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Supplementary materials for marginal additive hazards model for case-cohort studies with multiple disease outcomes: an application to the atherosclerosis risk in communities (ARIC) study

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Appendix A: Outline of the Proofs of Theorems 1 and 2

We assume the following conditions hold: Let $\{\boldsymbol{T}_i, \boldsymbol{C}_i, \boldsymbol{Z}_i(\cdot)\}, i = 1, \ldots, n$, be independent and identically distributed where $\boldsymbol{T}_i = (T_{i1}, \cdots, T_{iK})^T$, $\boldsymbol{C}_i = (C_{i1}, \cdots, C_{iK})^T$, and $\boldsymbol{Z}_i(\cdot) = \{\boldsymbol{Z}_{i1}(\cdot), \cdots, \boldsymbol{Z}_{iK}(\cdot)\}^T$ for $i = 1, \cdots, n$; $\Pr(Y_{ik}(\tau) > 0) > 0$ for $i = 1, \cdots, n$ and $k = 1, \cdots, K$; $|Z_{ijk}(0)| + \int_0^\tau |dZ_{ijk}(u)| < C_z < \infty$ almost surely for some constant C_z; The matrix $\boldsymbol{A}_k = \mathrm{E}\left(\int_0^\tau Y_{1k}(t)\{\boldsymbol{Z}_{1k}(t)^{\otimes 2} - [\mathrm{E}\{Y_{1k}(t)\boldsymbol{Z}_{1k}(t)\}/\,\mathrm{E}\{Y_{1k}(t)\}]^{\otimes 2}\}dt\right)$ is positive definite; $\int_0^\tau \lambda_{0k}(t)dt < \infty$, for all $k = 1, \ldots, K$.

The following additional conditions are also needed to ensure the desired asymptotic convergence of the generalized case-cohort samples: As $n \to \infty$, $\tilde{\alpha} = \tilde{n}/n$ converges to a constant $\alpha \in (0, 1)$; For all $k = 1, \ldots, K$, $\tilde{q}_k = \tilde{m}^{(k)}/(n^{(k)} - \tilde{n}^{(k)})$ converges to a constant q_k in $(0, 1]$; $n^{(k)}/n$ converges to a constant $p_k \in [0, 1]$ as $n \to \infty$.

The consistency of the estimators for the hazards regression parameters were shown via the inverse function theorem in Foutz (1977). The key steps to show the asymptotic normality involved the decomposition of the weighted estimating functions into three terms which are asymptotically independent plus some asymptotically negligible terms. In addition, asymptotic expansion of $\hat{\alpha}_k(t)$ and $\hat{q}_k(t)$ are needed to handle the time-varying weight functions. This was based on Lemmas 1 and 2 in this section, the strong embedding theorem (Shorack and Wellner, 1986), and the Kolmogorov-Centsov Theorem (Karatzas and Shereve, 1988). Each of the three terms were shown to be mutually independent and asymptotically normally distributed via some theories of modern empirical processes (van der Vaart and Wellner, 1996) and the asymptotic theory for sampling from finite population (Hájek, 1960). The Taylor expansion ensures the desired asymptotic normality of the estimators for the hazards regression parameters. The uniform consistency of the cumulative baseline hazards estimators and the weak convergence to a tight Gaussian processes were shown via similar arguments mentioned above.

The following lemmas will be frequently used in proving the theorems.

Lemma 1 Let $\boldsymbol{W}_n(t)$ and $G_n(t)$ be two sequences of bounded processes. For some constant τ, assume that the following conditions (a) - (c) hold where

(a) $\sup_{0\,\leq\,t\,\leq\,\tau} \|\boldsymbol{W}_n(t) - \boldsymbol{W}(t)\| \xrightarrow{p} 0$ for some bounded process $\boldsymbol{W}(t)$,

(b) $\boldsymbol{W}_n(t)$ is monotone on $[0, \tau]$ and

(c) $G_n(t)$ converges to a zero-mean process with continuous sample paths. Then

$$\sup_{0\,\leq\,t\,\leq\,\tau} \left\| \int_0^t \{\boldsymbol{W}_n(s) - \boldsymbol{W}(s)\} dG_n(s) \right\| \xrightarrow{p} 0, \quad \sup_{0\,\leq\,t\,\leq\,\tau} \left\| \int_0^t G_n(s) d\{\boldsymbol{W}_n(s) - \boldsymbol{W}(s)\} \right\| \xrightarrow{p} 0.$$

The proof of this lemma follows from the strong embedding theorem (Shorack and Wellner, 1986, p47-48), Lemma 1 of Lin *and others* (2000) and the triangular argument of a norm. More detailed proof can be found in Kang (2007, Lemma 2).

Lemma 2 Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ be a random vector containing $\tilde{n}$ ones and $n - \tilde{n}$ zeros, with each permutation equally likely. Let $B_i(t)$, $i = 1, \ldots, n$, be *i.i.d.* real-valued random processes on $[0, \tau]$ with $\mathrm{E}\{B_i(t)\} = \mu_B(t)$, $\mathrm{Var}\{B_i(0)\} < \infty$ and $\mathrm{Var}\{B_i(\tau)\} < \infty$. Let $\boldsymbol{B}(t) = \{B_1(t), \ldots, B_n(t))\}$ and $\boldsymbol{\xi}$ be independent. Suppose that almost all paths of $B_i(t)$ have finite variation. Then,

$$n^{-1/2} \sum_{i=1}^{n} \xi_i \{B_i(t) - \mu_B(t)\}$$

converges weakly in $\ell^\infty[0, \tau]$ to a zero-mean Gaussian process and therefore

$$n^{-1} \sum_{i=1}^{n} \xi_i \{B_i(t) - \mu_B(t)\}$$

converges in probability to 0 uniformly in t.

This lemma is an extension of the proposition in Kulich and Lin (2000). The proof of this lemma follows from Hájek (1960)'s central limit theorem for finite population sampling and Example 3.6.14 of van der Vaart and Wellner (1996). More detailed proof can be found in Kang (2007, Lemma 5).

Note that for our case, ξ_i is the subcohort membership indicator and η_{ik} is the sampling indicator for the ith subject with the kth disease outside the subcohort where both the sampling of the subcohort and the cases outside the subcohort were conducted by simple random sampling without replacement. Thus, it is clear that our ξ_i's and η_{ik}'s satisfy the conditions in Lemma 2.

The following asymptotic properties of the time-varying sampling probability estimators $\hat{\alpha}_k(t) = \frac{\sum_{i=1}^n \xi_i(1-\Delta_{ik})Y_{ik}(t)}{\sum_{i=1}^n (1-\Delta_{ik})Y_{ik}(t)}$ and $\hat{q}_k(t) = \frac{\sum_{i=1}^n \Delta_{ik}(1-\xi_i)\eta_{ik}Y_{ik}(t)}{\sum_{i=1}^n \Delta_{ik}(1-\xi_i)Y_{ik}(t)}$ will also be frequently used in proving the theorems. $\hat{\alpha}_k(t)$ and $\tilde{\alpha}$ converge to the same limit uniformly in t and

$$n^{1/2}\left\{\hat{\alpha}_k(t)^{-1} - \tilde{\alpha}^{-1}\right\} = \frac{1}{\tilde{\alpha}\,\mathrm{E}\left\{(1-\Delta_{1k})Y_{1k}(t)\right\}} n^{-1/2}\left\{\sum_{i=1}^n \left(1 - \frac{\xi_i}{\tilde{\alpha}}\right)(1-\Delta_{ik})Y_{ik}(t)\right\} + o_p(1). \tag{0.1}$$

This follows from the Taylor expansion of $\hat{\alpha}_k(t)^{-1}$ around $\tilde{\alpha}$, Lemma 2, Glivenko-Cantelli lemma, and Slutsky's theorem. By similar arguments, $\hat{q}_k(t)$ and $\tilde{q}_k$ converge to the same limit uniformly in t and

$$n^{1/2}\left\{\hat{q}_k(t)^{-1} - \tilde{q}_k^{-1}\right\} = \frac{1}{\tilde{q}_k(1-\tilde{\alpha})\,\mathrm{E}\left\{\Delta_{1k}Y_{1k}(t)\right\}} n^{-1/2}\left\{\sum_{i=1}^n \left(1 - \frac{\eta_{ik}}{\tilde{q}_k}\right)\Delta_{ik}(1-\xi_i)Y_{ik}(t)\right\} + o_p(1). \tag{0.2}$$

Proof of Theorem 1 We first consider the proof for the consistency of $\hat{\boldsymbol{\beta}}_{II}$. Based on a straightforward extension of Foutz (1977), one can show $\hat{\boldsymbol{\beta}}_{II}$ to be consistent for $\boldsymbol{\beta}_0$ provided: (i) $\partial n^{-1}\hat{\boldsymbol{U}}^{II}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}^T$ exists and is continuous in an open neighborhood $\mathcal{B}$ of $\boldsymbol{\beta}_0$, (ii)$\partial n^{-1}\hat{\boldsymbol{U}}^{II}(\boldsymbol{\beta}_0)/\partial\boldsymbol{\beta}_0^T$ is negative definite with probability going to one as $n \to \infty$, (iii) $\partial n^{-1}\hat{\boldsymbol{U}}^{II}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}^T$ converges to $\boldsymbol{A}$ in probability uniformly for $\boldsymbol{\beta}$ in an open neighborhood about $\boldsymbol{\beta}_0$, and (iv) $n^{-1}\hat{\boldsymbol{U}}^{II}(\boldsymbol{\beta}) \to 0$ in probability.

