Web-based supplementary materials for "Estimation of stratified mark-specific proportional hazards models with missing marks" by Yanqing Sun and Peter Gilbert.

**Web Appendix A**

In this and the rest of Appendices, the references and equation numbers not defined in the Appendices refer to the expressions in the main paper.

The following Lemma 1 and Lemma 2 are extensions of Theorem 5.7 and Theorem 5.9 of van der Vaart (1998), respectively, and will be used to prove the uniform consistency of $\hat\beta(v)$.

**Lemma 1.** *Let $Q_n(v,\theta)$ be random functions and let $Q(v,\theta)$ be a fixed function of $(v,\theta) \in [a,b]\times\Theta$, $\Theta \subset R^p$. Let $\beta(v)$ be a fixed function of $v \in [a,b]$ taking values in $\Theta$. Assume that $\sup_{v,\theta}|Q_n(v,\theta) - Q(v,\theta)| \xrightarrow{P} 0$ and that for every $\epsilon > 0$ there exists a $\eta > 0$ such that $\sup_{\|\theta-\beta(v)\|>\epsilon} Q(v,\theta) < Q(v,\beta(v)) - \eta$ for $v \in [a,b]$. Then for any sequence of estimators $\hat\beta(v)$, with $Q_n(v,\hat\beta(v)) > Q_n(v,\beta(v)) - o_p(1)$ uniformly in $v\in[a,b]$, we have $\hat\beta(v)\xrightarrow{P}\beta(v)$ uniformly in $v\in[a,b]$.*

PROOF. For every $\epsilon > 0$, there exists a $\eta > 0$ such that

$$\{\sup_v \|\hat\beta(v) - \beta(v)\| > \epsilon\} \subset \cup_v\{\|\hat\beta(v)-\beta(v)\| > \epsilon\} \subset \cup_v\{Q(v,\hat\beta(v)) < Q(v,\beta(v)) - \eta\}.$$

Since $Q_n(v,\hat\beta(v)) > Q_n(v,\beta(v)) - o_p(1) \xrightarrow{P} Q(v,\beta(v))$, uniformly in $v\in[a,b]$, we have $Q_n(v,\hat\beta(v)) > Q(v,\beta(v)) - o_p(1)$, uniformly in $v \in [a,b]$. It follows that

$$\begin{aligned}
&\cup_v\{Q(v,\hat\beta(v)) < Q(v,\beta(v)) - \eta\} \subset \cup_v\{Q(v,\hat\beta(v)) < Q_n(v,\hat\beta(v)) - \eta + o_p(1)\}\\
&= \{\inf_v (Q(v,\hat\beta(v)) - Q_n(v,\hat\beta(v))) < -\eta + o_p(1)\}\\
&= \{\sup_v (Q_n(v,\hat\beta(v)) - Q(v,\hat\beta(v))) > \eta - o_p(1)\}\\
&\subset \{\sup_v |Q_n(v,\hat\beta(v)) - Q(v,\hat\beta(v))| > \eta - o_p(1)\},
\end{aligned}$$

whose probability goes to 0 by the uniform convergence of $Q_n(v,\theta)$ to $Q(v,\theta)$. Hence $P\{\sup_v \|\hat\beta(v) - \beta(v)\| > \epsilon\} \to 0$. □

**Lemma 2.** *Let $\Psi_n(v,\theta)$ be random vector-valued functions and let $\Psi(v,\theta)$ be a fixed vector-valued function of $(v,\theta) \in [a,b]\times\Theta$, $\Theta\subset R^p$. Let $\beta(v)$ be a fixed function of $v\in[a,b]$ taking* *values in* $\Theta$. *Assume that* $\sup_{v,\theta}\|\Psi_n(v,\theta)-\Psi(v,\theta)\|\xrightarrow{P}0$ *and that for every* $\epsilon>0$ *there exists a* $\eta>0$ *such that* $\inf_{\|\theta-\beta(v)\|>\epsilon}\|\Psi(v,\theta)\|>\|\Psi(v,\beta(v))\|+\eta=\eta$ *for* $v\in[a,b]$. *Then for any sequence of estimators* $\hat\beta(v)$, *with* $\|\Psi_n(v,\hat\beta(v))\|=o_p(1)$ *uniformly in* $v\in[a,b]$, *we have* $\hat\beta(v)\xrightarrow{P}\beta(v)$ *uniformly in* $v\in[a,b]$.

PROOF. Lemma 2 follows from applying Lemma 1 to the function $Q_n(v,\theta)=-\|\Psi_n(v,\theta)\|$ and $Q(v,\theta)=-\|\Psi(v,\theta)\|$. □

**Lemma 3.** *Suppose that* $(W_i,X_i), i=1,\ldots,n$, *are independent random variables satisfying* $\sup_{1\le i\le n}E(X_i^4)\le M<\infty$, $\sup_{1\le i\le n}E(g_n^4(W_i))\to 0$, *and* $\sup_{i\neq j}E\{g_n(W_i)X_ig_n(W_j)X_j\}=o(n^{-1})$, *where* $g_n(\cdot)$ *is a random function. Then*

$$E\left\{\left(n^{-1/2}\sum_{i=1}^n g_n(W_i)X_i\right)^2\right\}\to 0.$$

PROOF OF LEMMA 3.

$$\begin{aligned}
&E\left\{\left(n^{-1/2}\sum_{i=1}^n g_n(W_i)X_i\right)^2\right\}\\
&=n^{-1}\sum_{i=1}^n\sum_{j=1}^n E\left\{g_n(W_i)X_ig_n(W_j)X_j\right\}\\
&=n^{-1}\sum_{i=1}^n E\left\{(g_n(W_i)X_i)^2\right\}+o(1)\\
&=n^{-1}\sum_{i=1}^n E\left\{E[(g_n(W_i))^2|W_i]E[(X_i)^2|W_i]\right\}+o(1)\\
&\le n^{-1}\sum_{i=1}^n\{E[(g_n(W_i))^4]\}^{1/2}\{E[(X_i)^4]\}^{1/2}+o(1)\\
&=o(1). \qquad \square
\end{aligned}$$

PROOF OF THEOREM 1. Let $\psi_{k0}$ be the true value of $\psi_k$ such that $r_k(W_{ki})=r_k(W_{ki},\psi_{k0})$ under the correctly specified model for $r(W_{ki})$. Then $\hat\psi_k\xrightarrow{P}\psi_{k0}$. Let $\psi_0=(\psi_{10},\ldots,\psi_{K0})$,

$$\eta_n(u,\theta,\hat\psi)=n^{-1}\sum_{k=1}^K\sum_{i=1}^{n_k}\int_0^u\int_0^\tau(Z_{ki}(t)-\tilde Z_k(t,\theta,\hat\psi_k))\frac{R_{ki}}{\pi_k(Q_{ki},\hat\psi_k)}N_{ki}(dt,du),$$

$$\xi_n(u,\theta,\psi_0) = n^{-1}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\int_0^u\int_0^\tau (Z_{ki}(t)-\bar{z}_k(t,\theta))\frac{R_{ki}}{\pi_k(Q_{ki},\psi_{k0})}\,N_{ki}(dt,du).$$

Under the conditions of Theorem 1, $\eta_n(v,\theta,\hat{\psi}) = \xi_n(v,\theta,\psi_0)+O_p(n^{-1/2})$, uniformly in $v \in [0,1]$ and

$$\begin{aligned} n^{-1}U_{ipw}(v,\theta,\hat{\psi}) &= \int_0^1 K_h(u-v)\eta_n(du,\theta,\hat{\psi}) \\ &= \int_0^1 K_h(u-v)\xi_n(du,\theta,\psi_0)+O_p(n^{-1/2}h^{-1}), \end{aligned} \qquad \text{(W.1)}$$

uniformly in $(v,\theta)\in[0,1]\times[-M,M]$, for $M>0$.

