DYNAMICS OF THE PARTITION EQUATION

The equation of motion

In order to derive the equation of motion for the cochlear partition with two degrees of freedom, we begin by considering the forces on the partition illustrated in the main text of this paper. The vertical acceleration per unit length of the partition is equal to the forces due to internal damping, stiffness, and the fluid pressure. Thus the general equation of motion is

$$\mathbf{M}\frac{\partial^2}{\partial t^2}\vec{A}(z,t) = -\mathbf{S}\vec{A}(z,t) - \mathbf{D}\frac{\partial}{\partial t}\vec{A}(z,t) + \vec{F}_f \tag{1}$$

where $\mathbf{M}, \mathbf{D}$, and $\mathbf{S}$ are matrices representing the mass, damping and stiffness per unit length as functions of z. $\vec{A}$ is a vector with the first component, A^{t} equal to the displacement of the tectorial membrane (TM) and the second, A^{b} that of the basilar membrane (BM). $\vec{F}_f$ is the force per unit length due to fluid pressure. This can be expressed as

$$W(z)\begin{bmatrix} -P^{\mathrm{u}}(x,0,z) \\ P^{\mathrm{l}}(x,0,z) \end{bmatrix}, \tag{2}$$

where P is the pressure, $^{\mathrm{u}}$ and $^{\mathrm{l}}$ indicate expressions for the upper and lower fluid compartments. The fluid on the top acts in the opposite direction from fluid on the bottom, thus the different sign. To make a solvable problem, we need to relate this expression to $\vec{A}$.

WKB expansion of the velocity potential

We begin by considering the equation for incompressible, irrotational fluid flow in the upper chamber of the cochlea

$$\nabla^2\Phi^{\mathrm{u}}(X,Y,Z,t) = 0 \tag{3}$$

where Φ is the fluid velocity potential field (ie velocity $= \nabla\Phi$). WKB theory separates "short" wavelengths from the "long" scale of the media they travel in, so that to O(1) waves operate in a locally homogeneous environment. In the case of the cochlea the dimensions of the cross section are relevant for determining wavelength, and are much smaller than the length of the cochlea. Therefore the ratio of these scales, the small number ϵ, is used as the expansion parameter. To elucidate this the width of the cochlea is expressed as a product of a constant $W_0 = W(0)$ and a normalized function, $w(z)$, so $\epsilon = W_0/L$. We then normalize X and Y by W_0 and Z by L. For continuity these normalized coordinates (x, y and z) are used throughout this derivation. However, we note that after separation the equations may be presented without normalization as is done in the main text of the paper.

Continuing, we use ∇_{T} to express the transverse gradient $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. Thus Eq. 3 becomes

$$\Big(\nabla_T^2 + \epsilon^2 \frac{\partial^2}{\partial z^2}\Big)\Phi^{\mathrm{u}}(x, y, z, t) = 0 \tag{4}$$

The solution is assumed using the WKB expansion

$$\Phi^{\mathrm{u}} = e^{i(\omega t - \frac{1}{\epsilon}\int_0^z k(\eta)\,\mathrm{d}\eta)}(\Phi_0^{\mathrm{u}}(x, y, z) + \epsilon\Phi_1^{\mathrm{u}}(x, y, z) + \ldots),$$

where ω is the frequency in radians. A^{t} and P^{u} are expanded likewise.

Solution of the O(1) Fluid and Partition Equations

Since the partition acts as one of the boundaries on the fluid chamber, solving the first order fluid equation will give us the needed relation between $\vec{P}_0(x, 0, z)$ and $\vec{A}_0(z)$. Here we will focus on deriving the solution in the upper half of the partition. The derivation for the lower half is similar with appropriate adjustments to signs and boundaries.

