692	0.00716	0.00358	0.01789	0.02326
	Xpd_751, Xpd_312, XPC PAT	0.09302	0.13059	0.05546	0.01968	0.03399	0.02147	0.00537	0.01789	0.00358
		0.03757	0.04830	0.02147	0.12343	0.15027	0.07156	0.01431	0.01252	0.00716
		0.00000	0.00537	0.00358	0.01789	0.02326	0.01789	0.01610	0.03578	0.01252
	Xpd_751, Xpd_312, Pack_years	0.09123	0.11628	0.07156	0.01610	0.04114	0.01789	0.01252	0.00537	0.00894
		0.02683	0.04472	0.03578	0.12880	0.12522	0.09123	0.00894	0.01789	0.00716
		0.00358	0.00179	0.00358	0.01610	0.02504	0.01789	0.01610	0.02504	0.02326
	Xpd_751, Xpd_312, XRCC3_241	0.09481	0.03936	0.01431	0.04472	0.11807	0.00894	0.00358	0.02504	0.02862
		0.14132	0.02683	0.01252	0.04114	0.18784	0.02326	0.00537	0.02326	0.02683
		0.04293	0.00894	0.00000	0.02147	0.03936	0.00179	0.00000	0.01073	0.00894
	Xpd_751, Xpd_312, XRCC1_399	0.12701	0.02862	0.01789	0.10376	0.03578	0.00537	0.04830	0.01073	0.00358
		0.04830	0.13059	0.00894	0.03757	0.17174	0.01789	0.02147	0.04293	0.00716
		0.00537	0.02504	0.02504	0.00358	0.02326	0.02147	0.00000	0.01073	0.01789
	Xpd_751, Xpd_312, XRCC1_194	0.23792	0.03399	0.00716	0.05367	0.01610	0.00537	0.01431	0.00537	0.00716
		0.09302	0.01073	0.00358	0.30054	0.03757	0.00716	0.02326	0.00716	0.00358
		0.00894	0.00000	0.00000	0.05009	0.00894	0.00000	0.04830	0.01431	0.00179
T_{IIG}	APE1, XRCC1_399, XRCC1_194	0.10018	0.01073	0.00358	0.08766	0.01252	0.00179	0.05367	0.00000	0.00179
		0.16816	0.03399	0.00358	0.17531	0.03399	0.01610	0.07156	0.00179	0.00537
		0.07156	0.02326	0.00179	0.08050	0.01073	0.00179	0.02147	0.00716	0.00000
	Xpd_751, Xpd_312, XRCC1_194	0.13239	0.02254	0.00845	0.15211	0.01972	0.00563	0.04507	0.00563	0.00563
		0.12958	0.00563	0.01127	0.19155	0.02535	0.01408	0.03662	0.02535	0.00563
		0.05352	0.01690	0.00000	0.03662	0.01127	0.00000	0.03099	0.00845	0.00000
	XRCC3_241, APE1, XRCC1_194	0.07513	0.00894	0.00358	0.17352	0.02862	0.00894	0.06261	0.01610	0.00000
		0.11807	0.01073	0.00358	0.19499	0.03220	0.01252	0.09123	0.00179	0.02326
		0.04830	0.00358	0.00000	0.04651	0.00894	0.00358	0.01968	0.00179	0.00179
	XRCC3_241, APE1, XRCC1_399	0.04789	0.07606	0.01408	0.10141	0.07887	0.02535	0.01408	0.04507	0.00845
		0.03380	0.06761	0.00563	0.07324	0.10986	0.05634	0.04225	0.05634	0.00563
		0.01690	0.01690	0.00563	0.04507	0.02817	0.00845	0.00845	0.00282	0.00563
	XRCC3_241, XRCC1_399, XRCC1_194	0.11831	0.03380	0.01127	0.17746	0.01127	0.01127	0.04507	0.00000	0.00282
		0.11268	0.02817	0.00845	0.18028	0.05070	0.00282	0.05634	0.00563	0.00563
		0.05915	0.01127	0.00000	0.04225	0.00000	0.00563	0.01690	0.00000	0.00282

Table D.2: Three SNP combinations used to calculate the empirical type I error rates of test T_{IIG}. In each combination, the three SNPs are strongly correlated to each other. In the Table, $P(A, B, C) = P(G_A, G_B, G_C)$ means the joint genotype probability of three SNPs A, B, and C, etc.

SNPs			**The Null Hypothesis H_0, Test Values and the Related P-values of Pearson χ^2 Statistic**							
A	**B**	**C**	$P(A,B,C) = P(A,B)P(C)$		$P(A,B,C) = P(A)P(B,C)$		$P(A,B,C) = P(B)P(A,C)$		$P(A,B,C) = P(A)P(B)P(C)$	
			χ^2	P-value	χ^2	P-value	χ^2	P-value	χ^2	P-value
APE1	XRCC1_399	XRCC1_194	27.72382	0.03410582	27.02576	0.04119771	40.75471	0.0006034846	32.56543	0.03763361
Xpd_751	Xpd_312	XRCC1_194	26.23323	0.05083355	27.22967	0.03899995	38.98301	0.001093773	34.63468	0.02213853
XRCC3_241	APE1	XRCC1_194	42.08023	0.0003838726	30.2027	0.01697913	75.70076	9.810734e-10	53.44553	6.98728e-05
XRCC3_241	APE1	XRCC1_399	29.35457	0.021652	24.76337	0.074107	32.45247	0.008726808	34.3638	0.02376652
XRCC3_241	XRCC1_399	XRCC1_194	25.38762	0.0632772	26.1351	0.05215666	42.28494	0.0003577711	31.26234	0.05182107

Appendix E Extra Type I Error Rates

In Table E.1, we presented the type I error rates of 1-way entropy loss test statistic T_{EL}. The test was reasonably robust.

Table E.1: Type I error rates of 1-way test statistic T_{EL} at a nominal level $\alpha = 0.01$. One di-allelic marker A is used in the simulation. Each of the entries was based on $100,000$ simulations.

Frequency	**Sample Sizes** $M = N$								
P_A	100	150	200	250	300	400	500	600	700
0.1	0.01632	0.01428	0.01410	0.01318	0.01271	0.01226	0.01115	0.01124	0.01116
0.2	0.01932	0.01430	0.01280	0.01243	0.01171	0.01137	0.01105	0.01063	0.01073
0.3	0.01479	0.01315	0.01232	0.01083	0.01135	0.01061	0.01007	0.01060	0.01072
0.4	0.00856	0.00797	0.00912	0.00858	0.00963	0.00920	0.00945	0.00967	0.00967
0.5	0.00366	0.00487	0.00609	0.00684	0.00713	0.00765	0.00825	0.00874	0.00853