One can write

$$\begin{aligned}\frac{\partial n^{-1}\hat{\boldsymbol{U}}^{II}(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}^T} &= -n^{-1}\sum_{i=1}^n\sum_{k=1}^K\int_0^\tau \{\boldsymbol{Z}_{ik}(t) - \bar{\boldsymbol{Z}}_k^\omega(t)\}\omega_{ik}(t)Y_{ik}(t)\boldsymbol{Z}_{ik}(t)^T dt \\ &= n^{-1}\sum_{i=1}^n\sum_{k=1}^K\int_0^\tau \omega_{ik}(t)Y_{ik}(t)\{\boldsymbol{Z}_{ik}(t)^{\otimes 2} - \bar{\boldsymbol{Z}}_k^\omega(t)^{\otimes 2}\}dt\end{aligned} \tag{0.3}$$

Then, (i) is clearly satisfied on the basis of (0.3) and by the continuity of each component. Now, (0.3) can be decomposed as the followings:

$$
\begin{aligned}
& n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau}\omega_{ik}(t)Y_{ik}(t)\{\boldsymbol{Z}_{ik}(t)^{\otimes 2}-\bar{\boldsymbol{Z}}_k^{\omega}(t)^{\otimes 2}\}dt \\
&= n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau}Y_{ik}(t)\{\boldsymbol{Z}_{ik}(t)^{\otimes 2}-\bar{\boldsymbol{Z}}_k^{\omega}(t)^{\otimes 2}\}dt + n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau}(1-\Delta_{ik})\left(\frac{\xi_i}{\tilde{\alpha}}-1\right)Y_{ik}(t) \\
&\times\{\boldsymbol{Z}_{ik}(t)^{\otimes 2}-\bar{\boldsymbol{Z}}_k^{\omega}(t)^{\otimes 2}\}dt \\
&+ n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau}\Delta_{ik}\left(\frac{\eta_{ik}}{\tilde{q}_k}-1\right)Y_{ik}(t)\{\boldsymbol{Z}_{ik}(t)^{\otimes 2}-\bar{\boldsymbol{Z}}_k^{\omega}(t)^{\otimes 2}\}dt \\
&+ n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau}(1-\Delta_{ik})\left\{\hat{\alpha}_k^{-1}(t)-\tilde{\alpha}^{-1}\right\}Y_{ik}(t)\{\boldsymbol{Z}_{ik}(t)^{\otimes 2}-\bar{\boldsymbol{Z}}_k^{\omega}(t)^{\otimes 2}\}dt \\
&+ n^{-1}\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau}(1-\xi_i)\Delta_{ik}\left\{\hat{q}_k^{-1}(t)-\tilde{q}_k^{-1}\right\}Y_{ik}(t)\{\boldsymbol{Z}_{ik}(t)^{\otimes 2}-\bar{\boldsymbol{Z}}_k^{\omega}(t)^{\otimes 2}\}dt
\end{aligned}
$$

It follows from Lemma 2 that, for $k = 1, \ldots, K$, $\bar{\boldsymbol{Z}}_k^{\omega}(t)$ uniformly converges to $\boldsymbol{e}_k(t)$ in t since $Y_{ik}(t)\boldsymbol{Z}_{ik}(t)^{\otimes d}(d = 0, 1,\ \text{and}\ 2)$ is of bounded variation. Applying Lemma 2 together with this uniform convergence implies that the first term on the right side of the above equality converges to $\boldsymbol{A}$ in probability as $n \to \infty$. Each of the rest terms on the right side of the above equality can be shown to converge to zero as $n \to \infty$. This follows from the uniform convergence of $\bar{\boldsymbol{Z}}_k^{\omega}(t)$ to $\boldsymbol{e}_k(t)$ in t, (0.1),(0.2), and Lemma 2.

Hence,

$$
-\frac{\partial n^{-1}\hat{\boldsymbol{U}}^{II}(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}^T} \xrightarrow{p} \boldsymbol{A} \text{ as } n \to \infty
$$

and, thus, (ii) and (iii) are satisfied.

For (iv), we can show that $n^{-1/2}\hat{\boldsymbol{U}}^{II}(\boldsymbol{\beta}_0)$ is asymptotically equivalent to

$$\begin{aligned}
&n^{-1/2}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathbf{M}_{z,ik}(\boldsymbol{\beta}_0,\tau)\\
&+n^{-1/2}\sum_{i=1}^{n}\sum_{k=1}^{K}(1-\Delta_{ik})\left(\frac{\xi_i}{\tilde{\alpha}}-1\right)\int_0^{\tau}\left[\boldsymbol{R}_{ik}(\boldsymbol{\beta}_0,t)-Y_{ik}(u)\frac{\mathrm{E}\{(1-\Delta_{1k})\boldsymbol{R}_{1k}(\boldsymbol{\beta}_0,t)\}}{\mathrm{E}\{(1-\Delta_{1k})Y_{1k}(t)\}}\right]dt\\
&+n^{-1/2}\sum_{i=1}^{n}\sum_{k=1}^{K}\Delta_{ik}(1-\xi_i)\left(\frac{\eta_{ik}}{\tilde{q}_k}-1\right)\left[\mathbf{M}_{z,ik}(\boldsymbol{\beta}_0,\tau)-\int_0^{\tau}Y_{ik}(t)\frac{\mathrm{E}\{\mathrm{d}\mathbf{M}_{z,1k}(\boldsymbol{\beta}_0,t)|\Delta_{1k}=1,\xi_1=0\}}{\mathrm{E}\{Y_{1k}(t)|\Delta_{1k}=1\}}\right].
\end{aligned} \tag{0.4}$$

Specifically, one can decompose $n^{-1/2}\hat{\boldsymbol{U}}^{II}(\boldsymbol{\beta}_0)$ into four parts:

$$\begin{aligned}
&n^{-1/2}\hat{\boldsymbol{U}}^{II}(\boldsymbol{\beta}_0)=n^{-1/2}\sum_{i=1}^{n}\sum_{k=1}^{K}\mathbf{M}_{z,ik}(\boldsymbol{\beta}_0,\tau)+n^{-1/2}\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau}\left\{\boldsymbol{e}_k(t)-\bar{\boldsymbol{Z}}_k^{\omega}(t)\right\}dM_{ik}(\boldsymbol{\beta}_0,t)\\
&+n^{-1/2}\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau}\{\omega_{ik}(t)-1\}d\mathbf{M}_{z,ik}(\boldsymbol{\beta}_0,t)+n^{-1/2}\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau}\{\omega_{ik}(t)-1\}\left\{\boldsymbol{e}_k(t)-\bar{\boldsymbol{Z}}_k^{\omega}(t)\right\}dM_{ik}(\boldsymbol{\beta}_0,t)
\end{aligned} \tag{0.5}$$

The second term on the right-hand side of (0.5) can be shown to converge to zero. Specifically, for fixed t, $\sum_{i=1}^{n} M_{ik}(\boldsymbol{\beta}_0,t)$ is a sum of i.i.d. zero-mean random variables. $M_{ik}(\boldsymbol{\beta}_0,t)$ can be shown to be of bounded variation and therefore can be written as a difference of two monotone functions in t. It then follows from the example of 2.11.16 of van der Vaart and Wellner (1996, p215) that $n^{-1/2}\sum_{i=1}^{n} M_{ik}(\boldsymbol{\beta}_0,t)$ converges weakly to a zero-mean Gaussian process, say $\mathcal{W}_{Mk}(t)$. It can be shown that $\mathrm{E}\{\mathcal{W}_{Mk}(t)-\mathcal{W}_{Mk}(s)\}^4\le C\{\Lambda_{0k}(t)-\Lambda_{0k}(s)\}^2$ for some constant $C>0$. Thus, by the conditions on $\Lambda_{0k}(t)$, $\exists$ a constant M, such that $\Lambda_{0k}(t)-\Lambda_{0k}(s)\le M(t-s)$ for $s\le t$. Therefore, by the Kolmogorov-Centsov Theorem (Karatzas and Shereve, 1988, p53), $\mathcal{W}_{Mk}(t)$ has continuous sample paths. In addition, $\bar{\boldsymbol{Z}}_k^{\omega}(t)$ can be written as a sum of two monotone functions in t. Hence, it follows from Lemma 1 that the second term on the right-hand side of (0.5) converges to 0.

By similar arguments, the fourth term on the right-hand side of (0.5) can be shown to converge to 0.

The third term on the right-hand side of (0.5) can be written as

$$
\begin{aligned}
& n^{-1/2}\sum_{i=1}^{n}\sum_{k=1}^{K}(1-\Delta_{ik})\left(\frac{\xi_i}{\tilde{\alpha}}-1\right)\mathbf{M}_{z,ik}(\boldsymbol{\beta}_0,\tau) \\
&+ n^{-1/2}\sum_{i=1}^{n}\sum_{k=1}^{K}(1-\Delta_{ik})\xi_i\int_0^{\tau}\left\{\hat{\alpha}_k^{-1}(t)-\tilde{\alpha}^{-1}\right\}d\mathbf{M}_{z,ik}(\boldsymbol{\beta}_0,t) \\
&+ n^{-1/2}\sum_{i=1}^{n}\sum_{k=1}^{K}\Delta_{ik}(1-\xi_i)\left(\frac{\eta_{ik}}{\tilde{q}_k}-1\right)\mathbf{M}_{z,ik}(\boldsymbol{\beta}_0,\tau) \\
&+ n^{-1/2}\sum_{i=1}^{n}\sum_{k=1}^{K}\Delta_{ik}(1-\xi_i)\eta_{ik}\int_0^{\tau}\left\{\hat{q}_k^{-1}(t)-\tilde{q}_k^{-1}\right\}d\mathbf{M}_{z,ik}(\boldsymbol{\beta}_0,t)
\end{aligned} \tag{0.6}
$$

The second term on the right side of (0.6) is asymptotically equivalent to

$$
n^{-1/2}\sum_{i=1}^{n}\sum_{k=1}^{K}(1-\Delta_{ik})\left(1-\frac{\xi_i}{\tilde{\alpha}}\right)\int_0^{\tau}Y_{ik}(t)\frac{\mathrm{E}\{(1-\Delta_{1k})\boldsymbol{R}_{1k}(\boldsymbol{\beta}_0,t)\}}{\mathrm{E}\{(1-\Delta_{1k})Y_{1k}(t)\}}dt \tag{0.7}
$$

by (0.1) and applying Lemma 2. Likewise, by (0.2) and applying Lemma 2, the last term on the right side of (0.6) is asymptotically equivalent to

$$
n^{-1/2}\sum_{i=1}^{n}\sum_{k=1}^{K}\Delta_{ik}(1-\xi_i)\left(1-\frac{\eta_{ik}}{\tilde{q}_k}\right)\int_0^{\tau}Y_{ik}(t)\frac{\mathrm{E}\{\mathrm{d}\mathbf{M}_{z,1k}(\boldsymbol{\beta}_0,t)|\Delta_{1k}=1,\xi_1=0\}}{\mathrm{E}\{Y_{1k}(t)|\Delta_{1k}=1\}} \tag{0.8}
$$