Because the vertical velocity of the fluid at the partition must be equal to that of the TM, $\frac{\partial \Phi_0^{\mathrm{u}}}{\partial y} = i\omega W_0 A_0^{\mathrm{t}}$ at $y = 0$, which is one of the first order boundary conditions (the factor W_0 is due to normalization). Furthermore, because they are impermeable and stationary, $\frac{\partial \Phi}{\partial n_{\mathrm{T}}} = 0$ at the other boundaries. This can be expanded in orders of ϵ in the same manner as above to find O(1) boundary conditions on these surfaces. The fluid and material properties are considered constant across the width of the cross section, and there is no first order change over the vertical walls, thus Φ_0 has no first order x dependence and we will remove it from most notation henceforward. Hence the O(1) equation and boundary conditions for Φ_0 are

$$\frac{\partial^2 \Phi_0^{\mathrm{u}}(y, z)}{\partial y^2} - k(z)^2 \Phi_0^{\mathrm{u}}(y, z) = 0 \tag{5}$$

$$\frac{\partial \Phi_0^{\mathrm{u}}}{\partial y} = \begin{cases} i\omega W_0 A_0^{\mathrm{t}}(z) & \text{at } y = 0 \\ 0 & \text{at } y = H/W_0 \end{cases} \tag{6}$$

The solution to Eq. 5 when considering the second boundary is $\Phi_0^{\rm u}(y,z) = C\cosh(k(z)(y - H/W_0))$. Using the first boundary condition to solve for the constant C, we have

$$\Phi_0^{\rm u}(y,z) = -i\omega W_0 A_0^{\rm t}(z)\frac{\cosh(k(z)(y-\frac{H}{W_0}))}{k(z)\sinh(k(z)\frac{H}{W_0})}. \tag{7}$$

The relationship between P and Φ is well known for linearized incompressible fluid flow to be $P = -\rho\frac{\partial\Phi}{\partial t}$ so

$$\begin{aligned} P_0^{\rm u}(y,z) &= -i\omega\rho\Phi_0^{\rm u}(y,z) \\ &= -\omega^2\rho W_0\frac{\cosh(k(z)(y-\frac{H}{W_0}))}{k(z)\sinh(k(z)\frac{H}{W_0})}A_0^{\rm t}(z). \end{aligned} \tag{8}$$

The fluid pressure in the lower compartment is,

$$P_0^{\rm l}(y,z) = \omega^2\rho W_0\frac{\cosh(k(z)(y+\frac{H}{W_0}))}{k(z)\sinh(k(z)\frac{H}{W_0})}A_0^{\rm b}(z). \tag{9}$$

The fluid loading $W_0P_0^l(0,z)$ can be expressed as $\omega^2 m_f A_0^{\rm t}$ where

$$m_f(z) = \rho W_0^2 w(z)/k(z)\coth(k(z)H/W_0), \tag{10}$$

the normalized version of m_f in the paper, leading to the first order homogeneous equation of motion

$$[-\omega^2(\mathbf{M} + m_f(z)\mathbf{I}) + i\omega\mathbf{D} + \mathbf{S}]\vec{A}_0(z) = 0. \tag{11}$$

Solution of O(ϵ) fluid equation

Scaling the eigenvectors at $z = 0$ can be done using boundary conditions at the base of the cochlea, as is explained in the main text. However, to find the amplitudes for all other z, we must take a less local approach and connect the values in subsequent position. This can be done with the O(ϵ) problem for each eigenvector. We begin by considering the upper chamber and collecting O(ϵ) terms from Eq. 4, arriving at

$$\frac{\partial^2\Phi_1^{\rm u}(y,z)}{\partial y^2} - k^2(z)\Phi_1^{\rm u}(y,z) = 2ik\Phi_0'^{\rm u}(y,z) + ik'\Phi_0^{\rm u}(y,z) \tag{12}$$

for each mode, where $'$ denotes derivatives in z. Furthermore on the boundary of the cross section

$$\frac{\partial\Phi_1^{\rm u}}{\partial n_{\rm T}} = \begin{cases} -i\omega W_0 A_1^{\rm t} & \text{at } y = 0 \\ 0 & \text{at } y = H/W_0 \\ -ik\Phi_0^{\rm u} w'(z)/2 & \text{at } x = \pm w(z)/2 \end{cases} \tag{13}$$

