Supporting Information

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SI Text

1. Cauchy Formula. The problem of determining the perimeter and the area of the convex hull of any 2D stochastic process $[x(\tau), y(\tau)]$ with $0 \le \tau \le t$ can be mapped to the problem of computing the statistics of the maximum and the time of occurrence of the maximum of the 1D component process $x(\tau)$ (1, 2, 3). This goal is achieved by resorting to a formula by Cauchy, which applies to any closed convex curve C.

A sketch of the method is illustrated in Fig. S1. Choose the coordinates system such that the origin is inside the curve C, and take a given direction θ. For fixed θ, consider a stick perpendicular to this direction, and imagine bringing the stick from infinity; stop on first touching the curve C. At this point, the distance $M(\theta)$ of the stick from the origin is called the support function in the direction θ. Intuitively, the support function measures how close one can get to the curve C in the direction θ, coming from infinity. After the support function $M(\theta)$ is known, then Cauchy equations (4, 5) give the perimeter L and the area A enclosed by C, namely

$$L = \int_0^{2\pi} M(\theta) d\theta$$

and

$$A = \frac{1}{2} \int_0^{2\pi} \left[M^2(\theta) - \left(M'(\theta)\right)^2 \right] d\theta, \qquad \textbf{[S1]}$$

where $M'(\theta) = dM/d\theta$. For example, for a circle of radius $R = r$, $M(\theta) = r$, and one recovers the standard equations: $L = 2\pi r$ and $A = \pi r^2$. When C is the convex hull associated with the process at time t, we first need to compute its associated support function $M(\theta)$. A crucial point is to realize that actually $M(\theta) = \max_{0 \le \tau \le t}[x(\tau)\cos(\theta) + y(\tau)\sin(\theta)]$ (1, 2). Furthermore, if the process is rotationally invariant, any average is independent of the angle θ. Hence, for the average perimeter, we can simply set $\theta = 0$ and write $\langle L(t) \rangle = 2\pi \langle M(0) \rangle$, where brackets denote the ensemble average over realizations. Similarly, for the average area, $\langle A(t) \rangle = \pi[\langle M^2(0) \rangle - \langle M'(0)^2 \rangle]$. Clearly, $M(0) = \max_{0 \le \tau \le t}[x(\tau)]$ is then the maximum of the 1D component process $x(\tau)$ for $\tau \in [0, t]$. Assuming that $x(\tau)$ takes its maximum value $x(t_m)$ at time $\tau = t_m$ (Fig. 4), then, $M(0) = x(t_m) = x_m(t)$, and $M'(0) = y(t_m)$. [Actually, t_m implicitly depends on θ; hence, formally, $M'(\theta) = -x(t_m)\sin(\theta) + y(t_m)\cos(\theta) + \frac{dt_m}{d\theta}\frac{dz_\theta(t)}{dt}\big|_{t=t_m}$. Nonetheless, because $z_\theta(t)$ is maximum at $t = t_m$, by definition, $dz_\theta(t)/dt|_{t=t_m} = 0$.] Now, by taking the average over Cauchy formulas and using isotropy, we simply have Eqs. **5** and **6** from the text for the mean perimeter and the mean area of the convex hull C at time t. Note that this argument is very general and applicable to any rotationally invariant 2D stochastic process. Because the branching Brownian motion with death is rotationally invariant, we can use these formulae.

2. Numerical Methods. ***Numerical integration.*** Eqs. **9** and **14** in the text have been integrated numerically by finite differences in the following way. Time has been discretized by setting $t = ndt$, and space has been discretized by setting $x = idx$, where dt and dx are small constants. For the sake of simplicity, here, we consider the case $R_0 = 1$. We, thus, have

$$Q_{n+1}(i) = Q_n(i) + \gamma dt[1 - Q_n(i)]^2 + D\frac{dt}{(dx)^2}[Q_n(i+1) - 2Q_n(i) + Q_n(i-1)] \qquad \textbf{[S2]}$$

and

$$T_{n+1}(i) = T_n(i) + 2\gamma dt\ T_n(i)[Q_n(i) - 1] + D\frac{dt}{(dx)^2}[T_n(i+1) - 2T_n(i) + T_n(i-1)] + \frac{dt}{dx}[T_n(i) - T_n(i-1)]. \qquad \textbf{[S2]}$$

As for the initial conditions, $Q_0(0) = 0$, $Q_0(i > 0) = 1$, and $T_0(i) = 0\ \forall i$. The boundary conditions at the origin are $Q_n(0) = 0$ and $T_n(0) = 0$. To implement the boundary condition at infinity, we impose $Q_n(i_{\max}) = 1$ and $T_n(i_{\max}) = 0\ \forall n$, where the large value $i_{\max}$ is chosen so that $T_n(i_{\max}) - T_n(i_{\max} - 1) < 10^{-7}$. We have verified that numerical results do not change when passing to the tighter condition $T_n(i_{\max}) - T_n(i_{\max} - 1) < 10^{-9}$.