By application of the Glivenko–Cantelli and Donsker theorems, similar to the proofs of Lemma 2 and Theorem 1 of Sun *et al.* (2009), $\xi_n(v,\theta,\psi_0)=\xi(v,\theta,\psi_0)+O_p(n^{-1/2})$, uniformly in $(v,\theta)\in[0,1]\times[-M,M]$, where

$$\begin{aligned} \xi(v,\theta,\psi_0) &= \sum_{k=1}^{K} p_k E\Big[\int_0^v\int_0^\tau (Z_{ki}(t)-\bar{z}_k(t,\theta))\frac{R_{ki}}{\pi_k(Q_{ki},\psi_{k0})}\,N_{ki}(dt,du)\Big] \\ &= \sum_{k=1}^{K} p_k\Big\{E\Big[\int_0^v\int_0^\tau (Z_{ki}(t)-\bar{z}_k(t,\theta))N_{ki}(dt,du)\Big] \\ &\quad -E\Big[\int_0^v\int_0^\tau (Z_{ki}(t)-\bar{z}_k(t,\theta))\Big(1-\frac{R_{ki}}{\pi_k(Q_{ki},\psi_{k0})}\Big)\,N_{ki}(dt,du)\Big]\Big\}. \end{aligned} \qquad \text{(W.2)}$$

The first expectation is zero under model (2) by Sun *et al.* (2009). Using the double expectation formula $E[\cdot]=E\{E[\cdot|V_{ki},W_{ki},\delta_{ki}]\}$ and the missing at random assumption (3), the second expectation is zero. Hence

$$n^{-1}U_{ipw}(v,\theta,\hat{\psi}) = \int_0^1 K_h(u-v)\xi(du,\theta,\psi_0)+O_p(n^{-1/2}h^{-1}) = o_p(1),$$

uniformly in $(v,\theta)\in[a,b]\times[-M,M]$ for $M>0$. By Lemma 2, $\hat{\beta}_{ipw}(v)\xrightarrow{P}\beta(v)$ uniformly in $v\in[a,b]$ as $nh^2\to\infty$. □

PROOF OF THEOREM 2.

By a Taylor expansion, $U_{ipw}(v,\hat{\beta}(v),\hat{\psi})-U_{ipw}(v,\beta(v),\hat{\psi}) = U'_{ipw}(v,\beta^*(v),\hat{\psi})\ (\hat{\beta}(v)-\beta(v))$, where $\beta^*(v)$ is on the line segment between $\hat{\beta}(v)$ and $\beta(v)$. Hence,

$$\begin{aligned} n^{1/2}h^{1/2}(\hat{\beta}^{ipw}(v)-\beta(v)) &= -(U'_{ipw}(v,\beta^*(v),\hat{\psi})/n)^{-1}n^{-1/2}h^{1/2}U_{ipw}(v,\beta(v),\hat{\psi}) \\ &= (\Sigma(v))^{-1}n^{-1/2}h^{1/2}U_{ipw}(v,\beta(v),\hat{\psi})+o_p(1), \end{aligned} \qquad \text{(W.3)}$$

where the last equality is obtained by the uniform consistency of $\hat{\beta}(v)$ on $v \in [a,b] \subset (0,1)$, the uniform consistency $n^{-1}U'_{ipw}(v, \beta^*(v), \hat{\psi}) \xrightarrow{P} -\Sigma(v)$ in $v \in [a,b]$, and the convergence in distribution of $n^{-1/2}h^{1/2}U_{ipw}(v, \beta, \hat{\psi})$ to $N(0, \nu_0 \Sigma^*(v))$ which we show next.

In the rest of the proof, we set $\beta = \beta(v)$ for simplicity. Under Condition A, using a second order Taylor expansion for $\lambda_k(t, u|Z_i(t))$ in the neighborhood of $v$, we have

$$
\begin{aligned}
& n^{-1/2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \int_0^1 \int_0^\tau K_h(u-v)[Z_{ki}(t) - \tilde{Z}_k(t, \beta, \hat{\psi}_k)] \frac{R_{ki}}{\pi_k(Q_{ki}, \hat{\psi}_k)} \\
& \quad Y_{ki}(t)[\lambda_k(t, u|Z_{ki}(t)) - \lambda_k(t, v|Z_{ki}(t))]\, dtdu \\
& = \frac{1}{2}\mu_2 n^{1/2} h^2 \vartheta(v) + o_p(n^{1/2}h^2),
\end{aligned}
$$

uniformly in $v \in [0,1]$. It follows that

$$
\begin{aligned}
& n^{-1/2} U_{ipw}(v, \beta, \hat{\psi}) \\
& = n^{-1/2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \int_0^1 \int_0^\tau K_h(u-v)[Z_{ki}(t) - \tilde{Z}_k(t, \beta, \hat{\psi}_{k0})] \frac{R_{ki}}{\pi_k(Q_{ki}, \hat{\psi}_k)} \\
& \quad [N_{ki}(dt, du) - Y_{ki}(t)\lambda_k(t, v|Z_{ki}(t))\, dtdu] \\
& = n^{-1/2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \int_0^1 \int_0^\tau K_h(u-v)[Z_{ki}(t) - \tilde{Z}_k(t, \beta, \hat{\psi}_k)] \frac{R_{ki}}{\pi_k(Q_{ki}, \hat{\psi}_{k0})} \\
& \quad [N_{ki}(dt, du) - Y_{ki}(t)\lambda_k(t, u|Z_{ki}(t))\, dtdu] + \frac{1}{2}\mu_2 n^{1/2} h^2 \vartheta(v) + o_p(n^{1/2}h^2),
\end{aligned}
\tag{W.4}
$$

uniformly in $v \in [0,1]$.

Let

$$
n^{-1/2} H_k(t, v) = {n_k}^{-1/2} \sum_{i=1}^{n_k} \frac{R_{ki}}{\pi_k(Q_{ki}, \hat{\psi}_k)} M_{ki}(t, v). \tag{W.5}
$$

By the first order Taylor expansion in $\hat{\psi}_k$,

$$
\begin{aligned}
n^{-1/2} H_k(t, v) &= {n_k}^{-1/2} \sum_{i=1}^{n_k} \frac{R_{ki}}{\pi_k(Q_{ki}, \psi_{k0})} M_{ki}(t, v) \\
& \quad - \frac{R_{ki}}{(\pi_k(Q_{ki}, \psi_{k0}))^2} \left( \frac{\partial \pi_k(Q_{ki}, \psi_{k0})}{\partial \psi_k} \right)^T M_{ki}(t, v)(\hat{\psi}_k - \psi_{k0}) + o_p(1) \\
&= {n_k}^{-1/2} \sum_{i=1}^{n_k} \frac{R_{ki}}{\pi_k(Q_{ki}, \psi_{k0})} M_{ki}(t, v) \\
& \quad - {n_k}^{-1} \sum_{i=1}^{n_k} \frac{R_{ki}}{(\pi_k(Q_{ki}, \psi_{k0}))^2} \left( \frac{\partial \pi_k(Q_{ki}, \psi_{k0})}{\partial \psi_k} \right)^T M_{ki}(t, v) \\
& \quad \times {n_k}^{-1/2} \sum_{i=1}^{n_k} (I_k^{\psi})^{-1} S_{ki}^{\psi} + o_p(1),
\end{aligned}
$$

