Supplementary Information for Network Cosmology

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Supplementary Methods

I. DATA AND SIMULATIONS

Here we describe the network data and methods used in Figs. 3&4. The considered real networks are paradigmatic examples of complex technological, social, and biological networks for which reliable large-scale data are available. We note that in all the three considered networks, links represent soft relational data instead of hard-wired network diagrams: economic/business relations in the Internet, trust relations between people, causal/correlation relations in the brain, and causal relations in causal sets.

A. Internet data

The Internet topology in Fig. 4 represents economic/business relations between *autonomous systems* or *ASs*. The AS is an organization or an individual owning a part of the Internet infrastructure. The network is extracted from the data collected by CAIDA's Archipelago Measurement Infrastructure (Ark) [1], `http://www.caida.org/projects/ark/`. The Ark infrastructure consists of a set of monitors continuously tracing IP-level paths to random destinations in the Internet. The union of the paths collected by all the monitors is then aggregated over a certain period of time, and each IP address in the collection is mapped to an AS owning this address, using the RouteViews BGP tables `http://www.routeviews.org/`. The resulting AS network has a power-law degree distribution with exponent $\gamma = 2.1$, and this result is stable over time [2, 3] and across different measurement methodologies [4, 5], `http://www.caida.org/research/topology/topo_comparison/`. The data and its further description are available at `http://www.caida.org/data/active/ipv4_routed_topology_aslinks_dataset.xml`. In Fig. 4, the data for June 2009 is used. The aggregation period is one month, and the number of monitors is 36.

B. Trust data

The trust network in Fig. 4 represents trust relations between people, extracted from the Pretty-Good-Privacy (PGP) data. PGP is a data encryption and decryption computer program that provides cryptographic privacy and authentication for data communication [6], `http://www.openpgp.org/`. In the PGP trust network, nodes are certificates consisting of public PGP keys and owner information. A directed link in the network pointing from certificate A to certificate B represents a digital signature by the owner of A, endorsing the owner/public key association of B. We use the PGP web of trust data collected and maintained by Jörgen Cederlöf `http://www.lysator.liu.se/~jc/wotsap/wots2/`. We post-process the data as follows. The directed network is first mapped to its undirected counterpart by taking into account only bi-directional trust links between the certificates. The largest connected component is then extracted from this undirected network. This way we strengthen the social aspect of the network, since we consider only pairs of users (owners of PGP keys) who have reciprocally signed each other's keys. Such filtering increases the probability that the connected users know each other, and makes the extracted network a reliable proxy to the underlying social network. The degree distribution in the resulting network is a power law with exponent $\gamma = 2.1$, and this result is stable over time [7], `http://pgp.cs.uu.nl/plot/`. In Fig. 4, the data snapshot taken on December 1, 2005, is used.

C. Brain data

The brain functional network in Fig. 4 represents causal/correlation relations between small areas in the human brain. This network is extracted from the fMRI measurements in [8]. In those experiments, the whole brains of different human subjects are split into $36 \times 64 \times 64 = 147456$ adjacent areas called *voxels*, each voxel of volume $3 \times 3.475 \times 3.475$ mm^3. The subjects are then asked to perform different tasks, during which the magnetic resonance activity $V(x,t)$ is recorded at each voxel x at time t. Time t is discrete: 400 recordings are made with the interval of 2.5 s. Given this data, and denoting by $\langle \cdot \rangle$ the time average, the correlation coefficient $r(x,x')$ for each pair of voxels is then computed:

$$r(x, x') = \frac{\langle V(x,t)V(x',t)\rangle - \langle V(x,t)\rangle\langle V(x',t)\rangle}{\sqrt{[\langle V(x,t)^2\rangle - \langle V(x,t)\rangle^2][\langle V(x',t)^2\rangle - \langle V(x',t)\rangle^2]}}. \tag{1}$$

To form a functional network out of this correlation data, two voxels x and x' are considered causally connected if the correlation coefficient between them exceeds a certain threshold r_c, $r(x,x') > r_c$. If r_c is too small or too large, then the resulting network is fully connected or fully disconnected. There exists however a unique percolation transition value of r_c corresponding to the onset of the giant component in the network. To find this value, we compute the sizes $|S_1|_{r_c}$ and $|S_2|_{r_c}$ of the largest and second largest components S_1 and S_2 in the network for different values of r_c. The value of threshold r_c corresponding to the largest values of $|\partial|S_1|/\partial r_c|_{r_c}$ and $|S_2|_{r_c}$ is then its percolation transition value. With threshold r_c set to this value, the network has a power-law degree distribution with exponent $\gamma = 2$ and exponential cutoffs, and this result is stable across different human subjects and different types of activity that they perform during measurements, see [8, 9] and `http://www.caida.org/publications/papers/2012/network_cosmology/supplemental/`. In Fig. 4, a specific dataset is used—*Set14*, see the URL—where the subject is at rest. The threshold value is $r_c = 0.7$.

D. De Sitter causet

The causet in Fig. 4 is generated by sprinkling a number of points over a patch in the $1 + 1$-dimensional de Sitter spacetime, and connecting each pair of points if they lie within the other's light cones. The point density is uniform in the de Sitter metric, and the size of the patch is such that the size of the generated causet and its average degree are close to those in the considered examples of complex networks.

In conformal time coordinates, see Section II A below, each spacetime point has two coordinates, spatial $\theta \in [0, 2\pi]$ and temporal $\eta \in (-\pi/2, \pi/2)$, with $\eta = -\pi/2$ and $\eta = \pi/2$ corresponding to the past and future infinities respectively. The spacetime patch that we consider is between $\eta = 0$ and $\eta = \eta_0 > 0$, where η_0 is determined below. This patch is illustrated in Fig. 2(a). To sample N points from this patch with uniform density, we sample N pairs of random numbers: N spatial coordinates θ drawn from the uniform distribution on $[0, 2\pi]$, and N temporal coordinates η drawn from the distribution

$$\rho(\eta|\eta_0) = \frac{\sec^2 \eta}{\tan \eta_0}. \tag{2}$$

Two spacetime points with coordinates (η, θ) and (η', θ') are then connected in the causet if $\Delta\theta < \Delta\eta$, where $\Delta\theta = \pi - |\pi - |\theta - \theta'||$ and $\Delta\eta = |\eta - \eta'|$ are the spatial and temporal distances between the points in this coordinate system.

To determine η_0 we first note that since the point density is uniform, the number of points N is proportional to the patch volume, and the proportionality coefficient is a constant point density δ. The volume of the patch is easy to calculate, see Section II A, where we can also calculate the average degree $\bar{k}$ in the resulting causets, so that we have:

$$N = 2\pi\delta a^2 \tan\eta_0, \tag{3}$$

$$\bar{k} = 4\delta a^2 \left(\frac{\eta_0}{\tan\eta_0} + \ln\sec\eta_0 - 1\right), \tag{4}$$

$$\frac{\bar{k}}{N} = \frac{2}{\pi} \cdot \frac{\eta_0/\tan\eta_0 + \ln\sec\eta_0 - 1}{\tan\eta_0}, \tag{5}$$

where a is the spacetime's pseudoradius determining its curvature. Given a target average degree $\bar{k}$ and number of nodes N in a causet, their ratio determines η_0 via the last equation. Sampling N points from this patch will then yield causets with expected average degree $\bar{k}$. The average degree in generated causets will not be exactly equal to
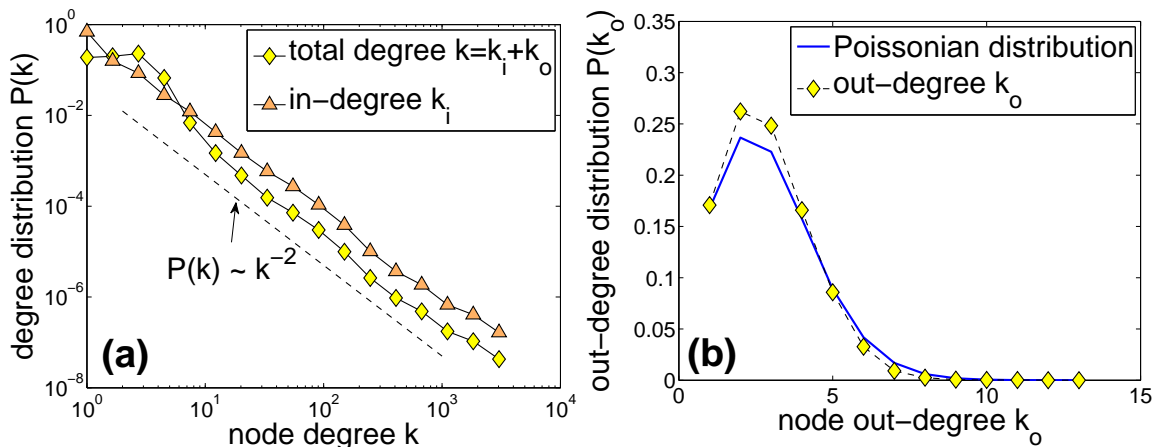
FIG. S1: (a) The total versus in-degree distributions in the de Sitter causet of Fig. 4. (b) The out-degree distribution in the same causet. The solid line shows the Poissonian distribution with the mean $\lambda = \bar{k}_o = \bar{k}/2 = 2.83$, where $\bar{k}_o$ is the average out-degree in the causet.

$\bar{k}$ since there will be nodes of degree 0 and Poissonian fluctuations of the numbers of nodes lying within light cones around their expected values. In Fig. 4, the exact number of sampled points is $N = 24586$ and target value of the average degree is $\bar{k} = 5.53$, so that the above equations yield $\eta_0 = \pi/2 - 3.86 \times 10^{-5}$ and $\delta a^2 = 0.151$. The resulting number of nodes excluding nodes of degree 0 and their average degree in the generated causet are reported in the caption of Fig. 4.

If we direct links in the causet from the future to the past, i.e. from nodes with larger η to nodes with smaller η, then the distributions of in-degrees k_i and total degrees $k = k_i + k_o$ are similar, Fig. S1(a), because the distribution of out-degrees k_o is not particularly interesting, close to Poissonian, Fig. S1(b). The semantics of this link direction in causal sets is similar to that in preferential attachment, where it is often convenient to consider links oriented from new to old nodes [10, 11]. In preferential attachment the out-degree distribution is not particularly interesting either, and is given by the delta-function $\delta(m)$, where $m = \bar{k}/2$ is the number of connections that each new node establishes. Figure 4 shows the degree distribution in the undirected causet, i.e. the distribution of total degrees $k = k_i + k_o$, since all the other networks shown in the figure are undirected (we consider only reciprocal trust relationships in the trust network). As a side note, the direction of links in different directed networks may have very different and network-specific semantics. In gene regulations [12] or the considered web of trust, for example, link directions show what genes regulate what other genes or who trusts whom, respectively, which may have little in common with the temporal link direction in preferential attachment or causal sets, showing what nodes are newer or older.

