Supporting Information for "Synchronization, quantum correlations and entanglement in oscillator networks"

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MASTER EQUATION FOR NODES

From the master equation in the basis of normal modes after the (post-trace) rotating wave approximation, given in Methods, we can derive an equivalent expression turning to the basis of the original oscillators by simply applying the change of basis matrix $\mathcal{F}$, defined by diagonalization of $\mathcal{H}$ (see Dissipation mechanism and synchronization). Rearranging terms one obtains:

$$\begin{aligned}\frac{d\rho(t)}{dt} &= -i[H_S,\rho(t)] - \\ &- \frac{1}{4}\sum_{jk} i\tilde{\Gamma}_{jk}\left([q_j,\{p_k,\rho(t)\}]-[p_k,\{q_j,\rho(t)\}]\right)+ \\ &+ \tilde{D}^a_{jk}[q_j,[q_k,\rho(t)]]-\tilde{D}^b_{jk}[p_j,[p_k,\rho(t)]]. \end{aligned} \tag{1}$$

Here we have introduced new master equation coefficients denoted by a tilde and defined from the previous ones as:

$$\tilde{\Gamma}_{jk} = \sum_n \mathcal{F}_{jn}\mathcal{F}_{kn}\Gamma_n \tag{2}$$

$$\tilde{D}^a_{jk} = \sum_n \mathcal{F}_{jn}\mathcal{F}_{kn}D_n \tag{3}$$

$$\tilde{D}^b_{jk} = \sum_n \mathcal{F}_{jn}\mathcal{F}_{kn}\frac{D_n}{\Omega_n^2} \tag{4}$$

valid for all the cases considered in the paper, namely common bath, local bath and separate baths, with the proper definitions of the untilded coefficients for each case. Note however that for the case of separate baths (assuming an Ohmic frequency spectral distribution with sharp cutoff in the bath) the damping coefficients in the master equation reduce simply to $\tilde{\Gamma}_{ij}=\gamma\delta_{ij}$, i.e. all the nodes in the network dissipate through their own bath at the same rate, determined by the equivalence of the separate baths. This further simplification in the case of separate baths marks its difference from the common or local bath cases, producing a different structure for the friction terms in the equations of motion, as we will see in the next sections of this Supporting Information.

EQUATIONS FOR THE FIRST- AND SECOND-ORDER MOMENTS

For Gaussian states, the full dynamics of the oscillators is embedded in the first- and second-order moments [1] and the former give the classical limit of this quantum system, obtained neglecting quantum fluctuations. From the master equation we obtain the evolution of the first-order moments:

$$\frac{d}{dt}\langle Q_n\rangle = \langle P_n\rangle - \frac{1}{2}\Gamma_n\langle Q_n\rangle \tag{5}$$

$$\frac{d}{dt}\langle P_n\rangle = -\Omega_n^2\langle Q_n\rangle - \frac{1}{2}\Gamma_n\langle P_n\rangle \tag{6}$$

where the first term corresponds to the free evolution of uncoupled oscillators and the second one is a damping term stemming from the influence of the bath. For the second order moments we obtain the more complicated expressions:

$$\begin{aligned}\frac{d}{dt}\langle Q_n Q_m\rangle &= \frac{1}{2}\langle\{Q_n, P_m\} + \{P_n, Q_m\}\rangle \\ &- \left(\frac{\Gamma_n + \Gamma_m}{2})\right)\langle Q_n Q_m\rangle + D_n\frac{\delta_{nm}}{2\Omega_n^2} \qquad (7)\\ \frac{d}{dt}\langle P_n P_m\rangle &= -\frac{\Omega_n^2}{2}\langle\{Q_n, P_m\}\rangle - \frac{\Omega_m^2}{2}\langle\{Q_m, P_n\}\rangle \\ &- \left(\frac{\Gamma_n + \Gamma_m}{2}\right)\langle P_n P_m\rangle + D_n\frac{\delta_{nm}}{2} \qquad (8)\\ \frac{d}{dt}\langle\{Q_n, P_m\}\rangle &= 2\langle P_n P_m\rangle - 2\Omega_m^2\langle Q_n Q_m\rangle \\ &- \left(\frac{\Gamma_n + \Gamma_m}{2})\right)\langle\{Q_n P_m\}\rangle \qquad (9)\end{aligned}$$

where the first two terms arise from the reduced motion of the free normal modes and the last ones are induced by the environmental action, which combines damping and diffusion effects.