where $S_{ki}^{\psi}$ and $I_k^{\psi}$ are defined in (17) and (18). Following the proof of Lemma 2 in Sun *et al.* (2009), it is easy to show that ${n_k}^{-1/2}H_k(t,v)$ converges weakly to a mean-zero Gaussian process. By Condition A, $\|\tilde{S}_k^{(j)}(t,\beta,\hat{\psi}_k) - s_k^{(j)}(t,\beta)\| = o_p(n^{-1/2+\delta})$, uniformly in $t$ for $j = 0, 1$, for $0 < \delta < 1/2$. Note that $h = n^{-\alpha}$ with $\alpha < 1/2$. We can choose $\delta > 0$ such that $\alpha+\delta < 1/2$. Thus $n^{-1/2+\delta}h^{-1/2} = o(h^{1/2})$. We have $h^{1/2}K_h(u-v)\|\tilde{S}_k^{(j)}(t,\beta,\hat{\psi}_k) - s_k^{(j)}(t,\beta)\|$ goes in probability to zero. Applying Lemma 2 of Gilbert *et al.* (2008),

$$\begin{aligned}
&n^{-1/2}h^{1/2}U_{ipw}(v,\beta(v),\hat{\psi}) \\
&= n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\int_0^1\int_0^\tau K_h(u-v)[Z_{ki}(t)-\bar{z}_k(t,\beta)]\frac{R_{ki}}{\pi_k(Q_{ki},\hat{\psi}_k)} \\
&\quad [N_{ki}(dt,du) - Y_{ki}(t)\lambda_k(t,u|Z_{ki}(t))\,dtdu] \\
&\quad +\frac{1}{2}\mu_2 n^{1/2}h^{5/2}\vartheta(v) + o_p(n^{1/2}h^{5/2}) + o_p(h^{1/2}) \qquad \text{(W.6)} \\
&= n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\int_0^1\int_0^\tau K_h(u-v)[Z_{ki}(t)-\bar{z}_k(t,\beta(u))]\frac{R_{ki}}{\pi_k(Q_{ki},\hat{\psi}_k)} \\
&\quad [N_{ki}(dt,du) - Y_{ki}(t)\lambda_k(t,u|Z_{ki}(t))\,dtdu] \\
&\quad +\frac{1}{2}\mu_2 n^{1/2}h^{5/2}\vartheta(v) + o_p(n^{1/2}h^{5/2}) + o_p(h^{1/2}) \\
&= n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\mathcal{A}_{ki} + h^{1/2}\sum_{k=1}^{K}(n_k/n)^{1/2}\mathcal{D}_k\sqrt{n_k}(\hat{\psi}_k-\psi_{k0}) \\
&+\frac{1}{2}\mu_2 n^{1/2}h^{5/2}\vartheta(v) + o_p(n^{1/2}h^{5/2}) + o_p(h^{1/2}),
\end{aligned}$$

where $\mathcal{A}_{ki}$ and $\mathcal{D}_k$ is defined in (19). Note that $\hat{\psi}_k - \psi_{k0} = n_k^{-1}\sum_{i=1}^{n_k}(I_k^{\psi})^{-1}S_{ki}^{\psi} + o_p({n_k}^{-1/2})$. We have

$$\begin{aligned}
&n^{-1/2}h^{1/2}U_{ipw}(v,\beta(v),\hat{\psi}) \\
&= n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\mathcal{A}_{ki} + n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\mathcal{D}_k(I_k^{\psi})^{-1}S_{ki}^{\psi} \qquad \text{(W.7)} \\
&+\frac{1}{2}\mu_2 n^{1/2}h^{5/2}\vartheta(v) + o_p(n^{1/2}h^{5/2}) + o_p(h^{1/2}).
\end{aligned}$$

It follows that

$$\begin{aligned}
&n^{-1/2}h^{1/2}U_{ipw}(v,\beta(v),\hat{\psi}) \\
&= n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\int_0^1\int_0^\tau K_h(u-v)[Z_{ki}(t)-\bar{z}_k(t,\beta(u))] \\
&\quad \Big[\frac{R_{ki}}{\pi_k(t,Z_{ki},\delta_{ki}A_{ki},\psi_{k0})}M_{ki}^*(dt,du) + \frac{R_{ki}}{\pi_k(t,Z_{ki},\delta_{ki}A_{ki},\psi_{k0})}\lambda_{ki}^*(t,u)Y_{ki}(t)\,dtdu
\end{aligned}$$

$$
\begin{aligned}
&-\frac{R_{ki}}{\pi_k(Q_{ki},\psi_{k0})}\lambda_k(t,u|Z_{ki}(t))Y_{ki}(t)\,dtdu\Big] \\
&+n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\mathcal{D}_k(I_k^{\psi})^{-1}S_{ki}^{\psi}+\frac{1}{2}\mu_2 n^{1/2}h^{5/2}\vartheta(v)+o_p(n^{1/2}h^{5/2})+o_p(h^{1/2}) \\
=\ & h^{1/2}\int_0^1 K_h(u-v)\tilde{W}_n(du)+h^{1/2}\int_0^1 K_h(u-v)\delta_n(du) \\
&+n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\mathcal{D}_k(I_k^{\psi})^{-1}S_{ki}^{\psi} \qquad \text{(W.8)} \\
&+\frac{1}{2}\mu_2 n^{1/2}h^{5/2}\vartheta(v)+o_p(n^{1/2}h^{5/2})+o_p(h^{1/2})+o_p(1)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{W}_n(v) &= n^{-1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\int_0^v\int_0^\tau [Z_{ki}(t)-\bar{z}_k(t,\beta(u))]\frac{R_{ki}}{\pi_k(t,Z_{ki},\delta_{ki}A_{ki},\psi_{k0})}M_{ki}^*(dt,du) \\
\delta_n(v) &= n^{-1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\int_0^v\int_0^\tau [Z_{ki}(t)-\bar{z}_k(t,\beta(u))] \\
&\quad \left\{\frac{R_{ki}}{\pi_k(t,Z_{ki},\delta_{ki}A_{ki},\psi_{k0})}\lambda_{ki}^*(t,u)-\frac{R_{ki}}{\pi_k(Q_{ki},\psi_{k0})}\lambda_k(t,u|Z_{ki}(t))\right\}Y_{ki}(t)\,dtdu.
\end{aligned}
$$

It is easy to see that the third term in (W.8) is $O_p(h^{1/2})$. Now we check that the expectation of each term in the summand of $\delta_n(v)$ is zero. Applying double expectation $E(\cdot) = E[E(\cdot|Q_{ki}, Y_{ki}(t))]$, we have

$$
E\left\{[Z_{ki}(t)-\bar{z}_k(t,\beta(u))]\frac{R_{ki}}{\pi_k(Q_{ki},\psi_{k0})}\lambda_k(t,u|Z_{ki}(t))Y_{ki}(t)\right\}=0.
$$

By (16), we have

$$
\begin{aligned}
&E\left\{[Z_{ki}(t)-\bar{z}_k(t,\beta(u))]\frac{R_{ki}}{\pi_k(t,Z_{ki},\delta_{ki}A_{ki},\psi_{k0})}\lambda_{ki}^*(t,u)Y_{ki}(t)\right\}dtdu \\
&=E\left\{[Z_{ki}(t)-\bar{z}_k(t,\beta(u))]\frac{R_{ki}}{\pi_k(t,Z_{ki},\delta_{ki}A_{ki},\psi_{k0})}N_{ki}(dt,du)\right\} \\
&=E\left\{[Z_{ki}(t)-\bar{z}_k(t,\beta(u))]\frac{R_{ki}}{\pi_k(W_{ki},\psi_{k0})}N_{ki}(dt,du)\right\} \\
&=E\left\{[Z_{ki}(t)-\bar{z}_k(t,\beta(u))]E\left[\frac{R_{ki}}{\pi_k(W_{ki},\psi_{k0})}\middle|W_{ki},\delta_{ki}\right]N_{ki}(dt,du)\right\} \\
&=E\left\{[Z_{ki}(t)-\bar{z}_k(t,\beta(u))]N_{ki}(dt,du)\right\}=0. \qquad \text{(W.9)}
\end{aligned}
$$

By Lemma 1 of Sun & Wu (2005), the process $\delta_n(v)$, $0 \le v \le 1$, converges weakly to a mean-zero Gaussian process and consequently, the second term in (W.8) converges to zero in probability.