In higher dimensions, pre-asymptotic effects become more prominent in causets of similar size and average degree. In particular, the exponent of the degree distribution is slightly below 2, see Fig. S2 comparing the causets in the $d = 1$ and $d = 3$ cases. The former causet is the same as in Figs. 4&S1, while the latter is obtained by a procedure similar to the one described above, except that instead of Eqs. (2-5), we have

$$\rho(\eta|\eta_0) = \frac{3 \sec^4 \eta}{(2 + \sec^2 \eta_0) \tan \eta_0}, \tag{6}$$

$$N = \frac{2}{3}\pi^2 \delta a^4 \left(2 + \sec^2 \eta_0\right) \tan \eta_0, \tag{7}$$

$$\bar{k} = \frac{4}{9}\pi \delta a^4 \frac{12\left(\eta_0/\tan \eta_0 + \ln \sec \eta_0\right) + (6 \ln \sec \eta_0 - 5) \sec^2 \eta_0 - 7}{2 + \sec^2 \eta_0}, \tag{8}$$

$$\frac{\bar{k}}{N} = \frac{2}{3\pi} \cdot \frac{12\left(\eta_0/\tan \eta_0 + \ln \sec \eta_0\right) + (6 \ln \sec \eta_0 - 5) \sec^2 \eta_0 - 7}{\left(2 + \sec^2 \eta_0\right)^2 \tan \eta_0}, \tag{9}$$

and each point has two additional angular coordinates, θ_1 and θ_2, which are random variables between 0 and π drawn from distributions $(2/\pi) \sin^2 \theta_1$ and $(1/2) \sin \theta_2$, see Section II A. The spatial distance $\Delta\theta$ between pairs of points is then computed using the spherical law of cosines, and two points are causally linked if $\Delta\theta < \Delta\eta$ as before. In Fig. S2, the $d = 3$ causet has target $N = 25441$ and $\bar{k} = 5.29$, yielding $\eta_0 = \pi/2 - 4.11 \times 10^{-2}$ and $\delta a^4 = 0.267$. The maximum-likelihood fit of the degree sequence in the $d = 3$ and $d = 1$ causets yields $\gamma = 1.65$ and $\gamma = 1.90$, while the least-square fit of the complementary cumulative distribution function for node degrees yields $\gamma = 1.77$ and $\gamma = 1.98$.
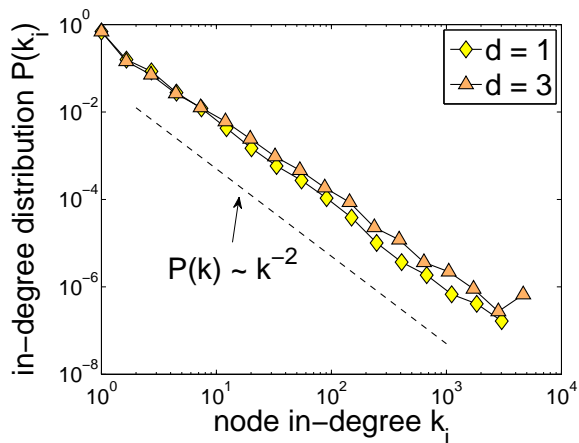
FIG. S2: The in-degree distribution of causets approximated by patches of 1+1 and 3+1 dimensional de Sitter spacetime. The causet sizes are $N = 23739$ and $N = 23732$, and their average degrees are $\bar{k} = 5.65$ and $\bar{k} = 5.25$ respectively.

E. Simulating the universe

The results in Section III below allow us to simulate the universe causet similarly to the de Sitter simulations described in the previous section. The main idea behind the universe simulations is that we can use the exact results for a flat universe, but apply them to a closed universe with an arbitrary finite number of nodes in its causet, since the real universe is almost flat, and since the degree distribution in both cases is the same.

Specifically, the causet in Fig. 3 is generated as follows. Unlike the previous section where only conformal time is used, here it is more convenient to begin with the rescaled time $\tau = t/a$. The current measurements of the universe yield, see Section III C, $\tau_0 = t_0/a = (2/3)\operatorname{arcsinh}\sqrt{\Omega_\Lambda/\Omega_M} = 0.8458$ as the best estimate for this rescaled time in the universe today. According to Eqs. (85,86), the scale factor is

$$R(\tau) = \alpha \sinh^{\frac{2}{3}} \frac{3}{2}\tau, \tag{10}$$

where α is a free parameter that we can set to whatever value we wish, since the degree distribution does not depend on it, see Section III E. We wish to set α to the value such that the generated causet would have a desired number of nodes

$$N = \frac{1}{3}\pi^2 \delta \alpha^3 \left(\sinh 3\tau_0 - 3\tau_0\right), \tag{11}$$

where δ, the node density, is yet another free parameter that does not affect the degree distribution according to Section III E. The scale factor in Eq. (10) means that the temporal coordinate that we assign to each of these N nodes is a random number $\tau \in [0, \tau_0]$ drawn from distribution

$$\rho(\tau|\tau_0) = \frac{6 \sinh^2 \frac{3}{2}\tau}{\sinh 3\tau_0 - 3\tau_0}. \tag{12}$$

Having times τ assigned, we then map them, for each node, to conformal times η via

$$\eta = \frac{2}{3\alpha} \int_0^{\frac{3}{2}\tau} \frac{dx}{\sinh^{\frac{2}{3}} x}. \tag{13}$$

The spatial coordinates θ, θ_1, and θ_2 are then assigned exactly as in the previous section, and the causet network is also formed exactly the same way, i.e. by future→past linking all node pairs whose temporal distance $\Delta\eta$ exceeds their spatial distance $\Delta\theta$.

Figure 3 shows the in-degree distribution for a causet generated with $\alpha = 2.01$, $a = 1$, and $\delta = 10^4$. The resulting number of nodes in the causet is $N = 10^6$. The analytic solution curves are obtained using the approximations for the in-degree distribution derived in Section III. The key equations are Eq. (70), yielding the Laplace approximation for the in-degree distribution, and Eq. (91) expressing conformal time as a function of the current value of the scale factor.
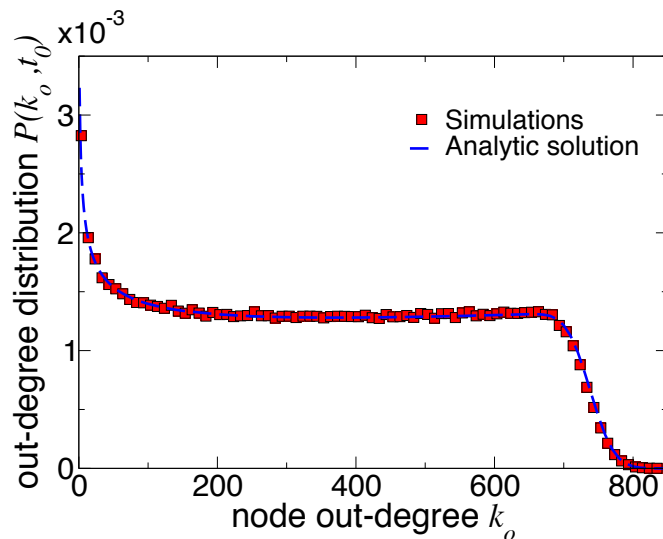
FIG. S3: Out-degree distribution in the simulated universe.

The precise steps to numerically compute the analytic solution for the in-degree distribution are listed in Section III G. The perfect match between the simulations and analytic solution in Fig. 3 confirms that the approximations used in Section III to derive the analytic solution yield very accurate results.

The out-degree distribution in the same simulated causet is shown in Fig. S3. The analytic solution is obtained by numeric evaluations of Eqs.(62-66). The distribution appears to be uniform over a wide range of degree values, which is quite different from the Poissonian out-degree distribution in the sparse de Sitter causet in Fig. S1(b) because $\delta a^{d+1} = 10^4 \gg 1$ in the universe simulations, whereas $\delta a^{d+1} < 1$ in the de Sitter simulations in the previous section. Depending on whether δa^{d+1} is smaller or larger than 1, (asymptotically) de Sitter causets have either power-law in-degree distributions with $\gamma = 2$ and Poissonian out-degree distributions, or double power-law in-degree distributions with $\gamma = 3/4$ and $\gamma = 2$, and non-trivial out-degree distributions of the type shown in Fig. S3.

Supplementary Notes

II. ASYMPTOTIC EQUIVALENCE BETWEEN CAUSAL SETS IN DE SITTER SPACETIME AND COMPLEX NETWORKS IN HYPERBOLIC SPACE

A. De Sitter spacetime

The $d+1$-dimensional de Sitter spacetime [13] is the exact solution of the Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{14}$$

for the empty universe with positive vacuum energy density. In these equations, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ the Ricci tensor, R the scalar curvature, $g_{\mu\nu}$ the metric tensor, Λ the cosmological constant, and all the notations are in the natural units with the speed of light $c = 1$. This spacetime can be represented as the one-sheeted $d+1$-dimensional hyperboloid of constant positive scalar R and Gaussian K curvatures [14]

$$-z_0^2 + z_1^2 + \ldots + z_{d+1}^2 = a^2 = \frac{d(d-1)}{2\Lambda} = \frac{d(d+1)}{R} = \frac{1}{K} \tag{15}$$

borrowing its metric from the $d+2$-dimensional ambient Minkowski space with metric

$$ds^2 = -dz_0^2 + dz_1^2 + \ldots + dz_{d+1}^2. \tag{16}$$

This hyperboloid in the $d=1$ case is visualized as the outer hyperboloid in Fig. 2. As a side note, the curvature of the same hyperboloid in the Euclidean metric is everywhere negative but not constant. For $d=3$, the Hubble constant H, vacuum energy density ρ_Λ, cosmological constant Λ, and the hyperboloid pseudoradius a, scalar curvature R, and Gaussian curvature K are all related by

$$H^2 = \frac{8}{3}\pi G\rho_\Lambda = \frac{\Lambda}{3} = \frac{1}{a^2} = \frac{R}{12} = K, \tag{17}$$

where G is the gravitational constant.

