

Appendix

Appendix A: Proof of Proposition 1

Consider the following equality

$$E \int_0^{t_f} e^T Re dt = EV(e(0)) - EV(e(t_f)) + E \int_0^{t_f} \left(e^T Re + \frac{dV(e)}{dt} \right) dt \quad (\text{A1})$$

By Ito formula (Zhang *et al.*, 2005), we get

$$\begin{aligned} dV(e) &= \left(\frac{\partial V(e)}{\partial e} \right)^T \left((F(x, s) + (C \otimes I_m)G(x, s) + Hv) dt + (F_w(x, s) + (C \otimes I_m)G_w(x, s)) dw \right) \\ &\quad + \frac{1}{2} \left(F_w(x, s) + (C \otimes I_m)G_w(x, s) \right)^T \frac{\partial^2 V(e)}{\partial e^2} \left(F_w(x, s) + (C \otimes I_m)G_w(x, s) \right) dt \end{aligned} \quad (\text{A2})$$

Substituting (A2) into (A1) and by the fact $EdW = 0$, $EV(e(t_f)) \geq 0$, we get

$$\begin{aligned} E \int_0^{t_f} e^T Re dt &\leq EV(e(0)) + E \int_0^{t_f} \left(e^T Re + \left(\frac{\partial V(e)}{\partial e} \right)^T (F(x, s) + (C \otimes I_m)G(x, s) + Hv) \right. \\ &\quad \left. + \frac{1}{2} \left(F_w(x, s) + (C \otimes I_m)G_w(x, s) \right)^T \frac{\partial^2 V(e)}{\partial e^2} \left(F_w(x, s) + (C \otimes I_m)G_w(x, s) \right) \right) dt \end{aligned} \quad (\text{A3})$$

By the fact that

$$\mathbf{a}^T \mathbf{b} = 2 \left(\frac{1}{2\rho} \mathbf{a} \right)^T (\rho \mathbf{b}) \leq \frac{1}{4\rho^2} \mathbf{a}^T \mathbf{a} + \rho^2 \mathbf{b}^T \mathbf{b} \quad (\text{A4})$$

for any $\rho > 0$ and vectors \mathbf{a} and \mathbf{b} . Then, we get the following inequality

$$\begin{aligned} E \int_0^{t_f} e^T Re dt &\leq EV(e(0)) \\ &\quad + E \int_0^{t_f} \left(e^T Re + \left(\frac{\partial V(e)}{\partial e} \right)^T (F(x, s) + (C \otimes I_m)G(x, s)) + \frac{1}{4\rho^2} \left(\frac{\partial V(e)}{\partial e} \right)^T HH^T \left(\frac{\partial V(e)}{\partial e} \right) \right. \\ &\quad \left. + \frac{1}{2} \left(F_w(x, s) + (C \otimes I_m)G_w(x, s) \right)^T \frac{\partial^2 V(e)}{\partial e^2} \left(F_w(x, s) + (C \otimes I_m)G_w(x, s) \right) \right) dt \end{aligned}$$

where $\mathbf{a} = H^T \frac{\partial V(e)}{\partial e}$ and $\mathbf{b} = v$

By the inequality in (19), we get

$$E \int_0^{t_f} e^T R e dt \leq EV(e(0)) + E \rho^2 \int_0^{t_f} v^T v dt \quad (\text{A5})$$

If coupled synthetic genetic network is free of extrinsic noise, i.e., $v(t) = 0$, then from (A5), we get

$$E \int_0^{t_f} e^T R e dt \leq EV(e(0)) \quad (\text{A6})$$

Since $EV(e(0))$ is a constant, from (A6), we get $E(e(t)) \rightarrow 0$ as $t_f \rightarrow \infty$ and intrinsic parameter fluctuation is tolerated.