Following the proof of Theorem 1 of Sun & Wu (2005), $\tilde{W}_n(v) \xrightarrow{\mathcal{D}} W(v)$, where $W(v)$ is a $p$-dimensional mean-zero Gaussian martingale with continuous sample path on $[0,1]$. The covariance matrix of $W(v)$ is $\int_0^v \Sigma^*(u)\,du$.

By the almost sure representation theorem (Shorack & Wellner, 1986, p.47), there exist $\tilde{W}_n^*(v)$ and $W^*(v)$ on some probability space with the same distributions and sample paths as $\tilde{W}_n(v)$ and $W(v)$, respectively, such that $\tilde{W}_n^*(v) \xrightarrow{\text{a.s.}} W^*(v)$ uniformly in $v \in [0,1]$. Hence $\int_0^1 K_h(u-v)\,\tilde{W}_n^*(du) = \int_0^1 K_h(u-v)\,W^*(du) + O_p(n^{-1/2}h^{-1})$ by integration by parts since $K(\cdot)$ has bounded variation. It follows that

$$h^{1/2}\int_0^1 K_h(u-v)\,\tilde{W}_n(du) \overset{\mathcal{D}}{=} h^{1/2}\int_0^1 K_h(u-v)\,\tilde{W}_n^*(du)$$
$$= h^{1/2}\int_0^1 K_h(u-v)\,W^*(du) + O_p(n^{-1/2}h^{-1/2}).$$

Since $W^*(v)$ is a Gaussian martingale with covariance matrix of $\int_0^v \Sigma^*(u)\,du$, $h^{1/2}\int_0^1 K_h(u-v)\,W^*(du)$ is a mean zero Gaussian random vector with covariance matrix equal to $h\int_0^1 K_h^2(u-v)\Sigma^*(u)\,du \to \nu_0\Sigma^*(v)$ as $h \to 0$. Hence, $h^{1/2}\int_0^1 K_h(u-v)\,\tilde{W}_n(du) \xrightarrow{\mathcal{D}} N(0, \nu_0\Sigma^*(v))$ as $h \to 0$, $nh \to \infty$. By the Slutsky theorem, $n^{-1/2}h^{1/2}U_{ipw}(v,\beta,\psi_0) - \frac{1}{2}\mu_2 n^{1/2}h^{5/2}\vartheta(v)$ converges weakly to $N(0,\nu_0\Sigma^*(v))$ as $nh^2 \to \infty$ and $nh^5 = O(1)$. It follows from (W.3) that $(nh)^{1/2}[\hat{\beta}^{ipw}(v) - \beta(v) + \frac{1}{2}\mu_2 h^2\vartheta(v)] \xrightarrow{\mathcal{D}} N(0, \nu_0\Sigma(v)^{-1}\Sigma^*(v)\Sigma(v)^{-1})$ as $nh^2 \to \infty$ and $nh^5 = O(1)$. $\square$

PROOF OF THEOREM 3.

Suppose $\hat{g}_k(a|t,u,z) \xrightarrow{P} g_k^*(t,u,z)$ under the parametric model $g_k(a|t,u,z,\gamma_k)$. Let

$$\rho_k^*(v,w) = \int_0^v \lambda_k(t,u|z)g_k^*(a|t,u,z)\,du \Big/ \int_0^1 \lambda_k(t,u|z)g_k^*(a|t,u,z)\,du,$$

$$\begin{aligned}&\eta_n(u,\theta,\hat{\psi},\hat{\rho}^{ipw}(\cdot))\\ &= n^{-1}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\int_0^u\int_0^\tau (Z_{ki}(t) - \bar{Z}_k(t,\theta))\\ &\quad\left\{\frac{R_{ki}}{\pi_k(Q_{ki},\hat{\psi}_k)}N_{ki}(dt,dx) + \left(1 - \frac{R_{ki}}{\pi_k(Q_{ki},\hat{\psi}_k)}\right)N_{ki}^x(dt)d(\hat{\rho}_k^{ipw}(x,W_{ki}))\right\},\end{aligned} \tag{W.10}$$

$$\begin{aligned}&\xi_n(u,\theta,\psi_0,\rho^*(\cdot))\\ &= n^{-1}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\int_0^u\int_0^\tau \Big(Z_{ki}(t) - \bar{z}_k(t,\theta)\Big)\\ &\quad\left\{\frac{R_{ki}}{\pi_k(Q_{ki},\psi_{k0})}N_{ki}(dt,dx) + \left(1 - \frac{R_{ki}}{\pi_k(Q_{ki},\psi_{k0})}\right)N_{ki}^x(dt)d(\rho_k^*(x,W_{ki}))\right\}.\end{aligned} \tag{W.11}$$

By the conditions of Theorem 3, $\eta_n(v,\theta,\hat{\psi},\hat{\rho}^{ipw}(\cdot)) = \xi_n(v,\theta,\psi_0,\rho^*(\cdot)) + O_p(n^{-1/2})$, uniformly in $v \in [0,1]$ and

$$\begin{aligned} n^{-1}U_{aug}(v,\theta,\hat{\psi},\hat{\rho}^{ipw}(\cdot)) &= \int_0^1 K_h(u-v)\eta_n(du,\theta,\hat{\psi},\hat{\rho}^{ipw}(\cdot)) \\ &= \int_0^1 K_h(u-v)\xi_n(du,\theta,\psi_0,\rho^*(\cdot)) + O_p(n^{-1/2}h^{-1}), \end{aligned}$$

uniformly in $(v,\theta) \in [0,1] \times [-M,M]$, for $M > 0$.