By combining (0.7) and (0.8), (0.6) is asymptotically equivalent to

$$
\begin{aligned}
& n^{-1/2}\sum_{i=1}^{n}\sum_{k=1}^{K}(1-\Delta_{ik})\left(\frac{\xi_i}{\tilde{\alpha}}-1\right)\int_0^{\tau}\left[\boldsymbol{R}_{ik}(\boldsymbol{\beta}_0,t)-Y_{ik}(t)\frac{\mathrm{E}\{(1-\Delta_{1k})\boldsymbol{R}_{1k}(\boldsymbol{\beta}_0,t)\}}{\mathrm{E}\{(1-\Delta_{1k})Y_{1k}(t)\}}\right]dt \\
&+ n^{-1/2}\sum_{i=1}^{n}\sum_{k=1}^{K}\Delta_{ik}(1-\xi_i)\left(\frac{\eta_{ik}}{\tilde{q}_k}-1\right)\left[\mathbf{M}_{z,ik}(\boldsymbol{\beta}_0,\tau)-\int_0^{\tau}Y_{ik}(t)\frac{\mathrm{E}\{\mathrm{d}\mathbf{M}_{z,1k}(\boldsymbol{\beta}_0,t)|\Delta_{1k}=1,\xi_1=0\}}{\mathrm{E}\{Y_{1k}(t)|\Delta_{1k}=1\}}\right].
\end{aligned}
$$

Combining the above results, we have shown that $n^{-1/2}\hat{\boldsymbol{U}}^{II}(\boldsymbol{\beta}_0)$ is asymptotically equivalent to (0.4). Under the regularity conditions, the first term on the right-hand side of (0.4) is asymptotically zero-mean normal with covariance matrix $\boldsymbol{Q}(\boldsymbol{\beta}_0)=\mathrm{E}\left\{\sum_{k=1}^{K}\mathbf{M}_{z,ik}(\boldsymbol{\beta}_0,\tau)\right\}^{\otimes 2}$ by Yin and Cai (2004).

The second and the third terms on the right-hand side of (0.4) can be shown to be asymptotically zero-mean normal with covariance matrix $\frac{1-\alpha}{\alpha}\boldsymbol{V}_1^{II}(\boldsymbol{\beta}_0)$ and $(1-\alpha)\sum_{k=1}^{K}\Pr(\Delta_{1k}=1)\left(\frac{1-q_k}{q_k}\right)\boldsymbol{V}_{2k}^{II}(\boldsymbol{\beta}_0)$ by Lemma 2, respectively. It follows from conditional expectation arguments that these three terms are mutually independent. Therefore, $n^{-1/2}\hat{\boldsymbol{U}}^{II}(\boldsymbol{\beta}_0)$ is asymptotically normally distributed with mean zero and with finite variance

$$\boldsymbol{Q}(\boldsymbol{\beta}_0) + \frac{1-\alpha}{\alpha}\boldsymbol{V}_1^{II}(\boldsymbol{\beta}_0) + (1-\alpha)\sum_{k=1}^{K}\Pr(\Delta_{1k}=1)\left(\frac{1-q_k}{q_k}\right)\boldsymbol{V}_{2k}^{II}(\boldsymbol{\beta}_0).$$

Hence $n^{-1}\hat{\boldsymbol{U}}^{II}(\boldsymbol{\beta})$ converges to zero in probability. Thus, (iv) is satisfied.

By (i),(ii),(iii) and (iv), it follows that there is a unique sequence $\hat{\boldsymbol{\beta}}_{II}$ *s.t.* $\hat{\boldsymbol{U}}^{II}(\hat{\boldsymbol{\beta}}_{II}) = 0$ with probability converging to one as $n \to 0$ and with $\hat{\boldsymbol{\beta}}_{II}$ converging in probability to $\boldsymbol{\beta}_0$ by extension of Theorem 2 (Foutz, 1977).

The asymptotic normality of $\hat{\boldsymbol{\beta}}_{II}$ follows from the consistency of $\hat{\boldsymbol{\beta}}_{II}$ and a Taylor series expansion of $\hat{\boldsymbol{U}}^{II}(\boldsymbol{\beta})$.

Proof of Theorem 2 One can make decomposition

$$\begin{aligned}
&n^{1/2}\{\hat{\Lambda}_{0k}^{II}(\hat{\boldsymbol{\beta}}_{II},t) - \Lambda_{0k}(t)\} \\
&= n^{1/2}\{\hat{\Lambda}_{0k}^{II}(\hat{\boldsymbol{\beta}}_{II},t) - \hat{\Lambda}_{0k}^{II}(\boldsymbol{\beta}_0,t)\} + n^{1/2}\{\hat{\Lambda}_{0k}^{II}(\boldsymbol{\beta}_0,t) - \Lambda_{0k}(t)\} \\
&= \int_0^t \frac{\sum_{i=1}^n \omega_{ik}(u)Y_{ik}(u)(\boldsymbol{\beta}_0 - \hat{\boldsymbol{\beta}}_{II})^T\boldsymbol{Z}_{ik}(u)}{\sum_{i=1}^n \omega_{ik}(u)Y_{ik}(u)}du + \int_0^t \frac{\sum_{i=1}^n dM_{ik}(\boldsymbol{\beta}_0,u)}{\sum_{i=1}^n \omega_{ik}(u)Y_{ik}(u)} \\
&+ \int_0^t \frac{\sum_{i=1}^n \{\omega_{ik}(u)-1\}dM_{ik}(\boldsymbol{\beta}_0,u)}{\sum_{i=1}^n \omega_{ik}(u)Y_{ik}(u)} \qquad (0.9)
\end{aligned}$$

By the uniform convergence of $\bar{\boldsymbol{Z}}_k^{\omega}(t)$ to $\boldsymbol{e}_k(t)$, the first term of (0.9) is asymptotically equivalent to $n^{1/2}\boldsymbol{r}_k(\boldsymbol{\beta}_0,t)^T\left(\hat{\boldsymbol{\beta}}_{II} - \boldsymbol{\beta}_0\right)$. Since $\left\{n^{-1}\sum_{i=1}^n \omega_{ik}(u)Y_{ik}(u)\right\}^{-1}$ can be written as a sum of two monotone functions in u and converges uniformly to $[\mathrm{E}\{Y_{1k}(u)\}]^{-1}$, where $\mathrm{E}\{Y_{1k}(u)\}$ is bounded away from 0, and $n^{-1/2}\sum_{i=1}^n M_{ik}(\boldsymbol{\beta}_0,u)$ converges to a zero-mean Gaussian process with continuous sample paths, it follows from Lemma 1 that the second term on the right-hand side of (0.9) is asymptotically equivalent to

$$\int_0^t \frac{1}{\mathrm{E}\{Y_{1k}(u)\}}d\left\{n^{-1/2}\sum_{i=1}^n M_{ik}(\boldsymbol{\beta}_0,u)\right\}$$

For the last term on the right-hand side of (0.9), it follows from (0.1), (0.2), and the uniform convergence of $\left\{n^{-1}\sum_{i=1}^n \omega_{ik}(u)Y_{ik}(u)\right\}^{-1}$ to $[\mathrm{E}\{Y_{1k}(u)\}]^{-1}$, where $\mathrm{E}\{Y_{1k}(u)\}$ is bounded away from 0 that

$$\int_0^t \frac{\sum_{i=1}^n \{\omega_{ik}(u)-1\} dM_{ik}(\boldsymbol{\beta}_0,u)}{\sum_{i=1}^n \omega_{ik}(u) Y_{ik}(u)}$$
$$= n^{-1/2}\sum_{i=1}^n \left(1-\frac{\xi_i}{\tilde{\alpha}}\right)(1-\Delta_{ik})\int_0^t Y_{ik}(u)\left[\boldsymbol{\beta}_0^T \boldsymbol{Z}_{ik}(u) - \frac{\mathrm{E}\left\{(1-\Delta_{1k})\, Y_{1k}(u)\boldsymbol{\beta}_0^T \boldsymbol{Z}_{1k}(u)\right\}}{\mathrm{E}\left\{(1-\Delta_{1k})\, Y_{1k}(u)\right\}}\right]\frac{du}{\mathrm{E}\{Y_{1k}(u)\}}$$
$$+ n^{-1/2}\sum_{i=1}^n \Delta_{ik}(1-\xi_i)\left(\frac{\eta_{ik}}{\tilde{q}_k}-1\right)\int_0^t \frac{1}{\mathrm{E}\{Y_{1k}(u)\}}\left[dM_{ik}(t) - Y_{ik}(u)\frac{\mathrm{E}\left\{dM_{1k}(\boldsymbol{\beta}_0,u)|\Delta_{1k}=1,\xi_1=0\right\}}{\mathrm{E}\left\{Y_{1k}(u)|\Delta_{1k}=1\right\}}\right]$$

Now by combining the above results and using the asymptotic expansion of $n^{1/2}(\hat{\boldsymbol{\beta}}_{II}-\boldsymbol{\beta}_0)$ where

$$n^{1/2}\left(\hat{\boldsymbol{\beta}}_{II}-\boldsymbol{\beta}_0\right) = \boldsymbol{A}^{-1}\left(n^{-1/2}\sum_{i=1}^n\sum_{k=1}^K \mathbf{M}_{z,ik}(\boldsymbol{\beta}_0,\tau)\right.$$
$$+ n^{-1/2}\sum_{i=1}^n\sum_{k=1}^K (1-\Delta_{ik})\left(\frac{\xi_i}{\tilde{\alpha}}-1\right)\int_0^\tau \left[\boldsymbol{R}_{ik}(\boldsymbol{\beta}_0,t) - Y_{ik}(t)\frac{\mathrm{E}\{(1-\Delta_{1k})\boldsymbol{R}_{1k}(\boldsymbol{\beta}_0,t)\}}{\mathrm{E}\{(1-\Delta_{1k})Y_{1k}(t)\}}\right) dt$$
$$+ n^{-1/2}\sum_{i=1}^n\sum_{k=1}^K \Delta_{ik}(1-\xi_i)\left(\frac{\eta_{ik}}{\tilde{q}_k}-1\right)$$
$$\left.\times\left[\mathbf{M}_{z,ik}(\boldsymbol{\beta}_0,t) - \int_0^\tau Y_{ik}(t)\frac{\mathrm{E}\{\mathrm{d}\mathbf{M}_{z,1k}(\boldsymbol{\beta}_0,t)|\Delta_{1k}=1,\xi_1=0\}}{\mathrm{E}\{Y_{1k}(t)|\Delta_{1k}=1\}}\right]\right) + o_p(1),$$