We also note that a common environment gives rise to a rather symmetric damping, also known as diffusive coupling (apart from an irrelevant change of sign) [2]. This kind of diffusive coupling is a typical phenomenological assumption when synchronization is modeled in classical systems [3]. This can be seen by looking at the first order moments, for which we obtain different expressions in the case of common, local and separate baths. In the first two cases we have:

$$\begin{aligned}\frac{d}{dt}\langle q_n\rangle &= \langle p_n\rangle - \frac{1}{2}\sum_k \tilde{\Gamma}_{nk}\langle q_k\rangle \qquad (10)\\ \frac{d}{dt}\langle p_n\rangle &= -\omega_n^2\langle q_n\rangle - \sum_k \lambda_{nk}\langle q_k\rangle - \frac{1}{2}\sum_k \tilde{\Gamma}_{nk}\langle p_k\rangle \qquad (11)\end{aligned}$$

while for the separate baths case the expression transforms into:

$$\begin{aligned}\frac{d}{dt}\langle q_n\rangle &= \langle p_n\rangle - \frac{1}{2}\tilde{\Gamma}\langle q_n\rangle \qquad (12)\\ \frac{d}{dt}\langle p_n\rangle &= -\omega_n^2\langle q_n\rangle - \sum_k \lambda_{nk}\langle q_k\rangle - \frac{1}{2}\tilde{\Gamma}\langle p_n\rangle \qquad (13)\end{aligned}$$

It is immediately seen that the presence of a common bath, a local bath or N separate (even if identical) baths leads to different friction terms in the dynamical equations. While the damping of oscillators in the common and local bath cases depends on all the network oscillators weighed by the effective couplings (κ_n^2) through the tilded damping coefficients of Eq. (2), in the separate bath case each oscillator decays independently from the rest of the network, being coupled only through the hamiltonian part of the dynamical evolution.

TUNING OF SYNCHRONIZATION FREQUENCY

The frequency of one node of a network ω_v can be tuned around the value $\omega_v = \bar{\omega}_v$ leading to a frozen normal mode. In Fig. 1, the behavior of the collective synchronization $\mathcal{S}$, the ratio between the two smallest damping rates R and the mean discord are compared at long times ($t = 8000/\omega_0$) . At the tuned frequency we observe the maximum value of quantum correlations, a large separation between the normal modes damping rates ($R \to 0$), and synchronization $\mathcal{S}$ reaching its maximum.

SYNCHRONIZATION TIME

Synchronization time is an estimation of the time necessary for the emergence of synchronization in any second order moment of the network oscillators. It can be obtained by considering the contributions of the different normal

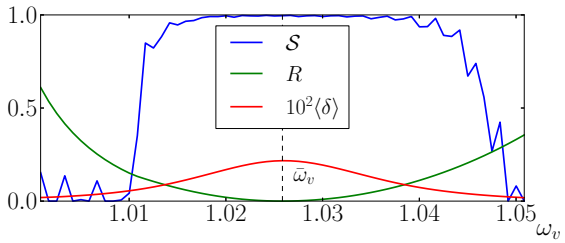

FIG. 1: Collective synchronization $\mathcal{S}$, ratio between the two smallest damping rates R and mean discord ($\times 10^2$) at long times ($8000/\omega_0$) for the same random network of 10 oscillators and probability connection $p = 0.6$ as in Figure 3 in the main manuscript . The dashed line indicates the tuning value $\omega_\nu = \bar{\omega}_\nu$ for which the mode σ decouples from the bath.