By application of the Glivenko–Cantelli and Donsker theorems, similar to the proofs of Lemma 2 and Theorem 1 of Sun *et al.* (2009), $\xi_n(v,\theta,\psi_0,\rho^*(\cdot)) = \xi(v,\theta,\psi_0,\rho^*(\cdot)) + O_p(n^{-1/2})$, uniformly in $(v,\theta) \in [0,1] \times [-M,M]$, with

$$\begin{aligned} &\xi(v,\theta,\psi_0,\rho^*(\cdot)) \\ &= \sum_{k=1}^K p_k E\bigg[\int_0^v \int_0^\tau (Z_{ki}(t) - \bar{z}_k(t,\theta)) \\ &\quad \bigg\{\frac{R_{ki}}{\pi_k(Q_{ki},\psi_{k0})} N_{ki}(dt,du) + \bigg(1 - \frac{R_{ki}}{\pi_k(Q_{ki},\psi_{k0})}\bigg) N_{ki}^x(dt) d(\rho_k^*(u,W_{ki}))\bigg\}\bigg] \\ &= \sum_{k=1}^K p_k \bigg[E\bigg\{\int_0^v \int_0^\tau (Z_{ki}(t) - \bar{z}_k(t,\theta)) N_{ki}(dt,du)\bigg\} \\ &\quad -E\bigg\{\int_0^v \int_0^\tau (Z_{ki}(t) - \bar{z}_k(t,\theta))\bigg(1 - \frac{R_{ki}}{\pi_k(Q_{ki},\psi_{k0})}\bigg) \\ &\quad N_{ki}^x(dt)[N_{ki}^v(du) - d(\rho_k^*(u,W_{ki}))]\bigg\}\bigg]. \end{aligned} \tag{W.12}$$

The first expectation is zero under model (2) by Sun *et al.* (2009). Using the double expectation formula $E[\cdot] = E\{E[\cdot|V_{ki},W_{ki},\delta_{ki}]\}$ and the missing at random assumption (3), the second expectation is zero if $r_k(W_{ki},\psi_k)$ or $g_k(a|t,v,z,\gamma_k)$ is correctly specified which leads to $\rho_k^*(v,w) = \rho_k(v,w)$. It follows that

$$n^{-1}U_{aug}(v,\theta,\hat{\psi},\hat{\rho}^{ipw}(\cdot)) = \int_0^1 K_h(u-v)\xi(du,\theta,\psi_0,\rho(\cdot)) + O_p(n^{-1/2}h^{-1}) = o_p(1),$$

uniformly in $(v,\theta) \in [a,b] \times [-M,M]$ for $M > 0$. By Lemma 2, $\hat{\beta}(v) \xrightarrow{P} \beta(v)$ uniformly in $v \in [a,b]$ as $nh^2 \to \infty$. □

PROOF OF THEOREM 4.

By a Taylor expansion, $U_{aug}(v,\hat{\beta}^{aug}(v),\hat{\psi},\hat{\rho}^{ipw}(\cdot)) - U_{aug}(v,\beta(v),\hat{\psi},\hat{\rho}^{ipw}(\cdot)) = U'_{aug}(v,\beta^*(v),\hat{\psi},\hat{\rho}^{ipw}(\cdot))\,(\hat{\beta}^{aug}(v) - \beta(v))$, where $\beta^*(v)$ is on the line segment between $\hat{\beta}(v)$ and $\beta(v)$. Hence,

$$n^{1/2}h^{1/2}(\hat{\beta}^{aug}(v) - \beta(v))$$

$$= -(U'_{aug}(v, \beta^*(v), \hat{\psi}, \hat{\rho}^{ipw}(\cdot))/n)^{-1} n^{-1/2} h^{1/2} U_{aug}(v, \beta(v), \hat{\psi}, \hat{\rho}^{ipw}(\cdot)). \quad \text{(W.13)}$$

By Condition A and the uniform consistency of $\hat{\beta}^{aug}(v)$ on $v \in [a, b] \subset (0, 1)$, $n^{-1} U'_{aug}(v, \beta^*(v), \hat{\psi}, \hat{\rho}^{ipw}(\cdot)) = -\Sigma(v) + o_p(1)$, uniformly in $v \in [a, b]$.

Next we study the asymptotic property of $n^{-1/2} h^{1/2} U_{aug}(v, \beta, \hat{\psi}, \hat{\rho}^{ipw}(\cdot))$. Let $g_n(W_{ki}, v) = \int_0^1 K_h(u - v) d(\hat{\rho}_k(u, W_{ki}) - \rho_k(W_{ki}, u))$ and

$$C = n^{-1/2} h^{1/2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} g_n(W_{ki}, v) \int_0^\tau (Z_{ki}(t) - \bar{Z}_k(t, \beta)) \left(1 - \frac{R_{ki}}{\pi_k(Q_{ki}, \hat{\psi}_k)}\right) N^x_{ki}(dt).$$

It follows that $g_n(W_{ki}, v) \xrightarrow{P} 0$ uniformly in $W_{ki}$ for $v \in [a, b]$. By the Taylor expansion of $\hat{\psi}_k$ around $\psi_{k0}$,

$$\begin{aligned} C &= n^{-1/2} h^{1/2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} g_n(W_{ki}, v) \zeta_{ki} \\ &\quad + n^{-1/2} h^{1/2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} g_n(W_{ki}, v) \int_0^\tau (Z_{ki}(t) - \bar{Z}_k(t, \beta)) N^x_{ki}(dt) \\ &\quad \frac{R_{ki}}{(\pi_k(Q_{ki}, \psi_{k0}))^2} \left(\frac{\partial \pi_k(Q_{ki}, \psi_{k0})}{\partial \psi_k}\right)^T (\hat{\psi}_k - \psi_{k0}) + o_p(1), \end{aligned}$$

where

$$\zeta_{ki} = \int_0^\tau (Z_{ki}(t) - \bar{Z}_k(t, \beta)) \left(1 - \frac{R_{ki}}{\pi_k(Q_{ki}, \psi_{k0})}\right) N^x_{ki}(dt).$$

The first term of $C$ is $o_p(h^{1/2})$ by Lemma 3. The second term of $C$ is at the order of $o_p(h^{1/2})$ since $n_k^{1/2}(\hat{\psi}_k - \psi_{k0}) = O_p(1)$, and $g_n(W_{ki}, v) \xrightarrow{P} 0$ uniformly in $W_{ki}$. Hence $C = o_p(h^{1/2})$.

Note that

$$\begin{aligned} & n^{-1/2} h^{1/2} U_{aug}(v, \beta, \hat{\psi}, \hat{\rho}^{ipw}(\cdot)) \\ &= n^{-1/2} h^{1/2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \int_0^1 \int_0^\tau K_h(u - v)(Z_{ki}(t) - \bar{Z}_k(t, \beta)) \Big\{ \frac{R_{ki}}{\pi_k(Q_{ki}, \hat{\psi}_k)} N_{ki}(dt, du) \\ &\quad + \left(1 - \frac{R_{ki}}{\pi_k(Q_{ki}, \hat{\psi}_k)}\right) N^x_{ki}(dt) d(\hat{\rho}_k(u, W_{ki})) \Big\} \\ &= n^{-1/2} h^{1/2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \int_0^1 \int_0^\tau K_h(u - v)(Z_{ki}(t) - \bar{Z}_k(t, \beta)) \Big\{ \frac{R_{ki}}{\pi_k(Q_{ki}, \hat{\psi}_k)} N_{ki}(dt, du) \\ &\quad + \left(1 - \frac{R_{ki}}{\pi_k(Q_{ki}, \hat{\psi}_k)}\right) N^x_{ki}(dt) d(\rho_k(u, W_{ki})) \Big\} + C \\ &= n^{-1/2} h^{1/2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \int_0^1 \int_0^\tau K_h(u - v)(Z_{ki}(t) - \bar{Z}_k(t, \beta)) \Big\{ \frac{R_{ki}}{\pi_k(Q_{ki}, \hat{\psi}_k)} M_{ki}(dt, du) \end{aligned}$$

$$+\left(1-\frac{R_{ki}}{\pi_k(Q_{ki},\hat{\psi}_k)}\right)E\{M_{ki}(dt,du)|Q_{ki}\}\Bigg\} \tag{W.14}$$

$$+n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\int_0^1\int_0^\tau K_h(u-v)(Z_{ki}(t)-\bar{Z}_k(t,\beta))Y_{ki}(t)\lambda_{ki}(t,u)\,dtdu+C.$$