we have

$$n^{1/2}\left\{\hat{\Lambda}_{0k}^{II}(\hat{\boldsymbol{\beta}}_{II},t)-\Lambda_{0k}(t)\right\}$$
$$= n^{-1/2}\sum_{i=1}^n \nu_{ik}(\boldsymbol{\beta}_0,t) + n^{-1/2}\sum_{i=1}^n\left(1-\frac{\xi_i}{\tilde{\alpha}}\right)\psi_{ik}^{II}(\boldsymbol{\beta}_0,t) + n^{-1/2}\sum_{i=1}^n \nu_{ik}^*(\boldsymbol{\beta}_0,t) + o_p(1)$$

where

$$\nu_{ik}(\boldsymbol{\beta}, t) = \boldsymbol{r}_k(t)^T \boldsymbol{A}^{-1} \sum_{m=1}^{K} \mathbf{M}_{z,im}(\boldsymbol{\beta}, \tau) + \int_0^t [\mathrm{E}\{Y_{1k}(u)\}]^{-1} dM_{ik}(\boldsymbol{\beta}, u),$$

$$\psi_{ik}^{II}(\boldsymbol{\beta}, t) = \boldsymbol{r}_k(t)^T \boldsymbol{A}^{-1} \sum_{m=1}^{K} (1-\Delta_{im}) \int_0^{\tau} \left[\boldsymbol{R}_{im}(\boldsymbol{\beta}, u) - \frac{Y_{im}(u)\,\mathrm{E}\left\{(1-\Delta_{1m})\boldsymbol{R}_{1m}(\boldsymbol{\beta}, u)\right\}}{\mathrm{E}\left\{(1-\Delta_{1m})Y_{1m}(u)\right\}}\right] du + (1-\Delta_{ik}) \int_0^t Y_{ik}(u) \left[\boldsymbol{\beta}^T \boldsymbol{Z}_{ik}(u) - \frac{\mathrm{E}\left\{(1-\Delta_{1k})\, Y_{1k}(u) \boldsymbol{\beta}^T \boldsymbol{Z}_{1k}(u)\right\}}{\mathrm{E}\left\{(1-\Delta_{1k})\, Y_{1k}(u)\right\}}\right] \frac{du}{\mathrm{E}\{Y_{1k}(u)\}} \quad \text{and}$$

$$\nu_{ik}^{*}(\boldsymbol{\beta}, t) = \boldsymbol{r}_k(t)^T \boldsymbol{A}^{-1} \sum_{m=1}^{K} \Delta_{im}(1-\xi_i) \left(\frac{\eta_{im}}{\tilde{q}_m} - 1\right) \boldsymbol{\zeta}_{im}^{(2)}(\boldsymbol{\beta}, t) + \Delta_{ik}(1-\xi_i) \left(\frac{\eta_{ik}}{\tilde{q}_k} - 1\right) \zeta_{ik}^{(1)}(\boldsymbol{\beta}, t),$$

$$\zeta_{ik}^{(1)}(\boldsymbol{\beta}, t) = \int_0^t \frac{1}{\mathrm{E}\{Y_{1k}(u)\}} \left[dM_{ik}(\boldsymbol{\beta}, u) - Y_{ik}(u) \frac{\mathrm{E}\left\{dM_{1k}(\boldsymbol{\beta}, u) | \Delta_{1k} = 1, \xi_1 = 0\right\}}{\mathrm{E}\left\{Y_{1k}(u) | \Delta_{1k} = 1\right\}}\right] \quad \text{and}$$

$$\boldsymbol{\zeta}_{ik}^{(2)}(\boldsymbol{\beta}, t) = \mathbf{M}_{z,ik}(\boldsymbol{\beta}, t) - \int_0^t Y_{ik}(u) \frac{\mathrm{E}\left\{d\mathbf{M}_{z,1k}(\boldsymbol{\beta}, u) | \Delta_{1k} = 1, \xi_1 = 0\right\}}{\mathrm{E}\left\{Y_{1k}(u) | \Delta_{1k} = 1\right\}}.$$

Now, let $\boldsymbol{W}^{(1)}(t) = \{W_1^{(1)}(t), \ldots, W_K^{(1)}(t)\}^T$ where $W_k^{(1)}(t) = n^{-1/2} \sum_{i=1}^n \nu_{ik}(\boldsymbol{\beta}_0, t)$, $\boldsymbol{W}^{(2)}(t) = \{W_1^{(2)}(t), \ldots, W_K^{(2)}(t)\}^T$ where $W_k^{(2)}(t) = n^{-1/2} \sum_{i=1}^n \left(1 - \frac{\xi_i}{\tilde{\alpha}}\right) \psi_{ik}^{II}(\boldsymbol{\beta}_0, t)$, and $\boldsymbol{W}^{(3)}(t) = \{W_1^{(3)}(t), \ldots, W_K^{(3)}(t)\}^T$ where $W_k^{(3)}(t) = n^{-1/2} \sum_{i=1}^n \nu_{ik}^{*}(\boldsymbol{\beta}_0, t)$ for $k = 1, \ldots, K$. Then, $\boldsymbol{W}^{(1)}(t)$ converges weakly to a zero-mean Gaussian process $\mathcal{W}^{(1)}(t) = \{\mathcal{W}_1^{(1)}(t), \ldots, \mathcal{W}_K^{(1)}(t)\}^T$ in $D[0, \tau]^K$ where the covariance function between $\mathcal{W}_j^{(1)}(t_1)$ and $\mathcal{W}_k^{(1)}(t_2)$ is $\mathrm{E}\{\nu_{1j}(\boldsymbol{\beta}_0, t_1)\nu_{1k}(\boldsymbol{\beta}_0, t_2)\}$ by Yin and Cai (2004, Thm.2). $\boldsymbol{W}^{(2)}(t)$ also can be shown to converge weakly to a zero-mean Gaussian process $\mathcal{W}^{(2)}(t) = \{\mathcal{W}_1^{(2)}(t), \ldots, \mathcal{W}_K^{(2)}(t)\}^T$ where the covariance function between $\mathcal{W}_j^{(2)}(t_1)$ and $\mathcal{W}_k^{(2)}(t_2)$ is $\frac{1-\alpha}{\alpha}\,\mathrm{E}\{\psi_{1j}^{II}(\boldsymbol{\beta}_0, t_1)\psi_{1k}^{II}(\boldsymbol{\beta}_0, t_2)\}$. This follows from Lemma 2, Cramer-Wold device and the marginal tightness of $W_k^{(2)}(t)$ for each k. It follows from the similar arguments that $\boldsymbol{W}^{(3)}(t)$ converges weakly to a zero-mean Gaussian process where the covariance function between $\mathcal{W}_j^{(3)}(t_1)$ and $\mathcal{W}_k^{(3)}(t_2)$ is

$$(1-\alpha)\left[I(j=k)\Pr(\Delta_{1k}=1)\left(\frac{1-q_k}{q_k}\right)\mathrm{Cov}\left\{\zeta_{1k}^{(1)}(\boldsymbol{\beta}_0,t_1),\zeta_{1k}^{(1)}(\boldsymbol{\beta}_0,t_2)|\Delta_{1k}=1,\xi_1=0\right\}\right.$$
$$+\Pr(\Delta_{1j}=1)\left(\frac{1-q_j}{q_j}\right)\mathrm{Cov}\left\{\zeta_{1j}^{(1)}(\boldsymbol{\beta}_0,t_1),\boldsymbol{r}_k(t_2)^T\boldsymbol{A}^{-1}\boldsymbol{\zeta}_{1j}^{(2)}(\boldsymbol{\beta}_0,t_2)|\Delta_{1j}=1,\xi_1=0\right\}$$
$$+\Pr(\Delta_{1k}=1)\left(\frac{1-q_k}{q_k}\right)\mathrm{Cov}\left\{\zeta_{1k}^{(1)}(\boldsymbol{\beta}_0,t_2),\boldsymbol{r}_j(t_1)^T\boldsymbol{A}^{-1}\boldsymbol{\zeta}_{1k}^{(2)}(\boldsymbol{\beta}_0,t_1)|\Delta_{1k}=1,\xi_1=0\right\}$$
$$+\sum_{m=1}^{K}\Pr(\Delta_{1m}=1)\left(\frac{1-q_m}{q_m}\right)$$
$$\left.\times\,\boldsymbol{r}_j(t_1)^T\boldsymbol{A}^{-1}\mathrm{Cov}\left\{\boldsymbol{\zeta}_{1m}^{(2)}(\boldsymbol{\beta}_0,t_1),\boldsymbol{\zeta}_{1m}^{(2)}(\boldsymbol{\beta}_0,t_2)|\Delta_{1m}=1,\xi_1=0\right\}\boldsymbol{A}^{-1}\boldsymbol{r}_k(t_2)\right].$$

It follows from the conditional expectation argument that these three terms are mutually independent. Therefore, $\boldsymbol{W}^{II}(t)=\boldsymbol{W}^{(1)}(t)+\boldsymbol{W}^{(2)}(t)+\boldsymbol{W}^{(3)}(t)$ converges weakly to a zero-mean Gaussian process $\mathcal{W}^{II}(t)=\mathcal{W}^{(1)}(t)+\mathcal{W}^{(2)}(t)+\mathcal{W}^{(3)}(t)$. This completes the proofs.