modes to the motion of second moments, which can schematically written (for the position) as:

$$\langle q_n^2\rangle(t) = \frac{\mathcal{F}_{n0}}{2}^2 e^{-\Gamma_0 t} g_0(2\Omega_0 t) + \frac{\mathcal{F}_{n1}}{2}^2 e^{-\Gamma_1 t} g_1(2\Omega_1 t) + \\ + \frac{\mathcal{F}_{n0}\mathcal{F}_{n1}}{2} e^{-\frac{\Gamma_0+\Gamma_1}{2}t} g_{01}(|\Omega_0+\Omega_1|t, |\Omega_0-\Omega_1|t) + ... \tag{14}$$

and the damping coefficients Γ_i are labeled in increasing order $(0, 1, 2, ..., N)$ from the minimum to the maximum (positive) value, while the g_i functions represent oscillating terms whose amplitude are determined by the initial conditions. Assuming that $\Gamma_0 >> \Gamma_1$, synchronization is achieved when the contributions other than the first one can be neglected in this dynamical evolution. By equating the maximum amplitudes of each mode contribution $1, 2, 3, ...$ to the first one (labeled by 0), the synchronization time can be estimated as the maximum over such possible times:

$$t^{(j)} \equiv \max_{\{k\}} \; 2\frac{\log \mathcal{F}_{jk} - \log \mathcal{F}_{j0}}{\Gamma_k - \Gamma_0}. \tag{15}$$

This expression corresponds to the minimum time for which the network oscillator (j) starts oscillating at the synchronization frequency Ω_0, the eigenfrequency of the less-damped mode.

DISSIPATION IN A LOCAL BATH

In Fig. 2A we show the ratio R between the two weakest effective couplings when varying the frequency in one node (ω_v, $v \neq 1$) comparing the cases of LB and CB. We see that depending on the frequency of the tuned node, the necessary condition for synchronization $R \sim 0$ can by satisfied in presence of both dissipation mechanisms. Still, even for identical networks, the "tuned" node produces synchronization for different frequencies (ω_v) depending on whether dissipation takes place in a CB or in a LB. Also we find that for a largely detuned oscillator ($|\omega_v - \omega_j| \gg 0$ $\forall j$), the rest of the network becomes rather insensitive to its frequency (see the behavior of R at small and large frequencies in Fig. 2A). This can be expected as for strong detuning the respective dynamics of the $N-1$ network and of the v oscillator tends to be effectively decoupled, the latter becoming one of the normal modes. We notice that for LB, in this large detuning limit, there is a normal mode v orthogonal to the dissipating one (here node d) so that it will not dissipate ($\kappa_v = 0$) leading to a vanishing ratio $R = \kappa_\sigma/\kappa_\eta$. Still, this node will be frozen but there is no synchronization of a whole cluster (conditions $\mathcal{F}_{d\sigma} = 0$ with $\mathcal{F}_{dj} \neq 0 \; \forall \; j \neq \sigma$ are not satisfied).

Fig. 2B shows that the measure $\mathcal{S}$ (see Methods) indicates synchronization for $\omega_v = \bar{\omega}_v$ when it includes contributions from all nodes excluding the lossy one, while synchronization is not evidenced when also this node is taken into account in the calculation of $\mathcal{S}$. Fig. 2B also shows that the time required for emergence of collective synchronization is larger for local dissipation than for dissipation through the center of mass (see CB in Fig. 3B of the main manuscript) by a factor of N, as expected as here we have one (instead of N) dissipation channels.

SYNCHRONIZATION OF A CLUSTER

The main conditions for emergence of synchronization in a localized cluster of a network can be derived as follows: first consider the existence of a normal mode Q_σ that involves only the cluster oscillators, i.e. :

$$Q_\sigma = \sum_{k \in \mathcal{C}_M} \mathcal{F}_{k\sigma} q_k \; , \; \mathcal{F}_{k\sigma} \neq 0 \tag{16}$$

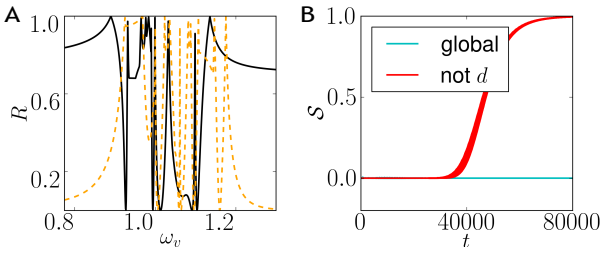

FIG. 2: (A) Ratio R between the less-damped modes for a random network (connection probability $p = 0.6$) of 10 nodes coupled to a common bath (black continuous line) and coupled to a local bath (color dashed line) when the tuning frequency ω_v is varied. (B) Collective synchronization factor $\mathcal{S}$ for a local bath when the dissipative oscillator is included into the factor $\mathcal{S}$ (line labeled 'global') or it is not included (line labeled 'not d'). Other parameters are $\gamma = 0.01\omega_0$, $T = 10\omega_0$ and $\Lambda = 50\omega_0$.