After $Q_n(i)$ and $T_n(i)$ are known, we use Eqs. **10** and **15** from the text to determine the average perimeter and area, respectively.

Monte Carlo simulations. The results of numerical integrations have been confirmed by running extensive Monte Carlo simulations. Branching Brownian motion with death has been simulated by discretizing time with a small dt: in each interval dt, with probability bdt, the walker branches and the current walker coordinates are copied to create a new initial point, which is then stored for being simulated in the next dt; with probability γdt, the walker dies and is removed, and with probability $1 - (b + \gamma)dt$, the walker diffuses: the x and y displacements are sampled from Gaussian densities of zero mean and SD $\sqrt{2Ddt}$, and the particle position is updated. The positions of all of the random walkers are recorded as a function of time, and the corresponding convex hull is then computed by resorting to the algorithm proposed in ref. 6.

Perimeter statistics. To complete the analysis of the convex hull statistics, in Figs. S2 and S3, we show the results for the perimeter.

3. Analysis of t_m. In the critical case $R_0 = 1$, the stationary joint probability density $P_\infty(x_m, t_m)$ satisfies (on setting $\partial P_t/\partial t = 0$ in Eq. **12** in the text)

$$\frac{\partial}{\partial t_m} P_\infty(x_m, t_m) = \left[D\frac{\partial^2}{\partial x_m^2} - \frac{2\gamma}{\left[1 + \sqrt{\frac{\gamma}{6D}}x_m\right]^2} \right] P_\infty(x_m, t_m). \qquad \textbf{[S4]}$$

For any $x_m > 0$, we have the condition $P_\infty(x_m, 0) = 0$. The boundary conditions for Eq. **S4** are $P_\infty(x_m \to \infty, t_m) = 0$ and $P_\infty(0, t_m) = q_\infty(0)\delta(t_m) = 2\sqrt{\gamma/(6D)}\delta(t_m)$. We first take the Laplace transform of Eq. **S4**, namely

$$\tilde{P}_\infty(x_m, s) = \int_0^\infty e^{-st_m} P_\infty(x_m, t_m) dt_m. \qquad \textbf{[S5]}$$

Hence, for all $x_m > 0$,

$$\frac{D}{s}\frac{\partial^2}{\partial x_m^2}\tilde{P}_\infty(x_m, s) = \left[1 + \frac{12}{\frac{s}{D}\left(\sqrt{\frac{6D}{\gamma}} + x_m\right)^2} \right] \tilde{P}_\infty(x_m, s), \qquad \textbf{[S6]}$$

where we have used the condition $P_\infty(x_m, 0) = 0$ for any $x_m > 0$. This second-order differential equation satisfies two boundary conditions: $\tilde{P}_\infty(\infty, s) = 0$ and $\tilde{P}_\infty(0, s) = 2\sqrt{\gamma/(6D)}$. The latter condition is obtained by Laplace transforming $P_\infty(0, t_m) = 2\sqrt{\gamma/(6D)}\delta(t_m)$. By setting

$$z = \left(\sqrt{\frac{6D}{\gamma}} + x_m\right)\sqrt{\frac{s}{D}}, \quad \textbf{[S7]}$$

we rewrite the equation as

$$\frac{\partial^2}{\partial z^2}\tilde{P}_\infty - \tilde{P}_\infty - \frac{12}{z^2}\tilde{P}_\infty = 0. \quad \textbf{[S8]}$$

On making the transformation $\tilde{P}_\infty(z) = \sqrt{z}F(z)$, the function $F(z)$ then satisfies the Bessel differential equation:

$$\frac{d^2}{dz^2}F(z) + \frac{1}{z}\frac{d}{dz}F(z) - \left[1 + \frac{49}{4z^2}\right]F(z) = 0. \quad \textbf{[S9]}$$