Appendix B: Explicit Forms of the Asymptotic Variances for $n^{1/2}(\hat{\boldsymbol{\beta}}_{II}-\boldsymbol{\beta}_0)$ and $n^{1/2}[\{\hat{\Lambda}_{01}^{II}(\hat{\boldsymbol{\beta}}_{II},t)-\Lambda_{01}(t)\},\ldots,\{\hat{\Lambda}_{0K}^{II}(\hat{\boldsymbol{\beta}}_{II},t)-\Lambda_{0K}(t)\}]^T$ and Their Consistent Estimators

In Theorem 1,

$$\boldsymbol{\Sigma}_{II}(\boldsymbol{\beta}_0)=\boldsymbol{A}^{-1}\left\{\boldsymbol{Q}(\boldsymbol{\beta}_0)+\frac{1-\alpha}{\alpha}\boldsymbol{V}^{II,(1)}(\boldsymbol{\beta}_0)+(1-\alpha)\sum_{k=1}^{K}\Pr(\Delta_{1k}=1)\left(\frac{1-q_k}{q_k}\right)\boldsymbol{V}_k^{II,(2)}(\boldsymbol{\beta}_0)\right\}\boldsymbol{A}^{-1}$$

where

$$\boldsymbol{A}=\sum_{k=1}^{K}A_k,\boldsymbol{Q}(\boldsymbol{\beta})=\mathrm{E}\left\{\sum_{k=1}^{K}\mathbf{M}_{z,1k}(\boldsymbol{\beta},\tau)\right\}^{\otimes 2},$$
$$\boldsymbol{V}^{II,(1)}(\boldsymbol{\beta})=\mathrm{Var}\left(\sum_{k=1}^{K}(1-\Delta_{1k})\int_0^{\tau}\left[\boldsymbol{R}_{1k}(\boldsymbol{\beta},t)-\frac{Y_{1k}(t)\,\mathrm{E}\left\{(1-\Delta_{1k})\boldsymbol{R}_{1k}(\boldsymbol{\beta},t)\right\}}{\mathrm{E}\left\{(1-\Delta_{1k})Y_{1k}(t)\right\}}\right]dt\right),$$
$$\boldsymbol{V}_k^{II,(2)}(\boldsymbol{\beta})=\mathrm{Var}\left[\mathbf{M}_{z,1k}(\boldsymbol{\beta},\tau)-\int_0^{\tau}Y_{1k}(t)\frac{\mathrm{E}\{d\mathbf{M}_{z,1k}(\boldsymbol{\beta},t)|\Delta_{1k}=1,\xi_1=0\}}{\mathrm{E}\{Y_{1k}(t)=1|\Delta_{1k}=1\}}\right],$$
$$\mathbf{M}_{z,ik}(\boldsymbol{\beta},t)=\int_0^{t}\{\boldsymbol{Z}_{ik}(u)-\boldsymbol{e}_k(u)\}\,dM_{ik}(\boldsymbol{\beta},u),$$
$$\boldsymbol{R}_{ik}(\boldsymbol{\beta},t)=Y_{ik}(t)\{\boldsymbol{Z}_{ik}(t)-\boldsymbol{e}_k(t)\}\{\lambda_{0k}(t)+\boldsymbol{\beta}^T\boldsymbol{Z}_{ik}(t)\},\text{ and }\boldsymbol{e}_k(t)=\frac{\mathrm{E}\{Y_{1k}(t)\boldsymbol{Z}_{1k}(t)\}}{\mathrm{E}\{Y_{1k}(t)\}}.$$

The matrices $\boldsymbol{A}$, $\boldsymbol{Q}(\boldsymbol{\beta}_0)$, $\frac{1-\alpha}{\alpha}\boldsymbol{V}_1^{II}(\boldsymbol{\beta}_0)$, and $(1-\alpha)\sum_{k=1}^{K}\Pr(\Delta_{1k}=1)\left(\frac{1-q_k}{q_k}\right)\boldsymbol{V}_{2k}^{II}(\boldsymbol{\beta}_0)$ can be consistently estimated by $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{Q}}(\hat{\boldsymbol{\beta}}_{II})$, $\frac{1-\tilde{\alpha}}{\tilde{\alpha}}\tilde{\boldsymbol{V}}_1^{II}(\hat{\boldsymbol{\beta}}_{II})$, and $(1-\tilde{\alpha})\sum_{k=1}^{K}\hat{\Pr}(\Delta_{1k}=1)\left(\frac{1-\tilde{q}_k}{\tilde{q}_k}\right)\hat{\boldsymbol{V}}_{II}(\hat{\boldsymbol{\beta}}_{II})$ where

$$
\begin{aligned}
&\hat{\boldsymbol{A}} = -n^{-1}\frac{\partial\hat{\boldsymbol{U}}^{II}(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}},\ \hat{\boldsymbol{Q}}(\boldsymbol{\beta}) = n^{-1}\sum_{i=1}^{n}\frac{\xi_i}{\tilde{\alpha}}\left(\sum_{k=1}^{K}\hat{\mathbf{M}}_{z,ik}(\boldsymbol{\beta},\tau)\right)^{\otimes 2},\\
&\hat{\boldsymbol{V}}_1(\boldsymbol{\beta}) = n^{-1}\sum_{i=1}^{n}\frac{\xi_i}{\tilde{\alpha}}\left(\sum_{k=1}^{K}(1-\Delta_{ik})\int_0^{\tau}\left[\hat{\boldsymbol{R}}_{ik}(\boldsymbol{\beta},t)-\frac{Y_{ik}(t)\hat{\mathrm{E}}\left\{(1-\Delta_{1k})\boldsymbol{R}_{1k}(\boldsymbol{\beta},t)\right\}}{\hat{\mathrm{E}}\left\{(1-\Delta_{1k})Y_{1k}(t)\right\}}\right]dt\right)^{\otimes 2},\\
&-\left\{n^{-1}\sum_{i=1}^{n}\frac{\xi_i}{\tilde{\alpha}}\left(\sum_{k=1}^{K}(1-\Delta_{ik})\int_0^{\tau}\left[\hat{\boldsymbol{R}}_{ik}(\boldsymbol{\beta},t)-\frac{Y_{ik}(t)\hat{\mathrm{E}}\left\{(1-\Delta_{1k})\boldsymbol{R}_{1k}(\boldsymbol{\beta},t)\right\}}{\hat{\mathrm{E}}\left\{(1-\Delta_{1k})Y_{1k}(t)\right\}}\right]dt\right)\right\}^{\otimes 2},\\
&\tilde{\boldsymbol{V}}_{2k}^{II}(\boldsymbol{\beta}) = (m^{(k)})^{-1}\sum_{i=1}^{n}\Delta_{ik}\left(1-\xi_i\right)\eta_{ik}\left[\hat{\mathbf{M}}_{z,ik}(\boldsymbol{\beta},\tau)-\int_0^{X_{ik}}\frac{\hat{\mathrm{E}}\left\{d\mathbf{M}_{z,1k}(\boldsymbol{\beta},t)|\Delta_{1k}=1,\xi_1=0\right\}}{\hat{\mathrm{E}}\left\{Y_{1k}(t)|\Delta_{1k}=1\right\}}\right]^{\otimes 2}\\
&-\left((m^{(k)})^{-1}\sum_{i=1}^{n}\Delta_{ik}\left(1-\xi_i\right)\eta_{ik}\left[\hat{\mathbf{M}}_{z,ik}(\boldsymbol{\beta},\tau)-\int_0^{X_{ik}}\frac{\hat{\mathrm{E}}\left(d\mathbf{M}_{z,1k}(\boldsymbol{\beta},t)|\Delta_{1k}=1,\xi_1=0\right)}{\hat{\mathrm{E}}\left(Y_{1k}(t)|\Delta_{1k}=1\right)}\right]\right)^{\otimes 2},
\end{aligned}
$$

$$
\begin{aligned}
&\hat{\mathbf{M}}_{z,ik}(\boldsymbol{\beta},\tau) = \left\{\Delta_{ik}-\sum_{j=1}^{n}\frac{\Delta_{jk}\omega_{jk}(X_{jk})Y_{ik}(X_{jk})}{\sum_{l=1}^{n}\omega_{lk}(X_{jk})Y_{lk}(X_{jk})}\right\}\left\{\boldsymbol{Z}_{ik}(X_{jk})-\bar{\boldsymbol{Z}}_k^{\omega}(X_{jk})\right\}\\
&+\int_0^{X_{ik}}\left\{\frac{Y_{jk}(t)\boldsymbol{\beta}^T\boldsymbol{Z}_{jk}(t)}{\sum_{l=1}^{n}\omega_{lk}(t)Y_{lk}(t)}-\boldsymbol{\beta}^T\boldsymbol{Z}_{ik}(t)\right\}\left\{\boldsymbol{Z}_{ik}(t)-\bar{\boldsymbol{Z}}_k^{\omega}(t)\right\}dt\\
&\hat{\boldsymbol{R}}_{ik}(\boldsymbol{\beta},t) = Y_{ik}(t)\left\{\boldsymbol{Z}_{ik}(t)-\bar{\boldsymbol{Z}}_k^{\omega}(t)\right\}\left[\frac{\sum_{j=1}^{n}\omega_{jk}(t)\left\{dN_{jk}(t)-Y_{jk}(t)\boldsymbol{\beta}^T\boldsymbol{Z}_{jk}(t)dt\right\}}{\sum_{l=1}^{n}\omega_{lk}(t)Y_{lk}(t)}+\boldsymbol{\beta}^T\boldsymbol{Z}_{ik}(t)\right],\\
&\hat{\mathrm{E}}\left\{(1-\Delta_{1k})\boldsymbol{R}_{1k}(\boldsymbol{\beta},t)\right\} = n^{-1}\sum_{i=1}^{n}\frac{\xi_i}{\tilde{\alpha}}(1-\Delta_{ik})\hat{\boldsymbol{R}}_{ik}(\boldsymbol{\beta},t), \hat{\Pr}(\Delta_{1k}=1) = \frac{n^{(k)}}{n},\\
&\hat{\mathrm{E}}\left\{(1-\Delta_{1k})Y_{1k}(t)\right\} = n^{-1}\sum_{i=1}^{n}(1-\Delta_{ik})Y_{ik}(t), \hat{\mathrm{E}}\left\{Y_{1k}(t)|\Delta_{1k}=1\right\} = n_k^{-1}\sum_{i=1}^{n}\Delta_{ik}Y_{ik}(t),\\
&\hat{\mathrm{E}}\left\{d\mathbf{M}_{z,ik}(\boldsymbol{\beta},t)|\Delta_{1k}=1,\xi_1=0\right\} = (m^{(k)})^{-1}\sum_{i=1}^{n}\Delta_{ik}\left(1-\xi_i\right)\eta_{ik}\left\{\boldsymbol{Z}_{ik}(t)-\bar{\boldsymbol{Z}}_k^{\omega}(t)\right\}d\hat{M}_{ik}(\boldsymbol{\beta},t),
\end{aligned}
$$

and $d\hat{M}_{ik}(\boldsymbol{\beta},t) = dN_{ik}(t)-\dfrac{Y_{ik}(t)\sum_{j=1}^{n}\omega_{jk}(t)\left\{dN_{jk}(t)-Y_{jk}(t)\boldsymbol{\beta}^T\boldsymbol{Z}_{jk}(t)dt\right\}}{\sum_{l=1}^{n}\omega_{lk}(t)Y_{lk}(t)}-Y_{ik}(t)\boldsymbol{\beta}^T\boldsymbol{Z}_{ik}(t)dt.$