We can solve for Φ_1^{u} on the partition using Green's formula along with Eqs. 12 and 13

$$\iint\limits_A (\Phi_0^{\mathrm{u}}\nabla_T^2\Phi_1^{\mathrm{u}} - \Phi_1^{\mathrm{u}}\nabla_T^2\Phi_0^{\mathrm{u}})\,\mathrm{d}A = \oint \Big(\Phi_0^{\mathrm{u}}\frac{\partial\Phi_1^{\mathrm{u}}}{\partial n_T} - \Phi_1^{\mathrm{u}}\frac{\partial\Phi_0^{\mathrm{u}}}{\partial n_T}\Big)\,\mathrm{d}S$$

$$\begin{aligned} iw(z)\int_0^{H/W_0}(k\Phi_0^{\mathrm{u}^2}(y,z))'\,\mathrm{d}y &= -w(z)\Phi_0^{\mathrm{u}}(0,z)A_1^{\mathrm{t}} - iw'(z)\int_0^{H/W_0} k\Phi_0^{\mathrm{u}^2}(y,z)\,\mathrm{d}y \\ &\quad + w(z)\Phi_1^{\mathrm{u}}(0,z)A_0^{\mathrm{t}} \\ i\Big[w(z)\int_0^{H/W_0}(k\Phi_0^{\mathrm{u}^2}(y,z))\,\mathrm{d}y\Big]' &= -w(z)\Phi_0^{\mathrm{u}}(0,z)A_1^{\mathrm{t}} + w(z)\Phi_1^{\mathrm{u}}(0,z)A_0^{\mathrm{t}} \end{aligned} \tag{14}$$

The integral can be evaluated, and for convenience we will express part of the solution as

$$G_0 = \frac{W_0 w(z)[\frac{2Hk(z)}{W_0} + \sinh(\frac{2Hk(z)}{W_0})]}{4k^2(z)\sinh(\frac{Hk(z)}{W_0})}. \tag{15}$$

By solving Eq. 14 for $\Phi_0^{\mathrm{u}}(0,z)$ and following a similar derivation to get $\Phi_0^{\mathrm{l}}(0,z)$ one arrives at the O(ϵ) partition equation

$$(\mathbf{\Gamma} - m_f\mathbf{I})\vec{A}_1 = \vec{F}_0 \tag{16}$$

The nonhomogeneous term in the upper chamber is

$$F_0^{\mathrm{u}} = \frac{-i\rho\omega^2}{A_0^{\mathrm{t}}(z)}\Big[A_0^{\mathrm{t}^2}(z)G_0(z)\Big]'. \tag{17}$$

and F_0^{l} can be obtained by replacing A_0^{t} with A_0^{b} in the above expression. The O(1) equation is the homogeneous version of this equation, so the determinant of $(\mathbf{\Gamma} - m_f\mathbf{I})$ is 0. The solvability condition for this case is

$$\vec{F}_0 \cdot \vec{\eta}_0 = 0 \tag{18}$$

where $\vec{\eta_0}$ is the eigenvector of the adjoint matrix $(\mathbf{\Gamma} - m_f\mathbf{I})^\dagger$. Since $(\mathbf{\Gamma} - m_f\mathbf{I})^\dagger\vec{\eta_0} = 0$ then $(\mathbf{\Gamma} - m_f\mathbf{I})^T\vec{\bar{\eta}}_0 = 0$, and because $(\mathbf{\Gamma} - m_f\mathbf{I})$ is symmetric, $\vec{\bar{\eta}}_0 = \vec{A}_0$. At this point it is convenient to separate the scaling of the eigenvectors from the relative contributions of the components by adopting the notation

$$\vec{A(z)} = \beta(z)\begin{bmatrix}\alpha(z)\\ 1\end{bmatrix}. \tag{19}$$

Thus substituting $\vec{A}_0$ into Eq. 18 and integrating we get

$$(\alpha_0^2(z)+1)\beta_0^2(z)G(z) = C_0 \tag{20}$$

where C_0 is the energy for each mode, which is constant in z. $C_0^{\pm}$ can be established with boundary conditions at $z = 0$, and $\beta(z)$ can then be determined for nonzero z.

The terms in Eq. 20 are equal to $\iint VP\mathrm{d}A$ for the upper and lower chambers, or the power flow. This derivation can be followed for both eigenmodes of the problem independently. Thus under most circumstances, energy is conserved within a mode across the entire cross section. However, if $\alpha_0(z) \to -i$, the value of $\beta(z)$ becomes undetermined. This corresponds to a point where the wavenumbers of both modes are similar, and the possibility of mode conversion exits. This phenomena is further discussed in the main text.