The general solution of this differential equation can be expressed as a linear combination of two independent solutions: $F(z) = AI_{7/2}(z) + BK_{7/2}(z)$, where $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions. Because $I_\nu(z) \sim e^z$ for large z, it is clear that, to satisfy the boundary condition $\tilde{P}_\infty(\infty, s) = 0$ [which means $F(z \to \infty) = 0$], we need to choose $A = 0$. Hence, we are left with $F(z) = BK_{7/2}(z)$, where the constant B is determined from the second boundary condition $\tilde{P}_\infty(0, s) = 2\sqrt{\gamma/(6D)}$. By reverting to the variable x_m, we finally get

$$\tilde{P}_\infty(x_m, s) = 2\sqrt{\frac{\gamma}{6D}}\sqrt{1 + \frac{\gamma}{6D}x_m}\,\frac{K_{7/2}\left[\left(\sqrt{\frac{6D}{\gamma}} + x_m\right)\sqrt{\frac{s}{D}}\right]}{K_{7/2}\left[\sqrt{\frac{6s}{\gamma}}\right]}. \quad \textbf{[S10]}$$

Now, we are interested in determining the Laplace transform of the marginal density $\tilde{p}_\infty(s) = \int_0^\infty e^{-st_m}p_\infty(t_m)dt_m$, where $p_\infty(t_m) = \int_0^\infty P_\infty(x_m, t_m)dx_m$. Taking Laplace transform of this last relation with respect to t_m gives $\tilde{p}_\infty(s) = \int_0^\infty \tilde{P}_\infty(x_m, s)dx_m$. After we know $\tilde{p}_\infty(s)$, we can invert it to obtain $p_\infty(t_m)$. Because we are interested only in the asymptotic tail of $p_\infty(t_m)$, it suffices to investigate the small s behavior of $\tilde{p}_\infty(s)$. Integrating Eq. **S10** over x_m and taking the $s \to 0$ limit, we obtain, after some straightforward algebra,

$$\tilde{p}_\infty(s) = 1 + \frac{3}{5\gamma}s\ln(s) + \cdots. \quad \textbf{[S11]}$$

We further note that

$$\int_0^\infty e^{-st_m}t_m^2 p_\infty(t_m)dt_m = \frac{d^2}{ds^2}\tilde{p}_\infty(s) \simeq \frac{3}{5\gamma s}, \quad \textbf{[S12]}$$

which can then be inverted to give the following asymptotic behavior for large t_m:

$$p_\infty(t_m) \simeq \frac{3}{5\gamma t_m^2}. \quad \textbf{[S13]}$$

Analogously as for $\langle x_m^2\rangle$, the moment $\langle t_m\rangle \to \infty$ because of the power-law tail $p_\infty(t_m) \propto t_m^{-2}$. Hence, we need to compute $\langle t_m\rangle$ for large but finite t: in this case, the time-dependent solution displays a scaling behavior:

$$p_t(t_m) \simeq p_\infty(t_m)g\left(\frac{t_m}{t}\right), \quad \textbf{[S14]}$$

where the scaling function $g(z)$ satisfies the conditions $g(z \ll 1) \simeq 1$ and $g(z \gg 1) = 0$. Much like in expression **17** in the text for the marginal density $q_t(x_m)$, we have a power-law tail of $p_t(t_m)$ for large t_m that has a cutoff at a scale $t_m^* \sim t$, and $g(z)$ is the cutoff function. As in the case of x_m, we do not need the precise form of $g(z)$ to compute the leading term of the first moment $\langle t_m\rangle = \int_0^\infty p_t(t_m)t_m dt_m$ for large t. Cutting off the integral at $t_m^* = c_1 t$ [where c_1 depends on the precise form of $g(z)$] and performing the integration gives

$$\langle t_m\rangle \simeq \int_0^t t_m p_\infty(t_m)dt_m \simeq \frac{3}{5\gamma}\ln t, \quad \textbf{[S15]}$$

which is precisely the result announced in expression **19** in the text.