In Theorem 2, The covariance function between $\mathcal{W}_j^{II}(t_1)$ and $\mathcal{W}_k^{II}(t_2)$ is

$$
\begin{aligned}
\phi_{jk}^{II}(t_1,t_2)(\boldsymbol{\beta}_0) = \; & \mathrm{E}\{\nu_{1j}(\boldsymbol{\beta}_0,t_1)\nu_{1k}(\boldsymbol{\beta}_0,t_2)\} + \frac{1-\alpha}{\alpha}\,\mathrm{E}\{\psi_{1j}^{II}(\boldsymbol{\beta}_0,t_1)\psi_{1k}^{II}(\boldsymbol{\beta}_0,t_2)\} + (1-\alpha) \\
& \times \left[I(j=k)\,\Pr(\Delta_{1k}=1)\left(\frac{1-q_k}{q_k}\right)\mathrm{Cov}\left\{\zeta_{1k}^{(1)}(\boldsymbol{\beta}_0,t_1),\zeta_{1k}^{(1)}(\boldsymbol{\beta}_0,t_2)|\Delta_{1k}=1,\xi_1=0\right\}\right. \\
& + \Pr(\Delta_{1j}=1)\left(\frac{1-q_j}{q_j}\right)\mathrm{Cov}\left\{\zeta_{1j}^{(1)}(\boldsymbol{\beta}_0,t_1),\boldsymbol{r}_k(t_2)^T\boldsymbol{A}^{-1}\boldsymbol{\zeta}_{1j}^{(2)}(\boldsymbol{\beta}_0,t_2)|\Delta_{1j}=1,\xi_1=0\right\} \\
& + \Pr(\Delta_{1k}=1)\left(\frac{1-q_k}{q_k}\right)\mathrm{Cov}\left\{\zeta_{1k}^{(1)}(\boldsymbol{\beta}_0,t_2),\boldsymbol{r}_j(t_1)^T\boldsymbol{A}^{-1}\boldsymbol{\zeta}_{1k}^{(2)}(\boldsymbol{\beta}_0,t_1)|\Delta_{1k}=1,\xi_1=0\right\} \\
& + \sum_{m=1}^{K}\Pr(\Delta_{1m}=1)\left(\frac{1-q_m}{q_m}\right) \\
& \left. \times\, \boldsymbol{r}_j(t_1)^T\boldsymbol{A}^{-1}\,\mathrm{Cov}\left\{\boldsymbol{\zeta}_{1m}^{(2)}(\boldsymbol{\beta}_0,t_1),\boldsymbol{\zeta}_{1m}^{(2)}(\boldsymbol{\beta}_0,t_2)|\Delta_{1m}=1,\xi_1=0\right\}\boldsymbol{A}^{-1}\boldsymbol{r}_k(t_2)\right].
\end{aligned}
$$

where

$$
\begin{aligned}
&\nu_{ik}(\boldsymbol{\beta},t) = \boldsymbol{r}_k(t)^T\boldsymbol{A}^{-1}\sum_{m=1}^{K}\mathbf{M}_{z,im}(\boldsymbol{\beta}) + \int_0^t [\mathrm{E}\{Y_{1k}(u)\}]^{-1}dM_{ik}(\boldsymbol{\beta},u), \\
&\psi_{ik}^{II}(\boldsymbol{\beta},t) = \boldsymbol{r}_k(t)^T\boldsymbol{A}^{-1}\sum_{m=1}^{K}\int_0^{\tau}(1-\Delta_{im})\left[\boldsymbol{R}_{im}(\boldsymbol{\beta},u) - \frac{Y_{im}(u)\,\mathrm{E}\left\{(1-\Delta_{1m})\boldsymbol{R}_{1m}(\boldsymbol{\beta},u)\right\}}{\mathrm{E}\left\{(1-\Delta_{1m})Y_{1m}(u)\right\}}\right]du \\
&+ (1-\Delta_{ik})\int_0^t \frac{1}{\mathrm{E}\{Y_{1k}(u)\}}\left[\boldsymbol{R}_{1k}(\boldsymbol{\beta},u) - Y_{ik}(u)\frac{\mathrm{E}\left\{(1-\Delta_{1k})\boldsymbol{R}_{1k}(\boldsymbol{\beta},u)\right\}}{\mathrm{E}\left\{(1-\Delta_{1k})Y_{1k}(u)\right\}}\right]du, \\
&\zeta_{ik}^{(1)}(\beta,t) = \int_0^t \frac{1}{\mathrm{E}\{Y_{1k}(u)\}}\left[dM_{ik}(\boldsymbol{\beta},u) - Y_{ik}(u)\frac{\mathrm{E}\left\{dM_{1k}(\boldsymbol{\beta},u)|\Delta_{1k}=1,\xi_1=0\right\}}{\mathrm{E}\left\{Y_{1k}(u)|\Delta_{1k}=1\right\}}\right], \\
&\boldsymbol{\zeta}_{ik}^{(2)}(\beta,t) = \mathbf{M}_{z,ik}(\boldsymbol{\beta},t) - \int_0^t Y_{ik}(u)\frac{\mathrm{E}\left\{d\mathbf{M}_{z,1k}(\boldsymbol{\beta},u)|\Delta_{1k}=1,\xi_1=0\right\}}{\mathrm{E}\left\{Y_{1k}(u)|\Delta_{1k}=1\right\}} \text{ and } \boldsymbol{r}_k(t) = -\int_0^t \boldsymbol{e}_k(u)du.
\end{aligned}
$$

$\phi_{jk}^{II}(t_1,t_2)(\boldsymbol{\beta}_0)$ can be consistently estimated by $\hat{\phi}_{jk}^{II}(t_1,t_2)(\hat{\boldsymbol{\beta}}_{II})$ where

$$
\begin{aligned}
&\hat{\phi}_{jk}^{II}(t_1,t_2)(\boldsymbol{\beta}) = n^{-1}\sum_{i=1}^{n}\frac{\xi_i}{\tilde{\alpha}}\hat{\nu}_{ij}(\boldsymbol{\beta},t_1)\hat{\nu}_{ik}(\boldsymbol{\beta},t_2) + \frac{1-\tilde{\alpha}}{\tilde{\alpha}}n^{-1}\sum_{i=1}^{n}\frac{\xi_i}{\tilde{\alpha}}\hat{\psi}_{ij}^{II}(\boldsymbol{\beta},t_1)\hat{\psi}_{ik}^{II}(\boldsymbol{\beta},t_2) + (1-\tilde{\alpha}) \\
&\times\left[I(j=k)\hat{\Pr}(\Delta_{1k}=1)\left(\frac{1-\tilde{q}_k}{\tilde{q}_k}\right)\hat{\mathrm{Cov}}\left\{\zeta_{1k}^{(1)}(\boldsymbol{\beta},t_1),\zeta_{1k}^{(1)}(\boldsymbol{\beta},t_2)|\Delta_{1k}=1,\xi_1=0\right\}\right. \\
&+ \hat{\Pr}(\Delta_{1j}=1)\left(\frac{1-\tilde{q}_j}{\tilde{q}_j}\right)\hat{\mathrm{Cov}}\left\{\zeta_{1j}^{(1)}(\boldsymbol{\beta},t_1),\;\boldsymbol{r}_k(t_2)^T\boldsymbol{A}^{-1}\boldsymbol{\zeta}_{1j}^{(2)}(\boldsymbol{\beta},t_2)|\Delta_{1j}=1,\xi_1=0\right\} \\
&+ \hat{\Pr}(\Delta_{1k}=1)\left(\frac{1-\tilde{q}_k}{\tilde{q}_k}\right)\hat{\mathrm{Cov}}\left\{\zeta_{1k}^{(1)}(\boldsymbol{\beta},t_2),\;\boldsymbol{r}_j(t_1)^T\boldsymbol{A}^{-1}\boldsymbol{\zeta}_{1k}^{(2)}(\boldsymbol{\beta},t_1)|\Delta_{1k}=1,\xi_1=0\right\} \\
&\left.+ \sum_{m=1}^{K}\hat{\Pr}(\Delta_{1m}=1)\left(\frac{1-\tilde{q}_m}{\tilde{q}_m}\right)\hat{\boldsymbol{r}}_j(t_1)^T\hat{\boldsymbol{A}}^{-1}\hat{\mathrm{Cov}}\left\{\boldsymbol{\zeta}_{1m}^{(2)}(\boldsymbol{\beta},t_1),\;\boldsymbol{\zeta}_{1m}^{(2)}(\boldsymbol{\beta},t_2)|\Delta_{1m}=1,\xi_1=0\right\}\hat{\boldsymbol{A}}^{-1}\hat{\boldsymbol{r}}_k(t_2)^T\right],
\end{aligned}
$$

$$\hat{\nu}_{ik}(\boldsymbol{\beta},t)=\hat{\boldsymbol{r}}_k(t)^T\hat{\boldsymbol{A}}^{-1}\sum_{m=1}^{K}\hat{\mathbf{M}}_{z,im}(\boldsymbol{\beta},t)+\int_0^t[\hat{\mathrm{E}}\{Y_{1k}(u)\}]^{-1}d\hat{M}_{ik}(\boldsymbol{\beta},u),$$

$$\hat{\psi}^{II}_{ik}(\boldsymbol{\beta},t)=\left(\hat{\boldsymbol{r}}_k(t)^T\hat{\boldsymbol{A}}^{-1}\sum_{m=1}^{K}(1-\Delta_{im})\int_0^{\tau}\left[\hat{\boldsymbol{R}}_{im}(u)-\frac{Y_{im}(u)\hat{\mathrm{E}}\left\{(1-\Delta_{1m})\boldsymbol{R}_{1m}(u)\right\}}{\hat{\mathrm{E}}\left\{(1-\Delta_{1m})Y_{1m}(u)\right\}}\right]du\right.$$
$$\left.+\ (1-\Delta_{ik})\int_0^t Y_{ik}(u)\left[\boldsymbol{\beta}^T\boldsymbol{Z}_{ik}(u)-\frac{\hat{\mathrm{E}}\left\{(1-\Delta_{1k})\,Y_{1k}(u)\boldsymbol{\beta}^T\boldsymbol{Z}_{1k}(u)\right\}}{\hat{\mathrm{E}}\left\{(1-\Delta_{1k})\,Y_{1k}(u)\right\}}\right]\frac{du}{\hat{\mathrm{E}}\{Y_{1k}(u)\}}\right),$$