De Sitter spacetime admits different natural coordinate systems with negative, zero, or positive *spatial* curvatures, which are not to be confused with the positive curvature of the whole *spacetime*. Here we use the standard coordinate system $(t, \theta_1, \ldots, \theta_d)$ with positive spatial curvature that covers the whole spacetime:

$$z_0 = a \sinh\frac{t}{a}, \tag{18}$$

$$z_1 = a \cosh\frac{t}{a} \cos\theta_1, \tag{19}$$

$$\vdots$$

$$z_d = a \cosh\frac{t}{a} \sin\theta_1 \ldots \sin\theta_{d-1} \cos\theta_d, \tag{20}$$

$$z_{d+1} = a \cosh\frac{t}{a} \sin\theta_1 \ldots \sin\theta_{d-1} \sin\theta_d, \tag{21}$$

where $t \in \mathbb{R}$ is the cosmological time of the universe, and $\theta_1, \ldots, \theta_{d-1} \in [0, \pi]$ and $\theta_d \in [0, 2\pi]$ are the standard angular coordinates on the unit d-dimensional sphere $\mathbb{S}^d$. The time t at spacetime point P is also the Minkowski length of the arc connecting P to the corresponding point at time $t=0$ belonging to the $z_0 = 0$ slice of the hyperboloid, see Fig. 2. In these coordinates the metric takes the Friedmann-Lemaître-Robertson-Walker (FLRW) form for an exponentially expanding homogeneous and isotropic universe with positive spatial curvature:

$$ds^2 = -dt^2 + a^2 \cosh^2\frac{t}{a}\left(d\theta^2 + \sin^2\theta\, d\Omega_{d-1}^2\right) = -dt^2 + a^2 \cosh^2\frac{t}{a}\, d\Omega_d^2, \text{ where} \tag{22}$$

$$d\Omega_d^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \ldots + \sin^2\theta_1 \ldots \sin^2\theta_{d-1}d\theta_d^2 \tag{23}$$

is the metric on $\mathbb{S}^d$. That is, at each time t, the universe is a sphere of exponentially growing radius $a\cosh(t/a)$ and volume

$$v = \sigma_d\left(a\cosh\frac{t}{a}\right)^d, \quad \text{where } \sigma_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \tag{24}$$
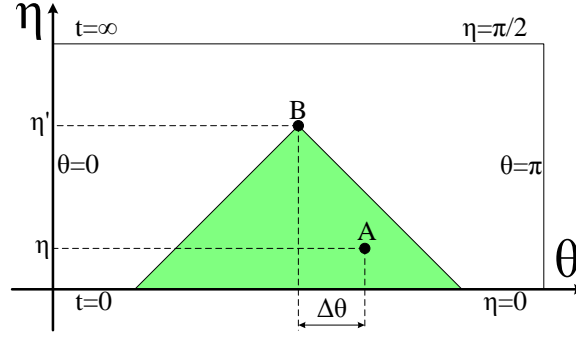
FIG. S4: Causal structure of the $d+1$-dimensional de Sitter spacetime. $d-1$ dimensions are suppressed, so that each point represents a $d-1$-sphere. $\Delta\theta$ is the distance between A and B on the d-dimensional unit sphere $\mathbb{S}^d$.

is the volume of $\mathbb{S}^d$. Figure 2 visualizes this foliation for $d=1$, in which case these time-slice spheres are circles, and their volume is the circle's circumference.

To study the causal structure of de Sitter spacetime, it is convenient to introduce conformal time $\eta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ via

$$\sec\eta = \cosh\frac{t}{a}. \tag{25}$$

These conformal-time coordinates are convenient because the metric becomes

$$ds^2 = a^2 \sec^2\eta \left(-d\eta^2 + d\theta^2 + \sin^2\theta \, d\Omega_{d-1}^2\right), \tag{26}$$

so that the light cone boundaries defined by $\Delta s = 0$ are straight lines at $45°$ with the (η, θ) axes, see Fig. S4. Therefore, in this figure, point A at time $\eta \in [0, \eta')$ lies in the past light cone of point B at time $\eta' \in (0, \pi/2)$ if the angular distance $\Delta\theta$ between A and B on $\mathbb{S}^d$ is less than the conformal time difference $\eta' - \eta$ between them:

$$\Delta\theta \;<\; \eta' - \eta = \operatorname{arcsec}\cosh\frac{t'}{a} - \operatorname{arcsec}\cosh\frac{t}{a}, \quad \text{or approximately} \tag{27}$$

$$\Delta\theta \;<\; 2\left(e^{-\frac{t}{a}} - e^{-\frac{t'}{a}}\right) \approx 2e^{-\frac{t}{a}}, \tag{28}$$

where the last approximation holds for $t' \gg t \gg 1$, and where we have used the approximation $\eta = \operatorname{arcsec}\cosh(t/a) \approx \pi/2 - 2e^{-t/a}$, which is valid for large t.

The volume form on de Sitter spacetime in the conformal and cosmological time coordinates is given by

$$dV \;=\; (a\sec\eta)^{d+1} d\eta \, d\Phi_d = \left(a\cosh\frac{t}{a}\right)^d dt \, d\Phi_d \approx \left(\frac{a}{2}\right)^d e^{\frac{d}{a}t} dt \, d\Phi_d, \quad \text{where} \tag{29}$$

$$d\Phi_d \;=\; \sin^{d-1}\theta_1 \sin^{d-2}\theta_2 \ldots \sin\theta_{d-1} d\theta_1 \, d\theta_2 \ldots d\theta_d \tag{30}$$

is the volume form on $\mathbb{S}^d$, i.e. $\int d\Phi_d = \sigma_d$.

B. Hyperbolic space

The hyperboloid model of the $d+1$-dimensional hyperbolic space [15] is represented by one sheet of the two-sheeted $d+1$-dimensional hyperboloid of constant negative scalar R and Gaussian K curvatures

$$-z_0^2 + z_1^2 + \ldots + z_{d+1}^2 = -b^2 = \frac{d(d+1)}{R} = \frac{1}{K} \tag{31}$$

borrowing its metric from the $d+2$-dimensional ambient Minkowski space with metric (16). This hyperboloid in the $d=1$ case is visualized as the inner hyperboloid in Fig. 2. As a side note, the curvature of the same hyperboloid in the Euclidean metric is everywhere positive but not constant. The standard coordinate system $(r, \theta_1, \ldots, \theta_d)$ on this

hyperboloid is given by

$$z_0 = b \cosh \frac{r}{b}, \tag{32}$$

$$z_1 = b \sinh \frac{r}{b} \cos \theta_1, \tag{33}$$

$$\vdots$$

$$z_d = b \sinh \frac{r}{b} \sin \theta_1 \ldots \sin \theta_{d-1} \cos \theta_d, \tag{34}$$

$$z_{d+1} = b \sinh \frac{r}{b} \sin \theta_1 \ldots \sin \theta_{d-1} \sin \theta_d, \tag{35}$$

where $r \in \mathbb{R}_+$ is the radial coordinate, and $\theta_1, \ldots, \theta_{d-1} \in [0, \pi]$ and $\theta_d \in [0, 2\pi]$ are the standard angular coordinates on the unit d-dimensional sphere $\mathbb{S}^d$. The radial coordinate r of point P on the hyperboloid is the Minkowski length of the arc connecting P to the hyperboloid vertex, which is the bottom of the inner hyperboloid in Fig. 2. As a side note, the Euclidean length r_E of the same arc is given by the incomplete elliptic integral of the second kind $r_E = -iE(ir, 2)$.

C. Hyperbolic model of complex networks

In the hyperbolic model of complex networks [7], networks grow over the $d + 1$ dimensional hyperbolic space of Gaussian curvature $K = -1/b^2$ according to the following rule in the simplest case. New nodes n are born one at a time, $n = 1, 2, 3, \ldots$, so that n can be called a network time. Each new node is located at a random position on $\mathbb{S}^d$. That is, the angular coordinates $(\theta_1, \ldots, \theta_d)$ for new nodes are drawn from the uniform distribution on $\mathbb{S}^d$. The radial coordinate of the new node is

$$r = 2\frac{b}{d} \ln \frac{n}{\nu}, \tag{36}$$

where ν is a parameter controlling the average degree in the network. Upon its birth, each new node connects to all the nodes lying within hyperbolic distance r from itself. In other words, the connectivity perimeter of new node n at time n is the hyperbolic ball of radius r centered at node n. The hyperbolic distance x between two points with radial coordinates r and r' located at angular distance $\Delta\theta$ is given by the hyperbolic law of cosines [15]:

$$x = b \operatorname{arccosh} \left(\cosh \frac{r}{b} \cosh \frac{r'}{b} - \sinh \frac{r}{b} \sinh \frac{r'}{b} \cos \Delta\theta \right) \approx r + r' + 2b \ln \frac{\Delta\theta}{2}. \tag{37}$$

Therefore new node n' connects to existing nodes $n < n'$ whose coordinates satisfy

$$x < r', \quad \text{or approximately} \tag{38}$$

$$r + 2b \ln \frac{\Delta\theta}{2} < 0. \tag{39}$$

This construction yields growing networks whose distribution $P(k)$ of node degrees k is a power law, $P(k) \sim k^{-\gamma}$, with $\gamma = 2$. Indeed, according to Eq. (36), the radial density of nodes at any given time scales with r as $\rho(r) \sim e^{\alpha r}$, where $\alpha = d/(2b)$. One can also calculate, see [16], the average degree of nodes at radial coordinate r, which is $\bar{k}(r) \sim e^{-\beta r}$, where $\beta = \alpha$. The probability that a node at r has degree k is given by the Poisson distribution with the mean equal to $\bar{k}(r)$. Taken altogether, these observations prove that $\gamma = \alpha/\beta + 1 = 2$. The networks in the model also have strongest possible clustering, i.e. the largest possible number of triangular subgraphs, for graphs with this degree distribution, and their average degree is

$$\bar{k} \approx 2^{d+1} \frac{v_d}{\sigma_d} \nu \ln n, \quad \text{where } v_d = \frac{\sigma_{d-1}}{d} \tag{40}$$

is the volume of the unit d-dimensional ball. The model and its extensions describe the large-scale structure and growth dynamics of different real networks, e.g. the Internet, metabolic networks, and social networks, with a remarkable accuracy [7].

D.   Duality between de Sitter causal sets and complex networks

To demonstrate the asymptotic equivalence between growing networks from the previous section and causal sets growing in de Sitter spacetime, we:

1. find a mapping of points in the hyperbolic space $\mathbb{H}^{d+1}$ to de Sitter spacetime $d\mathbb{S}^{d+1}$ such that:

2. the hyperbolic ball of new node $n \in \mathbb{H}^{d+1}$ is asymptotically identical to its past light cone in $d\mathbb{S}^{d+1}$ upon the mapping, and

3. the distribution of nodes after mapping is uniform in $d\mathbb{S}^{d+1}$.

1.   Mapping

A mapping that, as we prove below, satisfies the properties above is remarkably simple:

$$
\begin{align}
t &= r, \tag{41}\\
a &= 2b. \tag{42}
\end{align}
$$

That is, we identify radial coordinate r in $\mathbb{H}^{d+1}$ with time t in $d\mathbb{S}^{d+1}$, keeping all the angular coordinates the same—see Fig. 2 for illustration.

2.   Hyperbolic balls versus past light cones

The proof that the hyperbolic balls of new node connections map asymptotically to past light cones is trivial: inequalities (39) and (28) are identical with the mapping above. Figure 2(d-f) visualizes the approximation accuracy at different times.

3.   Uniform node density

The proof that the node density after mapping is uniform in $d\mathbb{S}^{d+1}$ is trivial as well. Indeed, according to Eq. (36) with $2b = a$, network time n is related to cosmological time t via

$$
\begin{align}
n &= \nu e^{\frac{d}{a}t}, \quad \text{so that} \tag{43}\\
dn &= \nu \frac{d}{a} e^{\frac{d}{a}t}\, dt. \tag{44}
\end{align}
$$

Using the last equation, we rewrite the de Sitter volume element in Eq. (29) as

$$
dV = \frac{2b^{d+1}}{\nu d} dn\, d\Phi_d. \tag{45}
$$

By construction in Section II C, the node density on $\mathbb{S}^d$ is uniform and equal to 1. Therefore the number of nodes dN in element $dn\, d\Phi_d$ is

$$
dN = \frac{1}{\sigma_d} dn\, d\Phi_d. \tag{46}
$$

Combining the last two equations, we obtain

$$
\begin{align}
dN &= \delta\, dV, \quad \text{where} \tag{47}\\
\delta &= \frac{\nu d}{2b^{d+1}\sigma_d}, \tag{48}
\end{align}
$$

meaning that nodes are distributed uniformly in $d\mathbb{S}^{d+1}$ with constant density δ, thus completing the proof.