Appendix B: Proof of Proposition 2

First, we use the following fuzzy interpolation system

$$de = \sum_{k=1}^N \mu_k(z) \left((I_N \otimes A_k + C \otimes B_k) e + Hv \right) dt + \left((I_N \otimes A_{w_k} + C \otimes B_{w_k}) e \right) dw$$

in (24) to replace (16). Following the proof of Proposition 1 in Appendix A, we get the following result

$$\begin{aligned} E \int_0^{t_f} e^T R e dt &\leq EV(e(0)) - EV(e(t_f)) \\ + E \int_0^{t_f} \left(e^T R e + \sum_{k=1}^L \mu_k(z) \left(\left(\frac{\partial V(e)}{\partial e} \right)^T (I_N \otimes A_k + C \otimes B_k) e + Hv \right) \right. \\ &\quad \left. + \frac{1}{2} e^T (I_N \otimes A_{w_k} + C \otimes B_{w_k}) \frac{\partial^2 V(e)}{\partial e^2} (I_N \otimes A_{w_k} + C \otimes B_{w_k}) e \right) dt \end{aligned} \quad (B1)$$

By the facts that $V(e(t_f)) \geq 0$, and

$$\left(\frac{\partial V(e)}{\partial e} \right)^T H v \leq \frac{1}{4\rho^2} \left(\frac{\partial V(e)}{\partial e} \right)^T H H^T \left(\frac{\partial V(e)}{\partial e} \right) + \rho^2 v^T v$$

we get the following result from (B1)

$$\begin{aligned} E \int_0^{t_f} e^T R e dt &\leq EV(e(0)) \\ + E \int_0^{t_f} \left(e^T R e + \sum_{k=1}^L \mu_k(z) \left(\left(\frac{\partial V(e)}{\partial e} \right)^T (I_N \otimes A_k + C \otimes B_k) e + \frac{1}{4\rho^2} \left(\frac{\partial V(e)}{\partial e} \right)^T H H^T \left(\frac{\partial V(e)}{\partial e} \right) \right. \right. \\ &\quad \left. \left. + \rho^2 v^T v + \frac{1}{2} e^T (I_N \otimes A_{w_k} + C \otimes B_{w_k}) \frac{\partial^2 V(e)}{\partial e^2} (I_N \otimes A_{w_k} + C \otimes B_{w_k}) e \right) dt \end{aligned} \quad (B2)$$

If we choose $V(e) = e^T P e$, then $\frac{\partial V(e)}{\partial e} = 2P e$, $\frac{\partial^2 V(e)}{\partial e^2} = 2P$, then we get the

following result from (B2)

$$\begin{aligned} E \int_0^{t_f} e^T R e dt &\leq E e(0)^T P e(0) \\ + E \sum_{k=1}^L \mu_k(z) \int_0^{t_f} e^T \left(R + P (I_N \otimes A_k + C \otimes B_k) + (I_N \otimes A_k + C \otimes B_k)^T P \right. \\ &\quad \left. + (I_N \otimes A_{w_k} + C \otimes B_{w_k})^T P (I_N \otimes A_{w_k} + C \otimes B_{w_k}) + \frac{1}{\rho^2} P H H^T P \right) e dt + \rho^2 v^T v dt \end{aligned} \quad (B3)$$

If

$$R + P(I_N \otimes A_k + C \otimes B_k) + (I_N \otimes A_k + C \otimes B_k)^T P + (I_N \otimes A_{wk} + C \otimes B_{wk})^T P (I_N \otimes A_{wk} + C \otimes B_{wk}) + \frac{1}{\rho^2} P H H^T P < 0 \quad (\text{B4})$$

then we get

$$E \int_0^{t_f} e^T R e dt \leq E e(0)^T P e(0) + E \rho^2 \int_0^{t_f} v^T v dt$$

which is (18) and will be reduced to (17) if $e(0) = 0$.

Therefore, if the LMIs in (B4) hold, then the noise filtering ability ρ on the synchronization of the coupled oscillation systems is achieved. By Schur complement, the inequalities in (B4) are equivalent to the LMIs in (25), i.e. if the LMIs in (25) have a common solution $P > 0$, then the synchronization of nonlinear stochastic coupled synthetic oscillation systems in (8) has a filtering level ρ against the extrinsic noises.