$$\hat{\mathrm{Cov}}\left\{\zeta^{(1)}_{1k}(\boldsymbol{\beta},t_1),\ \zeta^{(1)}_{1k}(\boldsymbol{\beta},t_2)|\Delta_{1k}=1,\xi_1=0\right\}=(m^k)^{-1}\sum_{i=1}^{n}\Delta_{ik}(1-\xi_i)\eta_{ik}\hat{\zeta}^{(1)}_{1k}(\boldsymbol{\beta},t_1)\hat{\zeta}^{(1)}_{1k}(\boldsymbol{\beta},t_2)$$
$$-\left\{(m^k)^{-1}\sum_{i=1}^{n}\Delta_{ik}(1-\xi_i)\eta_{ik}\hat{\zeta}^{(1)}_{1k}(\boldsymbol{\beta},t_1)\right\}\left\{\tilde{n}_{c,k}^{-1}\sum_{i=1}^{n}\Delta_{ik}(1-\xi_i)\eta_{ik}\hat{\zeta}^{(1)}_{1k}(\boldsymbol{\beta},t_2)\right\},$$

$$\hat{\mathrm{Cov}}\left\{\zeta^{(1)}_{1k}(\boldsymbol{\beta},t_1),\ \boldsymbol{r}_j(t)^T\boldsymbol{A}^{-1}\boldsymbol{\zeta}^{(2)}_{1k}(\boldsymbol{\beta},t_2)|\Delta_{1k}=1,\xi_1=0\right\}$$
$$=(m^k)^{-1}\sum_{i=1}^{n}\Delta_{ik}(1-\xi_i)\eta_{ik}\hat{\zeta}^{(1)}_{1k}(\boldsymbol{\beta},t_1)\left\{\hat{\boldsymbol{r}}_j(t)^T\hat{\boldsymbol{A}}^{-1}\hat{\boldsymbol{\zeta}}^{(2)}_{1k}(\boldsymbol{\beta},t_2)\right\}$$
$$-\left\{\tilde{n}_{c,k}^{-1}\sum_{i=1}^{n}\Delta_{ik}(1-\xi_i)\eta_{ik}\hat{\zeta}^{(1)}_{1k}(\boldsymbol{\beta},t_1)\right\}\left\{(m^k)^{-1}\sum_{i=1}^{n}\Delta_{ik}(1-\xi_i)\eta_{ik}\hat{\boldsymbol{r}}_j(t)^T\hat{\boldsymbol{A}}^{-1}\hat{\boldsymbol{\zeta}}^{(2)}_{1k}(\boldsymbol{\beta},t_2)\right\},$$

$$\hat{\mathrm{Cov}}\left\{\boldsymbol{\zeta}^{(2)}_{1k}(\boldsymbol{\beta},t_1),\boldsymbol{\zeta}^{(2)}_{1k}(\boldsymbol{\beta},t_2)|\Delta_{1k}=1,\xi_1=0\right\}=\tilde{n}_{c,k}^{-1}\sum_{i=1}^{n}\Delta_{ik}(1-\xi_i)\eta_{ik}\hat{\boldsymbol{\zeta}}^{(2)}_{1k}(\boldsymbol{\beta},t_1)\hat{\boldsymbol{\zeta}}^{(2)}_{1k}(\boldsymbol{\beta},t_2)^T$$
$$-\left\{(m^k)^{-1}\sum_{i=1}^{n}\Delta_{ik}(1-\xi_i)\eta_{ik}\hat{\boldsymbol{\zeta}}^{(2)}_{1k}(\boldsymbol{\beta},t_1)\right\}\left\{(m^k)^{-1}\sum_{i=1}^{n}\Delta_{ik}(1-\xi_i)\eta_{ik}\hat{\boldsymbol{\zeta}}^{(2)}_{1k}(\boldsymbol{\beta},t_2)\right\}^T,$$

$$\hat{\zeta}^{(1)}_{ik}(\boldsymbol{\beta},t)=\int_0^t\frac{1}{\hat{\mathrm{E}}\{Y_{1k}(u)\}}\left[d\hat{M}_{ik}(\boldsymbol{\beta},u)-Y_{ik}(u)\frac{\hat{\mathrm{E}}\left\{dM_{1k}(\boldsymbol{\beta},u)|\Delta_{1k}=1,\xi_1=0\right\}}{\hat{\mathrm{E}}\left\{Y_{1k}(u)|\Delta_{1k}=1\right\}}\right],$$
$$\hat{\boldsymbol{\zeta}}^{(2)}_{ik}(\boldsymbol{\beta},t)=\hat{\mathbf{M}}_{z,ik}(\boldsymbol{\beta},t)-\int_0^t Y_{ik}(u)\frac{\hat{\mathrm{E}}\left(d\mathbf{M}_{z,1k}(\boldsymbol{\beta},u)|\Delta_{1k}=1,\xi_1=0\right)}{\hat{\mathrm{E}}\left\{Y_{1k}(u)|\Delta_{1k}=1\right\}},$$

$$\hat{\mathrm{E}}\left(dM_{1k}(\boldsymbol{\beta},t)|\Delta_{1k}=1,\xi_1=0\right)=(m^{(k)})^{-1}\sum_{i=1}^{n}\Delta_{ik}\left(1-\xi_i\right)\eta_{ik}d\hat{M}_{ik}(\boldsymbol{\beta},t),$$
$$\hat{\boldsymbol{r}}_k(t)=-\int_0^t\bar{\boldsymbol{Z}}^{\omega}_k(u)du,\ \text{and}\ \hat{\mathrm{E}}\{Y_{1k}(u)\}=n^{-1}\sum_{i=1}^{n}Y_{ik}(u).$$

The asymptotic variance functions for $n^{1/2}(\hat{\boldsymbol{\beta}}_I-\boldsymbol{\beta}_0)$ and $n^{1/2}[\{\hat{\Lambda}^I_{01}(\hat{\boldsymbol{\beta}}_I,t)-\Lambda_{01}(t)\},\ldots,\{\hat{\Lambda}^I_{0K}(\hat{\boldsymbol{\beta}}_I,t)-$

$\Lambda_{0K}(t)\}]^T$ are the same as those in their time-varying counterparts except that

$$\frac{\mathrm{E}\{(1-\Delta_{1k})\boldsymbol{R}_{1k}(\boldsymbol{\beta},u)\}}{\mathrm{E}\{(1-\Delta_{1k})Y_{1k}(u)\}} \text{ in } \boldsymbol{V}^{II,(1)}(\boldsymbol{\beta}) \text{ and } \psi_{ik}^{II}(\boldsymbol{\beta},t), \frac{\mathrm{E}\left\{dM_{1k}(\boldsymbol{\beta},u)|\Delta_{1k}=1,\xi_1=0\right\}}{\mathrm{E}\left\{Y_{1k}(u)|\Delta_{1k}=1\right\}} \text{ in } \zeta_{ik}^{(1)}(\boldsymbol{\beta},t),$$
$$\text{and } \frac{\mathrm{E}\{d\mathbf{M}_{z,1k}(\boldsymbol{\beta},u)|\Delta_{1k}=1,\xi_1=0\}}{\mathrm{E}\{Y_{1k}(u)=1|\Delta_{1k}=1\}} \text{ in } \boldsymbol{V}_k^{II,(2)}(\boldsymbol{\beta}) \text{ and } \boldsymbol{\zeta}_{ik}^{(2)}(\boldsymbol{\beta},t)$$

will disappear in the asymptotic variances for $n^{1/2}(\hat{\boldsymbol{\beta}}_I-\boldsymbol{\beta}_0)$ and $n^{1/2}[\{\hat{\Lambda}_{01}^I(\hat{\boldsymbol{\beta}}_I,t)-\Lambda_{01}(t)\},\ldots,\{\hat{\Lambda}_{0K}^I(\hat{\boldsymbol{\beta}}_I,t)-\Lambda_{0K}(t)\}]^T$ since these terms are generated from the asymptotic expansion of $\hat{\alpha}_k(t)$ and $\hat{q}_k(t)$ ((0.1) and (0.2)) which exist only in the time-varying weighed estimators.

Appendix C: Explicit forms of the asymptotic variance functions for $n^{1/2}(\hat{\boldsymbol{\beta}}_{st}-\boldsymbol{\beta}_0)$ and $n^{1/2}[\{\hat{\Lambda}_{01}^{st}(\hat{\boldsymbol{\beta}}_{st},t)-\Lambda_{01}(t)\},\ldots,\{\hat{\Lambda}_{0K}^{st}(\hat{\boldsymbol{\beta}}_{st},t)-\Lambda_{0K}(t)\}]^T$