A causal set growing in de Sitter spacetime with node density δ in Eq. (48) is thus asymptotically equivalent to a growing complex network in Section II C with average degree $\bar{k}$ in Eq. (40). Combining these two equations, we can relate δ and $\bar{k}$ to each other:

$$\bar{k} \approx 2v_d \delta a^d t. \tag{49}$$

An important consequence of this asymptotic equivalence is that the degree distribution in both cases is the same power law with exponent $\gamma = 2$.

E. Lorentz invariance

Here we show that the described duality is Lorentz-invariant. The group of isometries of $d\mathbb{S}^{d+1}$ and $\mathbb{H}^{d+1}$ is $SO(d+1,1)$, i.e. the Lorentz group of the ambient Minkowski space $\mathbb{M}^{d+2}$. Any $g \in SO(d+1,1)$ preserves volumes, maps light cones in $d\mathbb{S}^{d+1}$ to light cones, and sends balls in $\mathbb{H}^{d+1}$ to balls of the same radii. The fact that the mapping $\Phi : d\mathbb{S}^{d+1} \mapsto \mathbb{H}^{d+1}$ described in the previous section is invariant under the $SO(d+1)$ subgroup of $SO(d+1,1)$ is immediately obvious. Consider now *any* element $g \in SO(d+1,1)$. This element defines a new coordinate system in $\mathbb{M}^{d+2}$, $\tilde{z}_i = \sum_{j=0}^{d+1} g_{ij} z_j$, where g_{ij} is the matrix representation of g, e.g. if g is the Lorentz boost with rapidity ϕ in the z_1 direction, then $g_{00} = g_{11} = \cosh \phi$, $g_{12} = g_{21} = -\sinh \phi$, and $g_{kk} = 1$, $g_{kl} = 0$ for $k, l > 1$. These new $\tilde{z}$-coordinates define new coordinates $(\tilde{t}, \tilde{\theta}_1, \ldots, \tilde{\theta}_d)$ and $(\tilde{r}, \tilde{\theta}_1, \ldots, \tilde{\theta}_d)$ on $d\mathbb{S}^{d+1}$ and $\mathbb{H}^{d+1}$ via the same Eqs. (18-21,32-35). The same g, considered as an active transformation, sends the patch between $t = 0$ and $t = t'$ in $d\mathbb{S}^{d+1}$ in Fig. 2 to *another* patch. As a side note, since $SO(d+1,1)$ is not compact, there exists no compact subset of $d\mathbb{S}^{d+1}$ (or $\mathbb{H}^{d+1}$) invariant under the action of $SO(d+1,1)$. Yet one can check that g sends the patch $t = 0$ and $t = t'$ to the patch between $\tilde{t} = 0$ and $\tilde{t} = t'$, the past light cone of point P at $t = t'$—to the past light cone of P's image $\tilde{P}$ at $\tilde{t} = t'$, the ball of radius r' centered at $r = 0$ in $\mathbb{H}^{d+1}$—to the ball of the same radius centered at $\tilde{r} = 0$, and so on. Therefore, mapping $\tilde{\Phi} : d\mathbb{S}^{d+1} \mapsto \mathbb{H}^{d+1}$ after the transformation induced by g is exactly the same as in Eq. (41), i.e. $\tilde{t} = \tilde{r}$, and under the action of $SO(d+1,1)$ this mapping transforms as $\tilde{\Phi} = g \circ \Phi \circ g^{-1}$.

III.   THE UNIVERSE AS A CAUSAL SET

De Sitter spacetime is the spacetime of a universe with positive vacuum (dark) energy density and no matter or radiation. Since the real universe does contain matter, its spacetime deviates from the pure de Sitter spacetime. At early times, matter dominates, leading to the Big Bang singularity at $t = 0$ that de Sitter spacetime lacks. At later times, the matter density decreases, while the dark energy density stays constant, so that it starts dominating, and the universe becomes asymptotically de Sitter. The universe today is at the crossover between the matter-dominated and dark-energy-dominated eras, since the matter and dark energy densities ρ_M and ρ_Λ are of the same order of magnitude, leading to rescaled cosmological time $\tau = t/a \sim 1$—the so-called "*why now?*" puzzle in cosmology [17, 18, 19, 20].

To quantify these deviations from pure de Sitter spacetime, and their effect on the structure of the causal set of the real universe, we calculate its degree distribution in this section. This task is quite challenging, and in what follows we first provide the exact analytic expression for the degree distribution, and then derive its approximations based on the measured properties of the universe. These approximations turn out to be remarkably accurate because according to the current measurements, the universe is almost flat.

We emphasize that the approximations in this section are based on the exact solution of the Einstein equations for a flat universe containing only matter and constant positive vacuum energy, i.e. constant cosmological constant Λ, and that we assume that the universe is governed by this solution at all times. This assumption is a simplification of reality for a number of reasons. For example, there are a plenty of cosmological scenarios, such as eternal inflation [21], in which the ultimate fate of the universe deviates from the asymptotic solution that we consider. There are also a variety of other models with non-constant Λ, yet the standard ΛCDM model with constant Λ is a baseline describing accurately many observed properties of the real universe [22], partly justifying our simplifying assumption. We also note that by relying on the exact solution for a universe containing only matter and Λ, we effectively neglect the earliest stages of universe evolution such as the radiation-dominated era or inflationary epoch. There is no consensus on how exactly the universe evolved at those earliest times. Since the radiation-dominated era ended soon after the Big Bang [23], these details are unlikely to have a profound effect on the universe causet's structure at much later times.

A.   Exact expression for the degree distribution

The metric in a homogeneous and isotropic universe takes the FLRW form:

$$ds^2 \;=\; -dt^2 + R^2(t) \left\{ d\chi^2 + \frac{1}{K} \sin^2 \left[\sqrt{K}\chi \right] \left(\sin^2 \theta \, d\theta^2 + d\phi^2 \right) \right\}, \; \text{where} \tag{50}$$

$$\frac{1}{K} \sin^2 \sqrt{K}\chi \;=\; \begin{cases} \sin^2 \chi & \text{if } K = 1 \text{ (closed universe with positive spatial curvature),} \\ \chi^2 & \text{if } K = 0 \text{ (flat universe with zero spatial curvature),} \\ \sinh^2 \chi & \text{if } K = -1 \text{ (open universe with negative spatial curvature).} \end{cases} \tag{51}$$

In the above expression, coordinates $\phi \in [0, 2\pi]$ and $\theta \in [0, \pi]$ are the standard angular coordinates on $\mathbb{S}^2$, while $\chi \in [0, \pi]$ if $K = 1$, or $\chi \in [0, \infty]$ if $K = 0, -1$, is the radial coordinate in a spherical, flat, or hyperbolic space. Finally, $t \in [0, \infty]$ is the cosmological time, and $R(t)$ is the scale factor, finding an appropriate approximation to which in the real universe is an important part of our approximations in subsequent sections. In this and the next sections, however, all the expressions are valid for *any* scale factor.

As with pure de Sitter, it is convenient to introduce conformal time η, related to cosmological time t via

$$\eta = \int_0^t \frac{dt'}{R(t')}. \tag{52}$$

In conformal time coordinates, the metric and volume form become

$$ds^2 \;=\; R^2(\eta) \left\{ -d\eta^2 + d\chi^2 + \frac{1}{K} \sin^2 \left[\sqrt{K}\chi \right] \left(\sin^2 \theta \, d\theta^2 + d\phi^2 \right) \right\}, \tag{53}$$

$$dV \;=\; \frac{1}{K} R^4(\eta) \sin^2 \left[\sqrt{K}\chi \right] \sin \theta \, d\eta \, d\chi \, d\theta \, d\phi. \tag{54}$$

As with pure de Sitter, we orient edges in the universe's causet from future to the past. Therefore if η is the current conformal time, then the average in- and out-degrees $\bar{k}_{i,o}(\eta'|\eta)$ of nodes born at conformal time $\eta' < \eta$ are simply

proportional to the volumes of their future and past light cones $V_{f,p}(\eta'|\eta)$:

$$\bar{k}_i(\eta'|\eta) = \delta V_f(\eta'|\eta), \tag{55}$$

$$\bar{k}_o(\eta'|\eta) = \delta V_p(\eta'|\eta), \tag{56}$$

where the coefficient of proportionality is the Planck-scale node density in the spacetime, which we take to be the inverse of the Planck 4-volume:

$$\delta = \frac{1}{t_P^4} = 1.184 \times 10^{173} \text{ s}^{-4}, \text{ where} \tag{57}$$

$$t_P = 5.391 \times 10^{-44} \text{ s} \tag{58}$$

is the Planck time.

Thanks to conformal time coordinates, the expressions for volumes $V_{f,p}(\eta'|\eta)$ are easy to write down:

$$V_f(\eta'|\eta) = \frac{1}{K} \int_{\eta'}^{\eta} d\eta'' \, R^4(\eta'') \int_0^{\eta''-\eta'} d\chi \, \sin^2\left[\sqrt{K}\chi\right] \int_0^{\pi} d\theta \, \sin\theta \int_0^{2\pi} d\phi, \tag{59}$$

$$V_p(\eta'|\eta) = \frac{1}{K} \int_0^{\eta'} d\eta'' \, R^4(\eta'') \int_0^{\eta'-\eta''} d\chi \, \sin^2\left[\sqrt{K}\chi\right] \int_0^{\pi} d\theta \, \sin\theta \int_0^{2\pi} d\phi. \tag{60}$$

Computing the three inner integrals, we obtain

$$\bar{k}_i(\eta'|\eta) = \delta\frac{\pi}{K} \int_{\eta'}^{\eta} \left\{ 2(\eta''-\eta') - \frac{1}{\sqrt{K}}\sin\left[2\sqrt{K}(\eta''-\eta')\right] \right\} R^4(\eta'') \, d\eta'', \tag{61}$$

$$\bar{k}_o(\eta'|\eta) = \delta\frac{\pi}{K} \int_0^{\eta'} \left\{ 2(\eta'-\eta'') - \frac{1}{\sqrt{K}}\sin\left[2\sqrt{K}(\eta'-\eta'')\right] \right\} R^4(\eta'') \, d\eta''. \tag{62}$$

As shown in [24], Lorentz invariance implies that nodes/events of the causet are distributed in spacetime according to a Poisson point process. Therefore, as explained in [25], to find the in- or out-degree distributions $P(k,\eta)$ at time η, we have to average the Poisson distribution

$$p(k|\eta',\eta) = \frac{1}{k!}\left[\bar{k}(\eta'|\eta)\right]^k e^{-\bar{k}(\eta'|\eta)}, \tag{63}$$

which is the probability that a node born at time η' has degree k, with the density of nodes born at time η'

$$\rho(\eta'|\eta) = \frac{R^4(\eta')}{\mathcal{N}(\eta)}, \text{ where} \tag{64}$$

$$\mathcal{N}(\eta) = \int_0^{\eta} R^4(\eta') d\eta' \tag{65}$$

is the time-dependent normalization factor. The result is

$$P(k,\eta) = \int_0^{\eta} p(k|\eta',\eta)\rho(\eta'|\eta) \, d\eta' = \frac{1}{\mathcal{N}(\eta)}\frac{1}{k!} \int_0^{\eta} \left[\bar{k}(\eta'|\eta)\right]^k e^{-\bar{k}(\eta'|\eta)} R^4(\eta') \, d\eta'. \tag{66}$$

The above expressions are valid for both in-degree (set $k \equiv k_i$ and $\bar{k}(\eta'|\eta) \equiv \bar{k}_i(\eta'|\eta)$) and out-degree ($k \equiv k_o$, $\bar{k}(\eta'|\eta) \equiv \bar{k}_o(\eta'|\eta)$) distributions. In what follows we focus on the in-degree distribution.