where the cluster is denoted by the ensemble $\mathcal{C}_M$ of M oscillators in the network. We impose that this mode is a normal mode of the cluster by writing:

$$(\Omega_\sigma^2 - \mathcal{H}_M)\mathbf{C}_\sigma = 0 \tag{17}$$

where $\mathcal{H}_M$ is the corresponding hamiltonian matrix of the cluster $\mathcal{C}_M$ and $\mathbf{C}_\sigma$ is a column vector containing the $\mathcal{F}_{k\sigma}$ coefficients of Eq. (16).

Now we can calculate (numerically in general and analytically for some particular situations) the coefficients of $\mathbf{C}_\sigma$ and the frequency Ω_σ in terms of the cluster parameters. This is done straightforwardly by using Eqs. (17) and the orthonormality condition for $\mathbf{C}_M$.

In order to have (asymptotically surviving) synchronization for both common and local bath we have to impose that $\kappa_s = 0$, that gives a relation among the cluster coefficients. Then by tuning only one parameter of the cluster we can obtain the desired synchronization. However note that in the case of a local bath this later condition is nothing but stating that the oscillator which is locally coupled to the bath cannot pertain to the cluster ensemble.

Finally we have to check that the normal mode which links the cluster involves only the cluster oscillators and induces a collective motion in the cluster (when $\kappa_\sigma \sim 0$) different to the rest of the network, that is:

$$(\Omega_\sigma^2 - \mathcal{H})\mathbf{C}_\sigma = 0 \tag{18}$$

where we complete $\mathbf{C}_\sigma$ with zeros in the positions of oscillators others than the cluster ones. Note that the last equation is equivalent to Eq. (17) when the next condition is fulfilled:

$$\sum_{k\in\mathcal{C}_M} \mathcal{F}_{k\sigma}\lambda_{kj} = 0 \ , \ \forall j \notin \mathcal{C}_M \tag{19}$$

that fixes a relation for the couplings between the cluster and any other oscillator outside. We can point out from the last equation that the cluster must be coupled with any other oscillator at least by a pair of couplings.

This analysis is valid in general even when the cluster is considered to be the whole network. The only variation in this case is that we do not have to ensure any more the condition (19) as long as we do not consider anything outside the cluster.

Three-oscillator motif

Finally we give here the analytical expressions for the synchronization of the three-oscillators linear motif, i.e. an open chain of three oscillators embedded in a bigger network. We are able to give the specific parameter relations that have to be fulfilled in order to obtain a non dissipative mode, i.e. making the effective coupling for a motif mode ($\kappa_\sigma = 0$).

By solving Eq. (17) for this particular case, we obtain:

$$\mathcal{F}_{a\sigma} = C\left(\frac{\lambda_{ac}}{\Omega_\sigma^2 - \omega_a^2}\right) \tag{20}$$

$$\mathcal{F}_{b\sigma} = C\left(\frac{\lambda_{bc}}{\Omega_\sigma^2 - \omega_b^2}\right) \tag{21}$$

$$\mathcal{F}_{c\sigma} = C \tag{22}$$

where $C^{-2} = 1 + \left(\frac{\lambda_{ac}}{\Omega_\sigma^2 - \omega_a^2}\right)^2 + \left(\frac{\lambda_{bc}}{\Omega_\sigma^2 - \omega_b^2}\right)^2$.

Now we can obtain a explicit expression for the effective coupling of the normal mode Q_σ to the heat bath as:

$$\kappa_\sigma = C\left(1 + \frac{\lambda_{ac}}{\Omega_\sigma^2 - \omega_a^2} + \frac{\lambda_{bc}}{\Omega_\sigma^2 - \omega_b^2}\right) \tag{23}$$

that enables a dissipation-free channel, i.e. no coupling with the bath ($\kappa_\sigma = 0$) when:

$$\frac{\lambda_{ac}}{\Omega_\sigma^2 - \omega_a^2} + \frac{\lambda_{bc}}{\Omega_\sigma^2 - \omega_b^2} = -1 \tag{24}$$