The asymptotic variance function for $n^{1/2}(\hat{\boldsymbol{\beta}}_{st}-\boldsymbol{\beta}_0)$ is

$$\boldsymbol{A}_{st}^{-1}\sum_{l=1}^{L}p_l\left\{\boldsymbol{Q}_l(\boldsymbol{\beta}_0)+\frac{1-\alpha_l}{\alpha_l}\boldsymbol{V}_l^{II,(1)}(\boldsymbol{\beta}_0)+(1-\alpha_l)\sum_{k=1}^{K}\Pr(\Delta_{1k}=1)\left(\frac{1-q_{lk}}{q_{lk}}\right)\boldsymbol{V}_{lk}^{II,(2)}(\boldsymbol{\beta}_0)\right\}\boldsymbol{A}_{st}^{-1}.$$

where

$$\boldsymbol{A}_{st}=\sum_{k=1}^{K}p_lA_{lk}, A_{lk}=\mathrm{E}_l\left[\int_0^\tau Y_{l1k}(t)\{\boldsymbol{Z}_{l1k}(t)^{\otimes 2}-\boldsymbol{e}_k^{st}(t)^{\otimes 2}\}dt\right], \boldsymbol{Q}_l(\boldsymbol{\beta})=\mathrm{E}_l\left\{\sum_{k=1}^{K}\mathbf{M}_{z,l1k}(\boldsymbol{\beta},\tau)\right\}^{\otimes 2},$$
$$\boldsymbol{V}_l^{II,(1)}(\boldsymbol{\beta})=\mathrm{Var}_l\left(\sum_{k=1}^{K}(1-\Delta_{l1k})\int_0^\tau\left[\boldsymbol{R}_{l1k}(\boldsymbol{\beta},t)-\frac{Y_{l1k}(t)\,\mathrm{E}_l\left\{(1-\Delta_{l1k})\boldsymbol{R}_{l1k}(\boldsymbol{\beta},t)\right\}}{\mathrm{E}_l\left\{(1-\Delta_{l1k})Y_{l1k}(t)\right\}}\right]du\right),$$
$$\boldsymbol{V}_{lk}^{II,(2)}(\boldsymbol{\beta})=\mathrm{Var}_l\left[\mathbf{M}_{z,l1k}(\boldsymbol{\beta},\tau)-\int_0^\tau Y_{l1k}(t)\frac{\mathrm{E}_l\{d\mathbf{M}_{z,l1k}(\boldsymbol{\beta},t)|\Delta_{l1k}=1,\xi_{l1}=0\}}{\mathrm{E}_l\{Y_{l1k}(t)=1|\Delta_{l1k}=1\}}\right],$$
$$\mathbf{M}_{z,lik}(\boldsymbol{\beta},t)=\int_0^t\left\{\boldsymbol{Z}_{lik}(u)-\boldsymbol{e}_k(u)\right\}dM_{lik}(\boldsymbol{\beta},t), \boldsymbol{R}_{lik}(\boldsymbol{\beta},t)=Y_{lik}(t)\{\boldsymbol{Z}_{lik}(t)-\boldsymbol{e}_k^{st}(t)\}\{\lambda_{0k}(t)+\boldsymbol{\beta}^T\boldsymbol{Z}_{lik}(t)\},$$
$$\boldsymbol{e}_k^{st}(t)=\frac{\sum_{l=1}^{L}p_l\,\mathrm{E}_l\left\{Y_{l1k}(t)\boldsymbol{Z}_{l1k}(t)\right\}}{\sum_{l=1}^{L}p_l\,\mathrm{E}_l\{Y_{l1k}(t)\}}, \alpha_l=\lim_{n\to\infty}\tilde{\alpha}_l, q_{lk}=\lim_{n\to\infty}\tilde{q}_{lk}, \text{ and } p_l=\lim_{n\to\infty}\frac{n_l}{n}.$$

Note that E_l, Var_l and Cov_l denote the expectation, the variance and the covariance within the lth stratum, respectively.

The covariance function between $\mathcal{W}^{II}_{st,j}(t_1)$ and $\mathcal{W}^{II}_{st,k}(t_2)$ is

$$\sum_{l=1}^{L} p_l \Bigg(\mathrm{E}_l\{\nu_{l1j}(\beta_0,t_1)\nu_{l1k}(\beta_0,t_2)\} + \frac{1-\alpha_l}{\alpha_l}\,\mathrm{E}_l\{\psi^{II}_{l1j}(\boldsymbol{\beta}_0,t_1)\psi^{II}_{l1k}(\boldsymbol{\beta}_0,t_2)\} + (1-\alpha_l)$$
$$\times \Bigg[I(j=k)\Pr(\Delta_{l1k}=1)\left(\frac{1-q_{lk}}{q_{lk}}\right)\mathrm{Cov}_l\left\{\zeta^{(1)}_{l1k}(\boldsymbol{\beta}_0,t_1),\zeta^{(1)}_{l1k}(\boldsymbol{\beta}_0,t_2)|\Delta_{l1k}=1,\xi_{l1}=0\right\},$$
$$+\Pr(\Delta_{l1j}=1)\left(\frac{1-q_{lj}}{q_{lj}}\right)\mathrm{Cov}_l\left\{\zeta^{(1)}_{l1j}(\boldsymbol{\beta}_0,t_1),\boldsymbol{r}^{st}_k(t_2)^T\boldsymbol{A}^{-1}_{st}\boldsymbol{\zeta}^{(2)}_{1j}(\boldsymbol{\beta}_0,t_2)|\Delta_{l1j}=1,\xi_{l1}=0\right\}$$
$$+\Pr(\Delta_{l1k}=1)\left(\frac{1-q_{lk}}{q_{lk}}\right)\mathrm{Cov}_l\left\{\zeta^{(1)}_{l1k}(\boldsymbol{\beta}_0,t_2),\boldsymbol{r}^{st}_j(t_1)^T\boldsymbol{A}^{-1}_{st}\boldsymbol{\zeta}^{(2)}_{l1k}(\boldsymbol{\beta}_0,t_1)|\Delta_{l1k}=1,\xi_{l1}=0\right\}$$
$$+\sum_{m=1}^{K}\Pr(\Delta_{l1m}=1)\left(\frac{1-q_{lm}}{q_{lm}}\right)$$
$$\times\ \boldsymbol{r}^{st}_j(t_1)^T\boldsymbol{A}^{-1}_{st}\,\mathrm{Cov}_l\left\{\boldsymbol{\zeta}^{(2)}_{l1m}(\boldsymbol{\beta}_0,t_1),\boldsymbol{\zeta}^{(2)}_{l1m}(\boldsymbol{\beta}_0,t_2)|\Delta_{l1m}=1,\xi_{l1}=0\right\}\boldsymbol{A}^{-1}_{st}\boldsymbol{r}^{st}_k(t_2)\Bigg]\Bigg).$$

where

$$\nu_{lik}(\boldsymbol{\beta},t)=\boldsymbol{r}^{st}_k(t)^T\boldsymbol{A}^{-1}_{st}\sum_{m=1}^{K}\mathbf{M}_{z,lim}(\boldsymbol{\beta})+\int_0^t[\sum_{l=1}^{L}p_l\,\mathrm{E}_l\{Y_{l1k}(u)\}]^{-1}dM_{lik}(\boldsymbol{\beta},u),$$
$$\psi^{II}_{ik}(\boldsymbol{\beta},t)=\boldsymbol{r}^{st}_k(t)^T\boldsymbol{A}^{-1}_{st}\sum_{m=1}^{K}\int_0^{\tau}(1-\Delta_{lim})\left[\boldsymbol{R}_{lim}(\boldsymbol{\beta},t)-\frac{Y_{lim}(u)\,\mathrm{E}_l\{(1-\Delta_{l1m})\boldsymbol{R}_{l1m}(\boldsymbol{\beta},t)\}}{\mathrm{E}_l\{(1-\Delta_{l1m})Y_{l1m}(t)\}}\right]dt,$$
$$+(1-\Delta_{lik})\int_0^t\frac{1}{\sum_{l=1}^{L}p_l\,\mathrm{E}_l\,Y_{l1k}(u)}\left[\boldsymbol{R}_{l1k}(\boldsymbol{\beta},u)-Y_{lik}(u)\frac{\mathrm{E}_l\{(1-\Delta_{l1k})\boldsymbol{R}_{l1k}(\boldsymbol{\beta},u)\}}{\mathrm{E}_l\{(1-\Delta_{l1k})Y_{l1k}(u)\}}\right]du$$
$$\zeta^{(1)}_{lik}(\boldsymbol{\beta},t)=\int_0^t\frac{1}{\sum_{l=1}^{L}p_l\,\mathrm{E}_l\,Y_{l1k}(u)}\left[dM_{lik}(\boldsymbol{\beta},u)-Y_{lik}(u)\frac{\mathrm{E}_l\{dM_{l1k}(\boldsymbol{\beta},u)|\Delta_{l1k}=1,\xi_{l1}=0\}}{\mathrm{E}_l\{Y_{l1k}(u)|\Delta_{l1k}=1\}}\right],$$
$$\boldsymbol{\zeta}^{(2)}_{lik}(\boldsymbol{\beta},t)=\mathbf{M}_{z,lik}(\boldsymbol{\beta},t)-\int_0^t Y_{lik}(u)\frac{\mathrm{E}_l\{d\mathbf{M}_{z,lik}(\boldsymbol{\beta},u)|\Delta_{l1k}=1,\xi_{l1}=0\}}{\mathrm{E}_l\{Y_{l1k}(u)|\Delta_{l1k}=1\}}\text{ and }\boldsymbol{r}^{st}_k(t)=-\int_0^t\boldsymbol{e}^{st}_k(u)du.$$

Appendix D: Plots for checking the marginal additive hazards assumption for the ARIC study data

[Fig. 1 about here.]

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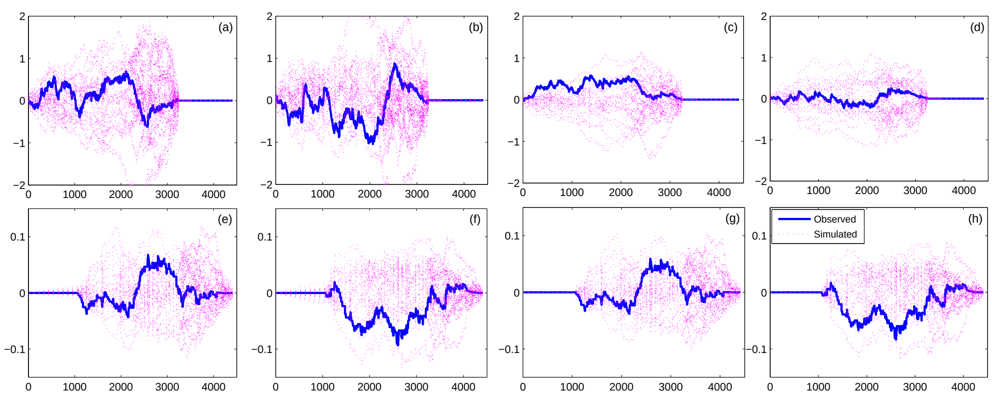

Fig. 1. Plots of the observed standardized score process (the bold, solid line) and the simulated score processes (the dashed lines) for the ARIC study data. (a) CRP2 for CHD; (b) CRP3 for CHD; (c) CRP2*(LDL-C < 130) for CHD; (d) CRP3*I(LDL-C < 130) for CHD; (e) CRP2 for stroke; (f) CRP3 for stroke; (g) CRP2*(t > 1,069) for stroke; (h) CRP3*(t > 1,069) for stroke