Since $h^{1/2}K_h(u-v)[\bar{Z}_k(t,\beta)-\bar{z}_k(t,\beta(u))]\xrightarrow{P}0$ uniformly in $u\in[a,b]$, applying Lemma 2 of Gilbert *et al.* (2008), we can replace $\bar{Z}_k(t,\beta)$ with $\bar{z}_k(t,\beta(u))$ in the first two integrations. Hence

$$\begin{aligned}
&n^{-1/2}h^{1/2}U_{aug}(v,\beta,\hat{\psi},\hat{\rho}^{ipw}(\cdot))\\
&=n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\int_0^1\int_0^\tau K_h(u-v)(Z_{ki}(t)-\bar{z}_k(t,\beta(u)))\Bigg\{\frac{R_{ki}}{\pi_k(Q_{ki},\hat{\psi}_k)}M_{ki}(dt,du)\\
&\quad+\left(1-\frac{R_{ki}}{\pi_k(Q_{ki},\hat{\psi}_k)}\right)E\{M_{ki}(dt,du)|Q_{ki}\}\Bigg\}\\
&\quad+n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\int_0^1\int_0^\tau K_h(u-v)(Z_{ki}(t)-\bar{Z}_k(t,\beta))Y_{ki}(t)\lambda_{ki}(t,u)\,dtdu+o_p(h^{1/2})\\
&\equiv I+II+III+o_p(h^{1/2}).
\end{aligned} \tag{W.15}$$

By $\lambda_{ki}(t,u)=\lambda_{ki}(t,v)+(\lambda_{ki}(t,u)-\lambda_{ki}(t,v))$ and applying Taylor expansion, the third term is

$$\begin{aligned}
III &= n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\int_0^1\int_0^\tau K_h(u-v)(Z_{ki}(t)-\bar{Z}_k(t,\beta))Y_{ki}(t)\lambda_{ki}(t,u)\,dtdu\\
&=\frac{1}{2}\mu_2 n^{1/2}h^{5/2}\vartheta(v)+o_p(n^{1/2}h^{5/2})+o_p(h^{1/2}).
\end{aligned}$$

Applying Taylor expansion at $\psi_{k0}$,

$$\begin{aligned}
I+II &= n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}(\mathcal{A}_{ki}+\mathcal{B}_{ki}) \qquad\qquad \text{(W.16)}\\
&\quad+n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\int_0^1\int_0^\tau K_h(u-v)(Z_{ki}(t)-\bar{z}_k(t,\beta(u)))\\
&\quad\Bigg\{\frac{-R_{ki}}{(\pi_k(Q_{ki},\psi_{k0}))^2}\frac{\partial\pi_k(Q_{ki},\psi_{k0})}{\partial\psi_k}M_{ki}(dt,du)\\
&\quad-\frac{-R_{ki}}{(\pi_k(Q_{ki},\psi_{k0}))^2}\frac{\partial\pi_k(Q_{ki},\psi_{k0})}{\partial\psi_k}E\{M_{ki}(dt,du)|Q_{ki}\}\Bigg\}(\hat{\psi}_k-\psi_{k0})+o_p(h^{1/2}).
\end{aligned}$$

By the missing at random assumption (3), $R_{ki}$ and $M_{ki}(t,v)$ are independent conditioning on $Q_{ki}$. It follows that

$$E\Bigg\{\frac{-R_{ki}}{(\pi_k(Q_{ki},\psi_{k0}))^2}\frac{\partial\pi_k(Q_{ki},\psi_{k0})}{\partial\psi_k}M_{ki}(dt,du)$$

$$+\frac{R_{ki}}{(\pi_k(Q_{ki},\psi_{k0}))^2}\frac{\partial\pi_k(Q_{ki},\psi_{k0})}{\partial\psi_k}\,E\{M_{ki}(dt,du)|Q_{ki}\}\Bigg\} = 0.$$

By the central limit theorem, the last term in (W.16) is $o_p(h^{1/2})$. Hence

$$I + II = n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}(\mathcal{A}_{ki} + \mathcal{B}_{ki}) + o_p(h^{1/2}).$$

By (W.15) and (W.16), we have

$$\begin{aligned} n^{-1/2}h^{1/2}U_{aug}(v,\beta,\hat\psi) &= n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}(\mathcal{A}_{ki} + \mathcal{B}_{ki}) \\ &\quad + \frac{1}{2}\mu_2 n^{1/2}h^{5/2}\vartheta(v) + o_p(n^{1/2}h^{5/2}) + o_p(h^{1/2}). \end{aligned} \tag{W.17}$$

Since $n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\mathcal{B}_{ki} = O_p(h^{1/2})$, it follows from (W.7) and (W.8) that

$$\begin{aligned} & n^{-1/2}h^{1/2}U_{aug}(v,\beta,\hat\psi,\hat\rho^{ipw}(\cdot)) \\ &= h^{1/2}\int_0^1 K_h(u-v)\tilde{W}_n(du) + h^{1/2}\int_0^1 K_h(u-v)\delta_n(du) \\ &+ \frac{1}{2}\mu_2 n^{1/2}h^{5/2}\vartheta(v) + O_p(h^{1/2}) + o_p(n^{1/2}h^{5/2}). \end{aligned} \tag{W.18}$$

By the Slutsky theorem, $n^{-1/2}h^{1/2}U_{aug}(v,\beta,\psi_0) - \frac{1}{2}\mu_2 n^{1/2}h^{5/2}\vartheta(v)$ converges weakly to $N(0,\nu_0\Sigma^*(v))$ as $nh^2\to\infty$ and $nh^5 = O(1)$. It follows from (W.18) that $(nh)^{1/2}[\hat\beta^{aug}(v) - \beta(v) + \frac{1}{2}\mu_2 h^2\Sigma^{-1}(v)\vartheta(v)]\xrightarrow{\mathcal{D}}N(0,\nu_0\Sigma(v)^{-1}\Sigma^*(v)\Sigma(v)^{-1})$ as $nh^2\to\infty$ and $nh^5 = O(1)$.

We now show that $\hat\beta^{aug}(v)$ is more efficient than $\hat\beta^{ipw}(v)$. By (W.17), we have,

$$\begin{aligned} & (nh)^{1/2}[\hat\beta^{aug}(v) - \beta(v) + \frac{1}{2}\mu_2 h^2\Sigma^{-1}(v)\vartheta(v)] \\ &= -(\Sigma(v))^{-1}n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}(\mathcal{A}_{ki} + \mathcal{B}_{ki}) + o_p(n^{1/2}h^{5/2}) + o_p(h^{1/2}). \end{aligned} \tag{W.19}$$

Let $\mathcal{O}_{ki} = \mathcal{D}_k(I_k^{\psi})^{-1}S_{ki}^{\psi}$. Then by (W.7),

$$\begin{aligned} n^{-1/2}h^{1/2}U_{ipw}(v,\beta,\hat\psi) &= n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\mathcal{A}_{ki} + n^{-1/2}h^{1/2}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\mathcal{O}_{ki} \\ &\quad + \frac{1}{2}\mu_2 n^{1/2}h^{5/2}\vartheta(v) + o_p(n^{1/2}h^{5/2}) + o_p(h^{1/2}). \end{aligned}$$