B.   First approximation using the Laplace method

Equations (66,61) give the exact solution for the in-degree distribution with an arbitrary scale factor $R(\eta)$, but it is difficult to extract any useful information from these expressions, even if we know the exact form of $R(\eta)$. The first step to get a better insight into the causet properties is to rewrite $\bar{k}_i(\eta'|\eta)$ as

$$\bar{k}_i(\eta'|\eta) = \bar{k}_i(0|\eta)F(\eta'|\eta), \quad \text{where } F(\eta'|\eta) = \frac{\bar{k}_i(\eta'|\eta)}{\bar{k}_i(0|\eta)}. \tag{67}$$

The in-degree of the oldest node $\bar{k}_i(0|\eta)$ is an astronomically large number, both because the node is old, so that its future light cone comprises a macroscopic portion of the 4-volume of the whole universe, and because the degree is proportional to huge δ. However, function $F(\eta'|\eta)$ is a monotonically decreasing function whose values lie in the interval $[0, 1]$. We can thus use the Laplace method to approximate the integral in Eq. (66) in the limit $\delta \gg 1$. Introducing $\eta^*(k_i)$, which is the solution of the transcendent equation

$$k_i = \bar{k}_i(\eta^*|\eta), \tag{68}$$

and function

$$\Phi(k_i, \eta) = \left| \frac{\partial \bar{k}_i(\eta'|\eta)}{\partial \eta'} \right|_{\eta'=\eta^*(k_i)}, \tag{69}$$

the result of this Laplace approximation reads:

$$P(k_i, \eta) = \begin{cases} \left(\dfrac{3}{\pi \delta} \right)^{\frac{1}{4}} \Gamma \left(\dfrac{5}{4} \right) \dfrac{R^3(\eta)}{\mathcal{N}(\eta)} & \text{if } k_i = 0, \\[2ex] \dfrac{1}{\mathcal{N}(\eta)} \dfrac{\sqrt{2\pi k_i} k_i^{k_i} e^{-k_i}}{k_i!} \dfrac{R^4[\eta^*(k_i)]}{\Phi(k_i, \eta)} \approx \dfrac{1}{\mathcal{N}(\eta)} \dfrac{R^4[\eta^*(k_i)]}{\Phi(k_i, \eta)} & \text{if } 1 \le k_i \le \bar{k}_i(0|\eta), \end{cases} \tag{70}$$

where we have also used Stirling's approximation $k! \approx \sqrt{2\pi k}(k/e)^k$. We see that the shape of the degree distribution is almost fully determined by $\eta^*(k_i)$, and that the distribution is a fast decaying function for degrees above $\bar{k}_i(0|\eta)$.

The expression for the in-degree distribution in Eq. (70) is now more tractable and gets ready to accept the scale factor $R(\eta)$ of the universe. Unfortunately, the exact expressions for the scale factor of a closed or open universe with matter and dark energy, although known [26], resist analytic treatment, and so does the integral for the average degree $\bar{k}_i(\eta'|\eta)$ in Eq. (61) that we are to use in Eq. (70). Therefore we next develop a series of approximations to the scale factor and average degree, based on the measured properties of the universe.

C. Measured properties of the universe

The current measurements of the universe [22] that are relevant to us here include:

$$\Omega_\Lambda = \frac{8\pi G}{3H_0^2} \rho_\Lambda \in [0.709, 0.741], \text{ (dark energy density)} \tag{71}$$

$$\Omega_M = \frac{8\pi G}{3H_0^2} \rho_M \in [0.2582, 0.2914], \text{ (matter density)} \tag{72}$$

$$\Omega_K = -\frac{K}{R_0^2 H_0^2} \in [-0.0133, 0.0084], \text{ (curvature density)}, \quad \text{where} \tag{73}$$

$$H_0 = \frac{\dot{R}_0}{R_0} \in [68.8, 71.6] \frac{\text{km}}{\text{s} \cdot \text{Mpc}} = [2.23, 2.32] \times 10^{-18} \text{ s}^{-1} \text{ is the Hubble constant,} \tag{74}$$

$$R_0 = R(t_0) \text{ is the scale factor at present time } t_0, \text{ i.e. the age of the universe,} \tag{75}$$

$$t_0 \in [13.65, 13.87] \text{ Gyr} = [4.308, 4.377] \times 10^{17} \text{ s, so that} \tag{76}$$

$$\Lambda = 3H_0^2 \Omega_\Lambda \in [1.06, 1.20] \times 10^{-35} \text{ s}^{-2}, \text{ is the cosmological constant, and} \tag{77}$$

$$a = \sqrt{\frac{3}{\Lambda}} = \frac{1}{\sqrt{\Omega_\Lambda} H_0} \in [5.00, 5.32] \times 10^{17} \text{ s, is the de Sitter pseudoradius.} \tag{78}$$

The range of values of Ω_Λ, Ω_M, H_0, and t_0 are given by the 95% confidence bounds in the universe measurements taken from the last column of Table 1 in [22], while the values of Ω_K come from Table 2 there: third row, last column. The matter density Ω_M is the sum of two contributions: the observable (baryon) matter density ($\Omega_b \in [0.0442, 0.0474]$), and dark matter density ($\Omega_c \in [0.214, 0.244]$). The radiation density is negligible. The universe thus consists of dark energy ($\approx 73\%$), dark matter ($\approx 23\%$), and observable matter ($\approx 4\%$). As a side note, the sum of all Ω's $\sum \Omega \in [0.95, 1.04]$ as expected, since $\sum \Omega = 1$ is the Einstein/Friedmann equation for a FLRW universe, which we recall in the next section. Much more important for us here is that the universe is almost flat, $\Omega_K \approx 0$.

D.   Approximations to the scale factor and average degree

In this section we utilize the approximate flatness of the universe to derive and quantify approximations to the scale factor of the universe and average degree in Eq. (61). The main idea is that since Ω_K is small, we can use the exact solution for the scale factor in a flat universe ($\Omega_K = 0$) with matter and dark energy (positive cosmological constant), which is actually quite simple [13].

We first recall that the 00-component of the Einstein equations in the FLRW metric is the first Friedmann equation:

$$\frac{\dot{R}^2 + K}{R^2} = \frac{8}{3}\pi G \left[\rho_\Lambda + \left(\frac{R_0}{R} \right)^3 \rho_M \right]. \tag{79}$$

This equation can be rewritten [27] in the following form:

$$H_0 t = \int_0^{R/R_0} \frac{dx}{x\sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3}}}. \tag{80}$$

Assuming non-vanishing positive dark energy and matter densities ($\Omega_\Lambda > 0$ and $\Omega_M > 0$), using rescaled time

$$\tau \equiv \frac{t}{a}, \tag{81}$$

and introducing two parameters

$$\alpha \equiv R_0 \sqrt[3]{\frac{\Omega_M}{\Omega_\Lambda}}, \text{ and} \tag{82}$$

$$\epsilon \equiv \frac{\Omega_K}{\sqrt[3]{\Omega_\Lambda \Omega_M^2}} \in [-0.0368, 0.0232], \tag{83}$$

the integral in Eq. (80) simplifies to

$$\tau = \int_0^r \frac{dx}{x\sqrt{1 + \epsilon x^{-2} + x^{-3}}}. \tag{84}$$

Solving it for $r \equiv r(\tau, \epsilon)$, we conclude that the solution for the scale factor becomes

$$R(t) = \alpha\, r\,(\tau; \epsilon)\,. \tag{85}$$

This is definitely not the only way to write down the scale factor in terms of the parameters of the universe. Yet written in this form, the rescaled scale factor $r(\tau; \epsilon)$ is a dimensionless function of its dimensionless arguments, and parameter ϵ is close to zero, so that $r(\tau; \epsilon) \approx r(\tau; 0)$, where $r(\tau, 0)$, i.e. the rescaled scale factor for a flat universe with positive matter and dark energy densities, is quite simple:

$$r(\tau; 0) = \sinh^{\frac{2}{3}} \frac{3}{2}\tau. \tag{86}$$

Figure S5(left) shows a good agreement between the numerical solution for $r(\tau; \epsilon)$ and the analytical expression for $r(\tau; 0)$ in Eq. (86) for the range of the values of ϵ allowed by measurements in Eq (83). Therefore in what follows we can use Eq. (86), even if the universe is "slightly closed" or "slightly open."