This last condition gives another different expression for the synchronization frequency in this regime:

$$\Omega_\sigma^2 = \tfrac{\omega_a^2+\omega_b^2}{2} - \tfrac{\lambda_{ac}+\lambda_{bc}}{2} \pm \\ \pm\sqrt{\left(\tfrac{\omega_a^2-\omega_b^2}{2}\right)^2 + \left(\tfrac{\lambda_{ac}+\lambda_{bc}}{2}\right)^2 - \tfrac{(\omega_a^2-\omega_b^2)(\lambda_{ac}-\lambda_{bc})}{2}} \tag{25}$$

where we have to check that Ω_σ^2 is real and positive, i.e. that $(\omega_a^2 - \omega_b^2)^2 + (\lambda_{ac} + \lambda_{bc})^2 > 2(\omega_a^2 - \omega_b^2)(\lambda_{ac} - \lambda_{bc})$.

From the explicit expression of Ω_σ and the previous equations a consistency relation follows for the selected natural frequencies and coupling between the a,b and c oscillators by substituting the expression of Ω_σ^2 in the equation:

$$\Omega_\sigma^2 - \omega_c^2 = \frac{\lambda_{ac}^2}{\Omega_\sigma^2 - \omega_a^2} + \frac{\lambda_{bc}^2}{\Omega_\sigma^2 - \omega_b^2} \tag{26}$$

whose solution for λ_{ac}, is:

$$\lambda_{ac} = \tfrac{\lambda_{bc}^2 - \lambda_{bc}(\omega_a^2 - \omega_b^2)}{2\lambda_{bc} - \omega_a^2 + \omega_c^2} \pm \\ \pm \tfrac{(\lambda_{bc} - \omega_a^2 + \omega_c^2)\sqrt{\lambda_{bc}^2 - (\omega_a^2 - \omega_b^2)(\omega_b^2 - \omega_c^2)}}{2\lambda_{bc} - \omega_a^2 + \omega_c^2} \tag{27}$$

corresponding to two different branches of solutions. These two branches intersect when we have that $\lambda_{bc} = \omega_a^2 - \omega_c^2$ or equivalently $\lambda_{ac} = \omega_b^2 - \omega_c^2$, in this case we have the simpler relation for the couplings $\lambda_{ac} - \lambda_{bc} = \omega_b^2 - \omega_a^2$ and here the mode Q_σ is degenerated, i.e. there are two non-dissipative normal modes with different frequencies. It is worth noting that when we have different branches it is necessary to impose the condition $\lambda_{bc}^2 > (\omega_a^2 - \omega_b^2)(\omega_b^2 - \omega_c^2)$ in order to obtain λ_{ac} real.

VIDEO S1 OF SYNCHRONIZATION IN A NETWORK

Synchronization process of a random network (with connection probability p=0.6) where links between 10 oscillators are shown by thin black lines. We start from an initial separable squeezed vacuum state in each node. Each squeezing factor has been sampled from a normal distribution with mean 2 and standard deviation 0.2. Natural frequencies of nodes are sampled from a uniform distribution from $0.9\omega_0$ to $1.2\omega_0$ (shown in vertical colourbar) and couplings around $0.1\omega_0$ with standard deviation $0.05\omega_0$. Time is scaled with a reference frequency $\omega_0 = 1$. Coloured links represents synchronization indicator $C_{\langle q_i^2\rangle\langle q_j^2\rangle}$ between pairs of nodes (shown in horizontal colourbar). Frequency of oscillator ν has been tuned by following the procedure described in the main paper in order to obtain collective synchronization in the whole network, reached at times of order t 3000. Finally, bath parameters employed for the simulation are $\gamma = 0.01\omega_0$, $T = 10\omega_0$ (Boltzmann constant is taken to be unity) and cutoff frequency $50\omega_0$.

[1] Gardiner, C. W. *Handbook of Stochastic Methods* (Springer, Berlin, 4th Ed., 2009).

[2] Giorgi, G. L., Galve, F., Manzano, G., Colet, P. & Zambrini, R. Quantum correlations and mutual synchronization. *Phys. Rev. A* **85,** 052101 (2012).

[3] Pikovsky, A., Rosenblum, M. & Kurths, J. *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge Univ. Press, Cambridge, 2001).