Thus,

$$(nh)^{1/2}[\hat\beta^{ipw}(v) - \beta(v) + \frac{1}{2}\mu_2 h^2\Sigma^{-1}(v)\vartheta(v)]$$

$$= -(\Sigma(v))^{-1} n^{-1/2} h^{1/2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} (\mathcal{A}_{ki} + \mathcal{O}_{ki}) + o_p(n^{1/2} h^{5/2}) + o_p(h^{1/2})$$
$$= -(\Sigma(v))^{-1} n^{-1/2} h^{1/2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} [(\mathcal{A}_{ki} + \mathcal{B}_{ki}) + (\mathcal{O}_{ki} - \mathcal{B}_{ki})] + o_p(n^{1/2} h^{5/2}) + o_p(h^{1/2}).$$

Next we show that $Cov\{\mathcal{A}_{ki} + \mathcal{B}_{ki}, \mathcal{O}_{ki} - \mathcal{B}_{ki}\} = 0$. This will be followed by

$$\begin{aligned}
& Cov\{(nh)^{1/2}[\hat{\beta}^{ipw}(v) - \beta(v) + \frac{1}{2}\mu_2 h^2 \Sigma^{-1}(v)\vartheta(v)]\} \\
&= Cov\{(nh)^{1/2}[\hat{\beta}^{aug}(v) - \beta(v) + \frac{1}{2}\mu_2 h^2 \Sigma^{-1}(v)\vartheta(v)]\} \\
&\quad + h\Sigma^{-1}(v) \sum_{k=1}^{K} \frac{n_k}{n} Cov\{\mathcal{O}_{k1} - \mathcal{B}_{k1}\} \Sigma^{-1}(v) + o_p(nh^5) + o_p(h).
\end{aligned}$$

By the missing at random assumption (3), conditional on $Q_{ki}$, $R_{ki}$ and $M_{ki}(t, v)$ are independent. We have

$$\begin{aligned}
& Cov(\mathcal{A}_{ki}, \mathcal{B}_{ki}) \\
= \;& E\Big\{ \int_0^1 \int_0^\tau \int_0^1 \int_0^\tau K_h(u_1 - v) K_h(u_2 - v) (Z_{ki}(t) - \bar{z}_k(t, \beta(u_1)) (Z_{ki}(t) - \bar{z}_k(t, \beta(u_2)))^T) \\
& \frac{R_{ki}}{\pi_k(Q_{ki}, \psi_{k0})} \Big(1 - \frac{R_{ki}}{\pi_k(Q_{ki}, \psi_{k0})}\Big) M_{ki}(dt, du_1) E\{M_{ki}(dt, du_2) | Q_{ki}\} \Big\} \\
= \;& E\Big\{ \int_0^1 \int_0^\tau \int_0^1 \int_0^\tau K_h(u_1 - v) K_h(u_2 - v) (Z_{ki}(t) - \bar{z}_k(t, \beta(u_1)) (Z_{ki}(t) - \bar{z}_k(t, \beta(u_2)))^T) \\
& E\Big[ \frac{R_{ki}}{\pi_k(Q_{ki}, \psi_{k0})} \Big(1 - \frac{R_{ki}}{\pi_k(Q_{ki}, \psi_{k0})}\Big) \Big| Q_{ki} \Big] E\{M_{ki}(dt, du_1) | Q_{ki}\} E\{M_{ki}(dt, du_2) | Q_{ki}\} \Big\}.
\end{aligned}$$

Note that

$$\begin{aligned}
& Cov(\mathcal{B}_{ki}, \mathcal{B}_{ki}) \\
= \;& E\Big\{ \int_0^1 \int_0^\tau \int_0^1 \int_0^\tau K_h(u_1 - v) K_h(u_2 - v) (Z_{ki}(t) - \bar{z}_k(t, \beta(u_1)) (Z_{ki}(t) - \bar{z}_k(t, \beta(u_2)))^T) \\
& E\Big[ \Big(1 - \frac{R_{ki}}{\pi_k(Q_{ki}, \psi_{k0})}\Big)^2 \Big| Q_{ki} \Big] E\{M_{ki}(dt, du_1) | Q_{ki}\} E\{M_{ki}(dt, du_2) | Q_{ki}\} \Big\}.
\end{aligned}$$

Since

$$E\Big[ \frac{R_{ki}}{\pi_k(Q_{ki}, \psi_{k0})} \Big(1 - \frac{R_{ki}}{\pi_k(Q_{ki}, \psi_{k0})}\Big) \Big| Q_{ki} \Big] = -E\Big[ \Big(1 - \frac{R_{ki}}{\pi_k(Q_{ki}, \psi_{k0})}\Big)^2 \Big| Q_{ki} \Big]$$

we have $Cov(\mathcal{A}_{ki} + \mathcal{B}_{ki}, \mathcal{B}_{ki}) = 0$.

By the conditional independence between $R_{ki}$ and $V_{ki}$ given $Q_{ki}$,

$$
\begin{aligned}
&Cov\{\mathcal{A}_{ki}+\mathcal{B}_{ki},\mathcal{O}_{ki}\}\\
&= E\bigg[\int_0^1\int_0^\tau K_h(u-v)(Z_{ki}(t)-\bar{z}_k(t,\beta(u)))\bigg\{\frac{R_{ki}}{\pi_k(Q_{ki},\psi_{k0})}M_{ki}(dt,du)\\
&\quad+\bigg(1-\frac{R_{ki}}{\pi_k(Q_{ki},\psi_{k0})}\bigg)E\{M_{ki}(dt,du)|Q_{ki}\}\bigg\}\mathcal{O}_{ki}\bigg]\\
&= E\bigg[\int_0^1\int_0^\tau K_h(u-v)(Z_{ki}(t)-\bar{z}_k(t,\beta(u)))\bigg\{E\bigg(\frac{R_{ki}}{\pi_k(Q_{ki},\psi_{k0})}\mathcal{O}_{ki}\bigg|Q_{ki}\bigg)E\{M_{ki}(dt,du)|Q_{ki}\}\\
&\quad+E\bigg(\bigg(1-\frac{R_{ki}}{\pi_k(Q_{ki},\psi_{k0})}\bigg)\mathcal{O}_{ki}\bigg|Q_{ki}\bigg)E\{M_{ki}(dt,du)|Q_{ki}\}\bigg\}\bigg]\\
&= E\bigg[\int_0^1\int_0^\tau K_h(u-v)(Z_{ki}(t)-\bar{z}_k(t,\beta(u)))E\{\mathcal{O}_{ki}|Q_{ki}\}\,E\{M_{ki}(dt,du)|Q_{ki}\}\\
&= 0.
\end{aligned}
$$

The equality holds since $E\{\mathcal{O}_{ki}|Q_{ki}\}=(n)^{-1/2}\mathcal{D}_k(I_k^\psi)^{-1}E\{S_{ki}^\psi|Q_{ki}\}=0.$ □

# Additional References


Shorack, G. R. & Wellner, J. A. (1986). *Empirical Processes with Applications to Statistics*. Wiley, New York.

Sun, Y. & Wu, H. (2005). Semiparametric time-varying coecients regression model for longitudinal data. *Scandinavian Journal of Statistics* **32**, 21–47.

van der Vaart, A. W. (1998). *Asymptotic Statistics*. Cambridge University Press, Cambridge.