To approximate the average degree $\bar{k}_i(\eta'|\eta)$ in Eq. (61), we first compute, using the scaling Eq. (85), the conformal time

$$\eta = \int_0^t \frac{dt'}{R(t')} = \sqrt{|\epsilon|} \int_0^\tau \frac{d\tau'}{r(\tau'; \epsilon)} = \sqrt{|\epsilon|} \int_0^{r(\tau;\epsilon)} \frac{dx}{x^2\sqrt{1 + \epsilon x^{-2} + x^{-3}}} \tag{87}$$

at the future infinity $t = \tau = r(\tau; \epsilon) = \infty$. For the extreme values of ϵ allowed by measurements, we obtain $\eta_\infty = 0.5403$ for the closed universe with $\epsilon = -0.0368$ and $\eta_\infty = 0.4260$ for the open universe with $\epsilon = 0.0232$. If $\epsilon \to 0$, then we also have $\eta_\infty \to 0$, yet if ϵ is exactly zero, then η_∞ is an undefined constant, since in the flat case, R_0 is a free parameter that can be set to an arbitrary value without affecting anything. If the universe is not exactly flat,
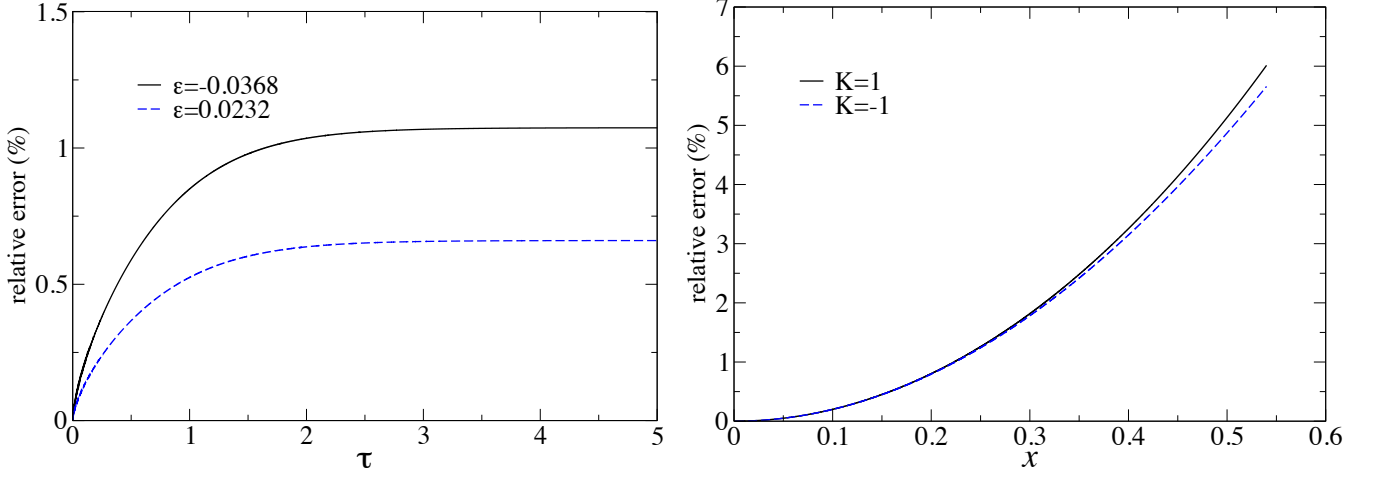
FIG. S5: Left: Relative error between functions $r(\tau; \epsilon)$ and $r(\tau; 0)$, $|r(\tau; \epsilon) - r(\tau; 0)|/r(\tau; \epsilon)$, for the two extremes of the 95% confidence interval of parameter ϵ in Eq. (83), as a function of rescaled time τ. In the worst case, the relative error is around 1%. Right: Relative error of approximation in Eq. (88) for $x \in [0, \eta_\infty]$.

then the above values of η_∞ allowed by measurements are well below $\pi/2$, so that we can safely replace expression $\left[2x - \sin(2\sqrt{K}x)/\sqrt{K}\right]/K$ in Eq. (61) by its Taylor expansion around zero,

$$\frac{1}{K}\left[2x - \frac{1}{\sqrt{K}}\sin(2\sqrt{K}x)\right] \approx \frac{4}{3}x^3. \tag{88}$$

Indeed the maximum possible relative error between the left and right hand sides in the last equation at $x = \eta_\infty$ is around 6%, see Fig. S5(right), whereas such error is zero for an exactly flat universe with $\epsilon = K = 0$, since the left and right hand sides are equal in this case. Therefore the approximation to the average degree in Eq. (61)

$$\bar{k}_i(\eta'|\eta) = \frac{4}{3}\pi\delta \int_{\eta'}^{\eta} (\eta'' - \eta')^3 R^4(\eta'') \, d\eta'' \tag{89}$$

is exact for flat universes, and almost exact for slightly closed or open universes with ϵ within the range of Eq. (83).

E. Scaling relations

To proceed to our final approximations to the degree distribution in the next section, we derive some scaling relations that tremendously simplify the calculations, provide important insights into the properties of the degree distribution, and suggest methods to simulate the causal set of the universe.

We begin with the scaling relation for conformal time η. Let us introduce the rescaled conformal time

$$\zeta \equiv \frac{\alpha}{a}\eta = \frac{\eta}{\sqrt{|\epsilon|}}, \tag{90}$$

and let notations $y = f_{1,2,\dots}(x)$ mean "y is a function of x." Given the definition of conformal time in Eq. (87), this rescaled conformal time is a function of rescaled cosmological time τ:

$$\zeta = f_1(\tau) = \int_0^{r(\tau;\epsilon)} \frac{dx}{x^2\sqrt{1 + \epsilon x^{-2} + x^{-3}}} \xrightarrow{\epsilon \to 0} 2\sqrt{r(\tau;0)} \, {}_2F_1\left(\frac{1}{6}, \frac{1}{2}, \frac{7}{6}, -r^3(\tau;0)\right), \tag{91}$$

where $r(\tau; 0)$ is given by Eq. (86), and ${}_2F_1$ is the hypergeometric function. This scaling means that

$$R(\eta) = \alpha f_2(\zeta), \tag{92}$$

which in turn implies that the average in-degree of nodes born at time η'

$$\bar{k}_i(\eta'|\eta) = q f_3(\zeta', \zeta) = q f_4(\tau', \tau), \text{ where} \tag{93}$$
$$q \equiv \delta a^4, \tag{94}$$

while the average in- or out-degree in the causet

$$\bar{k}_i(\eta) = \bar{k}_o(\eta) = \int_0^{\eta_\infty} \bar{k}_i(\eta'|\eta)\rho(\eta'|\eta)\, d\eta' = q f_5(\zeta) = q f_6(\tau). \tag{95}$$

Plugging scaling Eqs. (93,94) into the expression for the in-degree distribution in Eq. (70), we conclude that this distribution scales for $k_i \geq 1$ as

$$P(k_i, t) = \frac{1}{q} Q(\kappa_i, \tau), \text{ where} \tag{96}$$

$$\kappa_i \equiv \frac{k_i}{q}, \tag{97}$$

and the rescaled in-degree distribution $Q(\kappa_i, \tau)$ is some function of rescaled degree κ_i and time τ. This result is important for several reasons:

1. It defines a characteristic degree $q = \delta a^4$ and characteristic time a such that the degree distribution depends only on the rescaled dimensionless variables $\kappa_i = k_i/q$ and $\tau = t/a$. Therefore we are free to set $a = 1$ and $\delta = 1$, and the remaining task is to find an explicit form of the rescaled distribution $Q(\kappa_i, \tau)$ in Eq. (96).

2. Even though the scale factor in Eq. (85) does depend on α, the in-degree distribution $P(k_i, t)$ does not depend on α. This is important from the practical point of view, because we can carry out all the calculations setting $\alpha = 1$ as well.

3. Having set $\alpha = a = \delta = 1$, the in-degree distribution explicitly does not depend on any physical parameters of the universe. The causets in flat or open universes are graphs with obviously infinite numbers of nodes from the very beginning of the universe, while the causets of closed universes are always finite, and the number of nodes in them depend on time and on all the three parameters α, a, and δ. Yet the in-degree distributions in all these causets at a given time are all the same, and in particular, do not depend on the number of nodes at all. This somewhat unexpected result deserves further explanation. We note that parameter $\alpha = R_0 \sqrt[3]{\Omega_M/\Omega_\Lambda}$ depends on the physical properties of the universe. In particular, if the universe is closed, it defines, via R_0 in Eq. (73), the radius of the three-sphere, i.e. of the spatial part of the FLRW metric. Since the degree distribution does not depend on α, we can freely take the limit $\alpha \to \infty$ corresponding to a flat space, without affecting the degree distribution. In other words, we are free to choose a closed universe with $K = +1$ in Eq. (50), and this choice can be considered as a degree-distribution-preserving compactification of an infinite flat or hyperbolic space with an infinite number of nodes in it, into a compact spherical space with a finite number of nodes.

4. The previous point informs us how to simulate the universe causet on a computer. We cannot simulate infinite causets for flat or open universes, but since the degree distribution is the same in closed universes, we are free to simulate the latter. One can check that in a closed universe, the number of nodes in a causet grows with time as

$$N(t) = \delta a \alpha^3 f_7(\tau). \tag{98}$$

We can set $a = 1$, and since the in-degree distribution does not depend on the number of nodes N, then for any given time $t = a\tau = \tau$ and for any given value of δ, we can generate graphs with any desired size N by choosing the value of α accordingly. The degree distribution obtained from this simulation is then to be re-scaled according to Eq. (96) to obtain the rescaled distribution $Q(\kappa_i, \tau)$. Keeping fixed $t = \tau$ and N, we can also explore different regions of $Q(\kappa_i, \tau)$ by changing the value of δ.

We have to emphasize that the points above, and the scaling relations in this section, are valid for non-flat universes only when η_∞ is well below $\pi/2$, so that the approximation in Eq. (88) is valid for all $x \in [0, \eta_\infty]$. When this condition does not hold, e.g. when ϵ is large and the universe is far from being flat, then all these scaling relations break down. In particular, we cannot claim that in that case the in-degree distribution would still be the same. The latest measurements of the universe parameters indicate that the scaling laws derived in this section do apply to the real universe, but even if this were not the case, these laws would still be valid for small values of conformal time.

F. Final approximations to the degree distribution in the matter- and dark-energy-dominated eras

We now have all the material needed to derive the final approximations to the degree distribution. According to Section III D the exact solution for the scale factor in a flat universe is a good approximation for the scale factor in

the real universe within the measurement-allowed range of parameter values, so that we will use this approximation here, i.e. we set $\epsilon = 0$. Using the scaling results from Section III E, we also set $a = \alpha = \delta = 1$, which is equivalent to working with rescaled cosmological time τ, rescaled conformal time ζ, rescaled scale factor $r(\tau) = \sinh^{2/3}(3\tau/2)$, and rescaled in-degree κ_i. Since everything is rescaled in this section, we omit word "rescaled" in front of the variable names.

We consider two eras of the universe evolution. The first era characterizes the causal set of the universe at early times, when matter dominates. The second era deals with the aged universe at large times, when dark energy dominates. In the limit of infinite time (future infinity), we derive the exact solution for the degree distribution in a flat universe.

1. Matter-dominated era

In this era with $\tau \ll 1$, the scale factor and conformal time can be approximated by

$$r(\tau) = \left(\frac{3}{2}\tau\right)^{\frac{2}{3}} \tag{99}$$

$$\zeta = 2\left(\frac{3}{2}\tau\right)^{\frac{1}{3}}, \quad \text{so that} \tag{100}$$

$$r(\zeta) = \left(\frac{\zeta}{2}\right)^2. \tag{101}$$

Using these expressions in the rescaled version of Eq. (89), we obtain the average in-degree at time ζ of nodes born at time ζ':

$$\bar{\kappa}_i(\zeta'|\zeta) = \frac{\pi}{768}\zeta^{12}\left[\frac{1}{12} - \frac{3}{11}\frac{\zeta'}{\zeta} + \frac{3}{10}\left(\frac{\zeta'}{\zeta}\right)^2 - \frac{1}{9}\left(\frac{\zeta'}{\zeta}\right)^3 + \frac{1}{1980}\left(\frac{\zeta'}{\zeta}\right)^{12}\right]. \tag{102}$$

The maximum average in-degree, i.e. the average in-degree of the oldest nodes, is

$$\bar{\kappa}_i(0|\zeta) = \frac{\pi}{9216}\zeta^{12} = 9\pi\tau^4. \tag{103}$$

We note that if we reinsert constants a and δ, and cosmological time t using Eq. (93), the average in-degree of the oldest nodes becomes

$$\bar{k}_i(0|t) = 9\pi\left(\frac{t}{t_P}\right)^4, \tag{104}$$

where t_P is the Planck time, see Eq. (57). As expected, this in-degree does not depend on a and, consequently, on the cosmological constant Λ.