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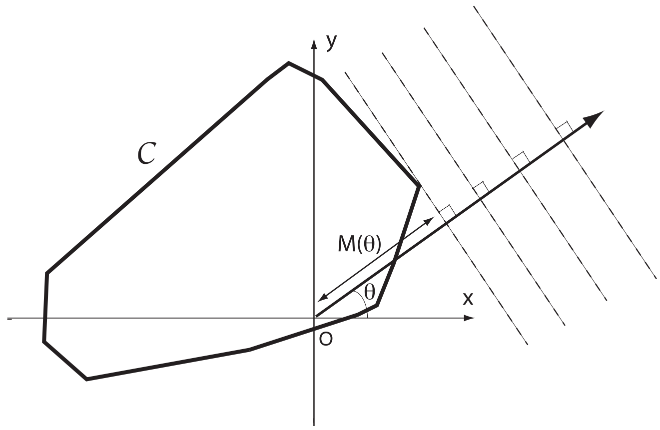

Fig. S1. Cauchy construction of the 2D convex hull, with support function $M(\theta)$ representing the distance along the direction θ.

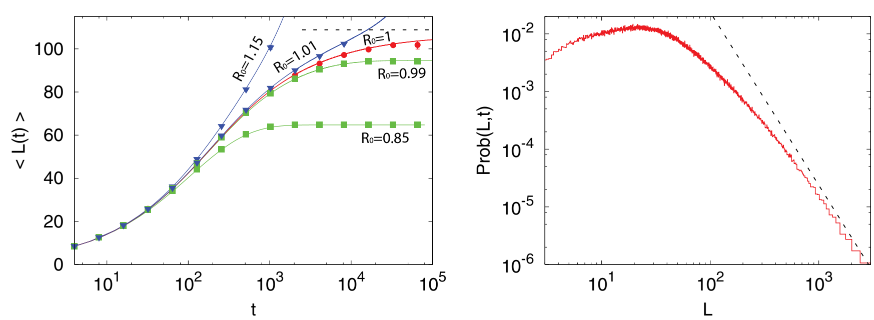

Fig. S2. (*Left*) The average perimeter $\langle L(t)\rangle$ of the convex hull as a function of the observation time. For the parameter values, we have chosen $D = 1/2$ and $b = R_0\gamma = 0.01$. We considered five different values of R_0. We have obtained these results by two different methods. (*i*) One method is by the numerical integration of Eq. **S9** and using Eq. **10** in the text (with the choices $dt = 0.003125$ and $dx = 0.1768$). These results are displayed as solid lines. (*ii*) Another method is by Monte Carlo simulations of the 2D branching Brownian motion with death with the same parameters and the choice of the Monte Carlo time step $dt = 0.25$ with the results averaged over 10^5 samples. Monte Carlo simulations are displayed as symbols. The dashed lines represent the asymptotic limits as given in Eq. **1** in the text for the critical case $R_0 = 1$. (*Right*) Distribution of the perimeter of the convex hull for the critical case $R_0 = 1$, with $\gamma = 0.01$ and $D = 1/2$, as obtained by Monte Carlo simulations with time step $dt = 1$ and $t = 4 \times 10^5$. The number of realizations is 2×10^6. The dashed line in *Right* corresponds to the power-law L^{-3} (up to an arbitrary prefactor).

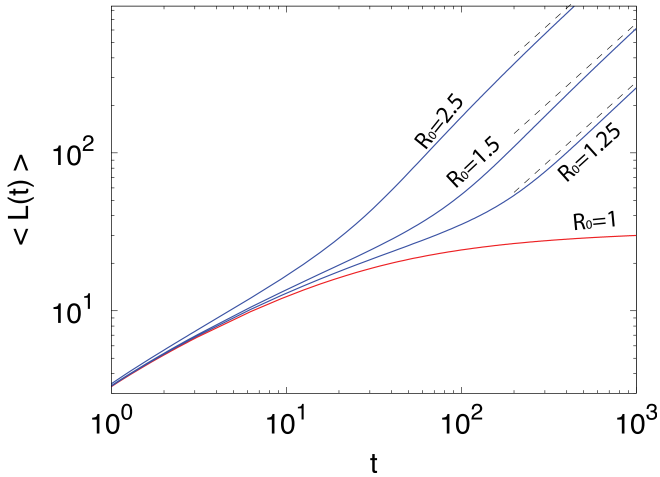

Fig. S3. The time behavior of the average perimeter in the supercritical regime for different values of $R_0 > 1$. Dashed lines represent the asymptotic scaling as in Eq. **3** in the text. The red curve corresponds to the critical regime.