For degrees well below the maximum degree, $\kappa_i \ll 9\pi\tau^4$, the Taylor expansion of Eq. (102) around $\zeta' = \zeta$ yields

$$\bar{\kappa}_i(\zeta'|\zeta) \approx \frac{\pi}{3072}\zeta^{12}\left(\frac{\zeta'}{\zeta} - 1\right)^4. \tag{105}$$

Using this expression in the rescaled version of Eq. (70), we obtain

$$Q(\kappa_i, \tau) = \frac{3}{4}\left(\frac{3}{\pi}\right)^{\frac{1}{4}}\frac{\kappa_i^{-3/4}}{\tau} \quad \text{if } \kappa_i \ll 9\pi\tau^4, \tag{106}$$

and, after reinserting all the physical constants,

$$P(k_i, t) = \frac{3}{4}\left(\frac{3}{\pi}\right)^{\frac{1}{4}}\frac{t_P}{t}k_i^{-3/4} \quad \text{if } 1 \leqslant k_i \ll 9\pi\left(\frac{t}{t_P}\right)^4, \tag{107}$$

with a soft cut-off at $9\pi(t/t_P)^4$. We thus observe that the degree distribution does not depend on the cosmological constant Λ, and that it is a power law $P(k_i, \tau) \sim k_i^{-\gamma}$ with exponent $\gamma = 3/4$.

2. Dark-energy-dominated era

We now analyze the future fate of the universe at $\tau \gg 1$, which is slightly more intricate. To begin, we derive a couple of expressions that we use in simulations and that are valid for any τ. We first recall that, according to Eq. (91), conformal time is related to the current value of the scale factor r via

$$\zeta(r) = 2\sqrt{r} \; {}_2F_1\left(\frac{1}{6}, \frac{1}{2}, \frac{7}{6}, -r^3\right), \tag{108}$$

which is a monotonously increasing function of r. Therefore we can use scale factor value r as a measure of time instead of ζ. Using this observation in Eq. (89), we write the average in-degree at time r of nodes born at time r' as

$$\bar{\kappa}_i(r'|r) = \frac{4\pi}{3} \int_{r'}^{r} [\zeta(x) - \zeta(r')]^3 \frac{x^2 dx}{\sqrt{1+x^{-3}}}. \tag{109}$$

Similarly, function $\Phi(\kappa_i, r)$ in Eq. (69) becomes

$$\Phi(\kappa_i, r) = 4\pi \int_{r^*(\kappa_i)}^{r} [\zeta(x) - \zeta(r^*(\kappa_i))]^2 \frac{x^2 dx}{\sqrt{1+x^{-3}}}, \tag{110}$$

where $r^*(\kappa_i)$ is the solution of equation $\bar{\kappa}_i(r^*|r) = \kappa_i$.

Assuming now that time is large, we see from Eq. (109) that the maximum average in-degree scales at $r \gg 1$ as

$$\bar{\kappa}_i(0|r) = \frac{4\pi}{3} \int_0^r \zeta^3(x) \frac{x^2 dx}{\sqrt{1+x^{-3}}} \approx \frac{4\pi}{3} \zeta_\infty^3 \int_0^r x^2 dx = \frac{4\pi}{9}(\zeta_\infty r)^3, \quad \text{where} \tag{111}$$

$$\zeta_\infty = \zeta(\infty) = \int_0^\infty \frac{dx}{x^2\sqrt{1+x^{-3}}} = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{7}{6}\right) \tag{112}$$

is the conformal time at the future infinity. Since r grows exponentially with time in this era, so does the average in-degree of the oldest nodes according to Eq. (111). In the long time limit, and for degrees well below the maximum average degree, $\kappa_i \ll \bar{\kappa}_i(0|r)$, we have $r > r' \gg 1$. Keeping the first two terms in the Taylor expansion of Eq. (108) for $r \gg 1$, we approximate conformal time by $\zeta(r) \approx \zeta_\infty - 1/r$. Inserting this approximation into Eqs. (109,110), and neglecting x^{-3} there, we obtain the implicit expression for the in-degree distribution at the future infinity,

$$Q(\kappa_i, \infty) = \frac{9}{4\pi} \frac{1}{x(\kappa_i)^3 [x(\kappa_i) - 1]^3}, \tag{113}$$

where $x(\kappa_i)$ is the solution of equation

$$\kappa_i = \frac{2\pi}{9}\left[(x-1)(2x^2 - 7x + 11) - 6\ln x\right], \tag{114}$$

in the region $x \geq 1$. The last two equations give a nearly exact solution for the asymptotic in-degree distribution in a flat universe, because all the approximations that we have made so far become exact in the $t, r \to \infty$ limit. The only approximation that is not rigorously exact is the one due to the Laplace method in Section III B, yet given the astronomical value of node density δ in Eq. (57) used in this approximation, it can be also considered exact. We also note that the distribution is properly normalized since $\int_1^\infty Q(x, \infty)(\kappa_i)'_x dx = 1$.

Finally, we find approximations to this exact solution for small and large degrees. If $\kappa_i \ll 1$, then the solution of Eq. (114) scales as $x(\kappa_i) = 1 + (3\kappa_i/\pi)^{1/4}$, whereas for $\kappa_i \gg 1$, the scaling is $x(\kappa_i) = (9\kappa_i/4\pi)^{1/3}$. Substituting these scalings into Eq. (113), and neglecting insignificant terms there, we obtain

$$Q(\kappa_i, \infty) \approx \begin{cases} \dfrac{3}{4}\left(\dfrac{3}{\pi}\right)^{\frac{1}{4}} \kappa_i^{-\frac{3}{4}} & \text{if } \kappa_i \ll c, \\[4mm] \dfrac{4\pi}{9}\kappa_i^{-2} & \text{if } \kappa_i \gg c, \end{cases} \tag{115}$$

where the crossover degree value $c = 1.66$ is given by equating the two approximations above. Figure S6 shows the exact numerical solution for the in-degree distribution at $t = \infty$, and juxtaposes it against these two approximations. The match is remarkable.
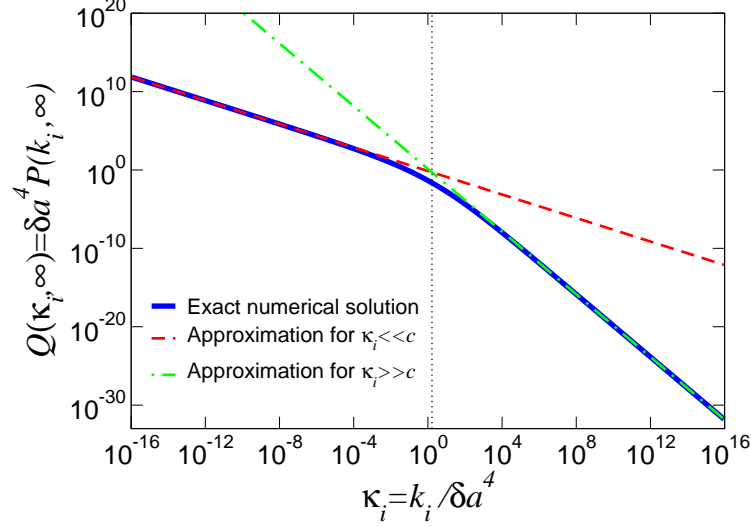
FIG. S6: Exact numerical solution versus approximations for the in-degree distribution in the universe causet at the future infinity. The solid line shows the numeric solution in Eqs. (113,114), while the dashed(-dotted) lines are the two approximations in Eq. (115) for $\kappa_i \ll c$ and $\kappa_i \gg c$. The crossover degree value c is shown by the dotted vertical line.

G. Numeric evaluation of the analytic solution

Given the results in the previous sections, the precise steps to numerically evaluate the analytic solution for the in-degree distribution in the universe at any given time are:

1. For a given rescaled age of the universe τ, e.g. $\tau = \tau_0$, compute the current value of the rescaled scale factor

$$r(\tau) = \sinh^{\frac{2}{3}} \frac{3}{2}\tau, \tag{116}$$

   and the normalization factor

$$\mathcal{N}(\tau) = \frac{1}{6}\left(\sinh 3\tau - 3\tau\right). \tag{117}$$

2. According to Eq. (91), define function

$$\zeta(r) = 2\sqrt{r} \; {}_2F_1\left(\frac{1}{6}, \frac{1}{2}, \frac{7}{6}, -r^3\right), \tag{118}$$

   where ${}_2F_1$ is the hypergeometric function.

3. Generate a (log-spaced) sequence of rescaled in-degrees κ_i, and for each value of κ_i in the sequence, find numerically the solution $r^*(\kappa_i) \in [0, r(\tau)]$ of equation

$$\kappa_i = \frac{4\pi}{3} \int_{r^*(\kappa_i)}^{r(\tau)} \left[\zeta(x) - \zeta(r^*(\kappa_i))\right]^3 \frac{x^2 dx}{\sqrt{1 + x^{-3}}}. \tag{119}$$

4. With $r^*(\kappa_i)$ and $r(\tau)$ at hand, compute numerically the integral

$$\Phi(\kappa_i, r(\tau)) = 4\pi \int_{r^*(\kappa_i)}^{r(\tau)} \left[\zeta(x) - \zeta(r^*(\kappa_i))\right]^2 \frac{x^2 dx}{\sqrt{1 + x^{-3}}}. \tag{120}$$

5. Finally, according to Eq. (70), the value of the rescaled in-degree distribution at κ_i is given by

$$Q(\kappa_i, \tau) = \frac{[r^*(\kappa_i)]^4}{\mathcal{N}(\tau)\Phi(\kappa_i, r(\tau))}. \tag{121}$$

H. Universality of $\gamma = 3/4$ scaling for small degrees

The portion of the degree distribution with exponent $\gamma = 3/4$ for small degrees below $\sim \delta a^4$ remaining in the universe causet even at long times may appear as a paradox at the first glance. Indeed, at long times, the universe is in its accelerating era dominated by dark energy, and the scale factor grows exponentially, versus polynomial growth in the matter-dominated era. Yet the degree distribution for small degrees behaves exactly the same way as in a matter-dominated universe, i.e. it has the same exponent $\gamma = 3/4$.

The intuitive explanation of this paradox is as follows [28]. The degree distribution in the range of small degrees is shaped by spacetime quanta born at times η' near the current time η. The future horizon radii of these nodes are smaller than the Hubble radius $1/H_0 \sim a$. Therefore these nodes do not yet "feel" the acceleration of the universe. For them the universe expands as if it was matter-dominated.

To formalize this intuition we check analytically that exponent $\gamma = 3/4$ for degrees $k_i \ll \delta a^4$ is universal for any scale factor. Keeping only the first term in the Taylor expansion of $\bar{k}_i(\eta'|\eta)$ at $\eta' = \eta$, we approximate

$$\bar{k}_i(\eta'|\eta) \approx \frac{1}{3}\pi\delta R^4(\eta)(\eta' - \eta)^4. \tag{122}$$

Using this approximation to solve equation $k_i = \bar{k}_i(\eta'|\eta)$ for $\eta^*(k_i)$, and substituting the solution into Eq. (70), we obtain

$$P(k_i, \eta) \approx \left(\frac{3}{\pi\delta}\right)^{1/4} \frac{R^3(\eta)}{4\mathcal{N}(\eta)} k_i^{-3/4}, \tag{123}$$

The degree distribution for small degrees is thus a power law with universal exponent $\gamma = 3/4$, while the scale factor is relegated to normalization. The scale factor is important only for old spacetime quanta with large degrees, where its exponential growth in asymptotically de Sitter spacetimes is responsible for the emergence of exponent $\gamma = 2$.

IV. RELATED WORK

The idea of replacing continuum spacetime with a graph or network appears in many approaches to quantum gravity, as it is a natural way to describe a discrete geometry. Causal Dynamical Triangulations (CDTs), for example, are formulated in terms of a simplicial triangulation of spacetime, which can be regarded as a (fixed valence) graph in which adjacent simplices are connected by an edge [29, 30**?**]. Another popular approach to quantum gravity is Loop Quantum Gravity, where the quantum states of geometry are described naturally in terms of *spin networks*, which are graphs embedded into a three dimensional manifold. The edges and vertices in these graphs are colored with various mathematical structures, see [31] and references therein. There are a multitude of other approaches whose mathematical formulations have a similar discrete network-like character, such as Wolfram's evolving networks [32], "quantum graphity" [33], D'Ariano's causal networks [34], Requardt's lumpy networks [35], etc.

Most descriptions of quantum spacetime geometry in terms of a graph structure are of a purely *spatial* character, in that the description of space*time* arises from temporal evolution of the graph whose edges usually connect spatially nearest neighbors. The intuition behind this idea is clear, being similar to the idea of approximating a continuous space by a fine lattice embedded in it. However, the precise physical meaning of the "time" in which the network evolves, and the manner in which the Lorentz invariant nature of special relativity can emerge from such discreteness, are often unclear [24]. Causal sets and CDTs are different in that they are described in terms of a graph-like structure which has a fundamentally space*time* character.

The degree of a node is an effective measure of its age in growing complex networks. This simple observation led us here to establishing a connection between these networks and the causal networks of discrete spacetime. The notion of degree as a measure of time is reminiscent of unimodular gravity [36].

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[1] Kc Claffy, Y Hyun, K Keys, M Fomenkov, and Dmitri Krioukov. Internet Mapping: from Art to Science. In *Proceedings of the 2009 Cybersecurity Applications & Technology Conference for Homeland Security (CATCH 2009), Washington, DC, March 3-4, 2009*, pages 205–211. IEEE Computer Society, 2009. `doi:10.1109/CATCH.2009.38`.

[2] G Siganos, M Faloutsos, P Faloutsos, and C Faloutsos. Power-Laws and the AS-Level Internet Topology. *IEEE ACM T Network*, 11(4):514–524, 2003. `doi:10.1109/TNET.2003.815300`.

[3] Amogh Dhamdhere and K Dovrolis. Ten Years in the Evolution of the Internet Ecosystem. In Konstantina Papagiannaki and Zhi-Li Zhang, editors, *Proceedings of the 8th ACM SIGCOMM Conference on Internet Measurement (IMC 2008), Vouliagmeni, Greece, October 20-22, 2008*, pages 183–196. ACM, 2008. `doi:10.1145/1452520.1452543`.

[4] B Zhang, R Liu, D Massey, and Lixia Zhang. Collecting the Internet AS-level Topology. *Comput Commun Rev*, 35(1):53–61, 2005. `doi:10.1145/1052812.1052825`.

[5] Pria Mahadevan, Dmitri Krioukov, Marina Fomenkov, Bradley Huffaker, Xenofontas Dimitropoulos, Kc Claffy, and Amin Vahdat. The Internet AS-Level Topology: Three Data Sources and One Definitive Metric. *Comput Commun Rev*, 36(1):17–26, 2006. `doi:10.1145/1111322.1111328`.

[6] J Callas, L Donnerhacke, H Finney, D Shaw, and R Thayer. OpenPGP Message Format. IETF, RFC 4880, 2007.

[7] Fragkiskos Papadopoulos, Maksim Kitsak, M. Ángeles Serrano, Marián Boguñá, and Dmitri Krioukov. Popularity versus similarity in growing networks. *Nature*, 489:537–540, September 2012. `doi:10.1038/nature11459`.

[8] Victor Eguíluz, Dante Chialvo, Guillermo Cecchi, Marwan Baliki, and A. Vania Apkarian. Scale-Free Brain Functional Networks. *Phys Rev Lett*, 94(1):018102, January 2005. `doi:10.1103/PhysRevLett.94.018102`.

[9] Daniel Fraiman, Pablo Balenzuela, Jennifer Foss, and Dante Chialvo. Ising-like dynamics in large-scale functional brain networks. *Phys Rev E*, 79(6):061922, June 2009. `doi:10.1103/PhysRevE.79.061922`.

[10] P L Krapivsky, S Redner, and F Leyvraz. Connectivity of Growing Random Networks. *Phys Rev Lett*, 85(21):4629–4632, 2000. `doi:10.1103/PhysRevLett.85.4629`.

[11] S N Dorogovtsev, J F F Mendes, and A N Samukhin. Structure of Growing Networks with Preferential Linking. *Phys Rev Lett*, 85(21):4633–4636, 2000. `doi:10.1103/PhysRevLett.85.4633`.

[12] Nabil Guelzim, Samuele Bottani, Paul Bourgine, and François Képès. Topological and causal structure of the yeast transcriptional regulatory network. *Nat Genet*, 31(1):60–3, May 2002. `doi:10.1038/ng873`.

[13] Jerry B. Griffiths and Jií Podolský. *Exact Space-Times in Einstein's General Relativity*. Cambridge University Press, Cambridge, 2009.

[14] M. Postnikov. *Geometry VI: Riemannian Geometry*. Springer-Verlag, Berlin, 2001.

[15] Francis Bonahon. *Low-Dimensional Geometry*. AMS, Providence, 2009.

[16] Dmitri Krioukov, Fragkiskos Papadopoulos, Maksim Kitsak, Amin Vahdat, and Marián Boguñá. Hyperbolic Geometry of Complex Networks. *Phys Rev E*, 82:36106, 2010. `doi:10.1103/PhysRevE.82.036106`.

[17] Jaume Garriga, Mario Livio, and Alexander Vilenkin. Cosmological constant and the time of its dominance. *Phys Rev D*, 61(2):023503, December 1999. `doi:10.1103/PhysRevD.61.023503`.

[18] Rafael Sorkin, Arttu Rajantie, Carlo Contaldi, Paul Dauncey, and Horace Stoica. Is the cosmological "constant" a nonlocal quantum residue of discreteness of the causal set type? In *AIP Conf Proc*, volume 957, pages 142–153. AIP, 2007.

doi:10.1063/1.2823750.

[19] John Barrow and Douglas Shaw. New Solution of the Cosmological Constant Problems. *Phys Rev Lett*, 106(10):101302, March 2011. doi:10.1103/PhysRevLett.106.101302.

[20] Daniel Harlow, Stephen Shenker, Douglas Stanford, and Leonard Susskind. Tree-like structure of eternal inflation: A solvable model. *Phys Rev D*, 85(6):063516, March 2012. doi:10.1103/PhysRevD.85.063516.

[21] Matthew Kleban. Cosmic bubble collisions. *Class Quant Grav*, 28(20):204008, October 2011. doi:10.1088/0264-9381/28/20/204008.

[22] E Komatsu, K M Smith, J Dunkley, C L Bennett, B Gold, G Hinshaw, N Jarosik, D Larson, M. R. Nolta, L Page, D N Spergel, M Halpern, R S Hill, A Kogut, M Limon, S S Meyer, N. Odegard, G S Tucker, J L Weiland, E Wollack, and E L Wright. Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation. *Astrophys J Suppl S*, 192(2):18, February 2011. doi:10.1088/0067-0049/192/2/18.

[23] R P Woodard. How far are we from the quantum theory of gravity? *Rep Prog Phys*, 72(12):126002, December 2009. doi:10.1088/0034-4885/72/12/126002.

[24] Luca Bombelli, Joe Henson, and Rafael Sorkin. Discreteness without symmetry breaking: a theorem. *Mod Phys Lett A*, 24(32):2579–2587, 2009. doi:10.1142/S0217732309031958.

[25] Marián Boguñá and Romualdo Pastor-Satorras. Class of Correlated Random Networks with Hidden Variables. *Phys Rev E*, 68:36112, 2003. doi:10.1103/PhysRevE.68.036112.

[26] Dilwyn Edwards. Exact Expressions for the Properties of the Zero-Pressure Friedmann Models. *Mon Not R Astr Soc*, 159:51–66, 1972.

[27] Steven Weinberg. *Cosmology*. Oxford University Press, New York, 2008.

[28] Jaume Garriga. Private communications, 2012.

[29] Jan Ambjø rn, J. Jurkiewicz, and R. Loll. Reconstructing the Universe. *Phys Rev D*, 72(6):064014, September 2005. doi:10.1103/PhysRevD.72.064014.

[30] Jan Ambjø rn, Jerzy Jurkiewicz, and Renate Loll. The Self-Organizing Quantum Universe. *Sci Am*, (July):42–49, 2008. doi:10.1038/scientificamerican0708-42.

[31] Carlo Rovelli and Simone Speziale. On the geometry of loop quantum gravity on a graph. *Phys Rev D*, 82:44018, 2010. doi:10.1103/PhysRevD.82.044018.

[32] Steven Wolfram. *A New Kind of Science*. Wolfram Media, Champaign, IL, 2002.

[33] Tomasz Konopka, Fotini Markopoulou, and Simone Severini. Quantum graphity: A model of emergent locality. *Phys Rev D*, 77(10):104029, May 2008. doi:10.1103/PhysRevD.77.104029.

[34] Giacomo Mauro D'Ariano and Alessandro Tosini. Emergence of Space-Time from Topologically Homogeneous Causal Networks. September 2011. arXiv:1109.0118.

[35] Manfred Requardt. Scale Free Small World Networks and the Structure of Quantum Space-Time. August 2003. arXiv:0308089.

[36] Alan Daughton, Jorma Louko, and Rafael Sorkin. Initial Conditions and Unitarity in Unimodular Quantum Cosmology. In R B Mann and R G McLenaghan, editors, *Proceedings of the 5th Canadian Conference on General Relativity and Relativistic Astrophysics, University of Waterloo, 13-15 May, 1993*, pages 181–185, Singapore, 1994. World Scientific.