**Web Appendix B**

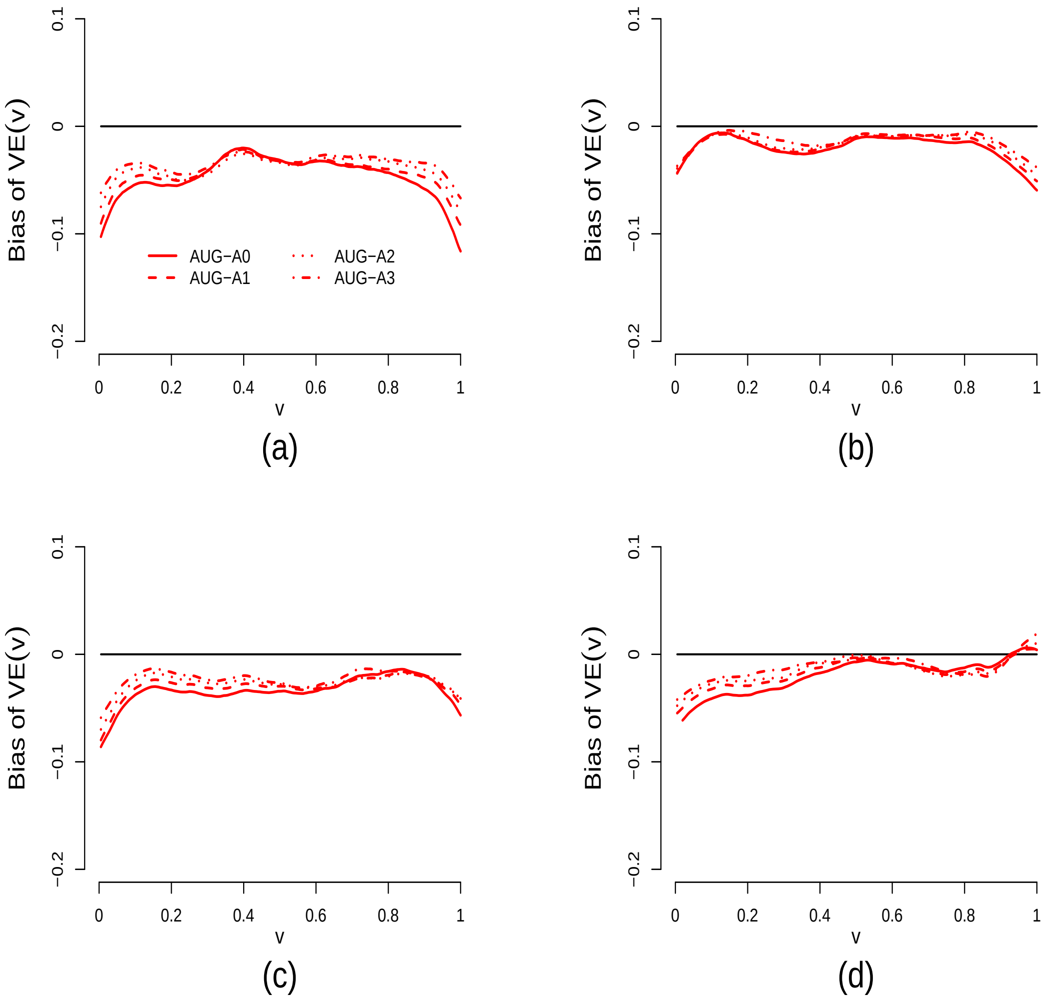


Figure 1: Bias of estimation of VE($v$) using the AUG procedure based on 500 simulations for $n = 500$, $b_1 = 0.1$ and $h = b_2 = 0.15$: (a) for model (M1), (b) for model (M2), (c) for model (M3) and (d) for model (M4). AUG-A0 is the AUG estimator corresponding to $\rho = 0$, AUG-A1 for $\rho \approx 0.78$, AUG-A2 for $\rho \approx 0.92$ and AUG-A3 for $\rho \approx 0.98$, where $\rho$ is the correlation coefficient between $A_{ki}$ and $V_{ki}$.

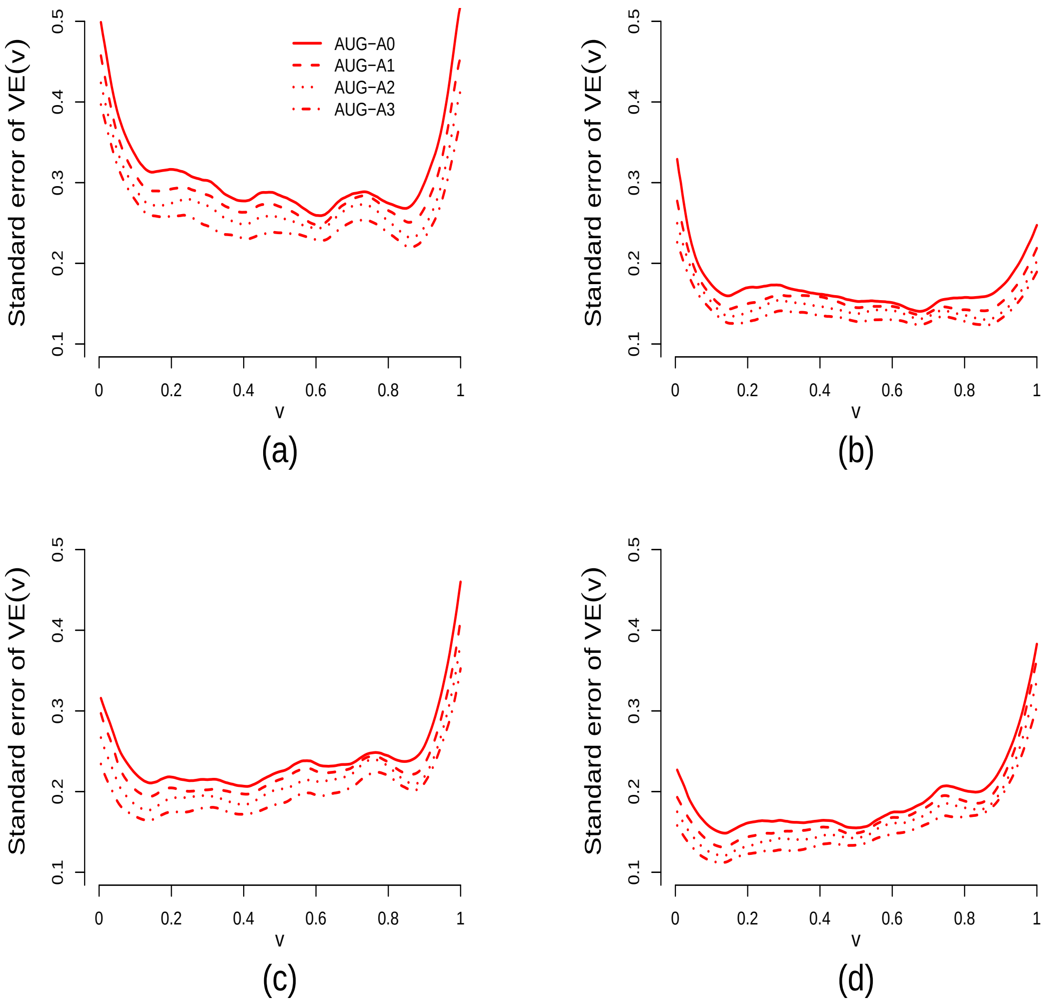


Figure 2: Average standard errors of the estimates for VE$(v)$ using the AUG procedure based on 500 simulations for $n = 500$, $b_1 = 0.1$ and $h = b_2 = 0.15$: (a) for model (M1), (b) for model (M2), (c) for model (M3) and (d) for model (M4). AUG-A0 is the AUG estimator corresponding to $\rho = 0$, AUG-A1 for $\rho \approx 0.78$, AUG-A2 for $\rho \approx 0.92$ and AUG-A3 for $\rho \approx 0.98$, where $\rho$ is the correlation coefficient between $A_{ki}$ and $V_{ki}$.

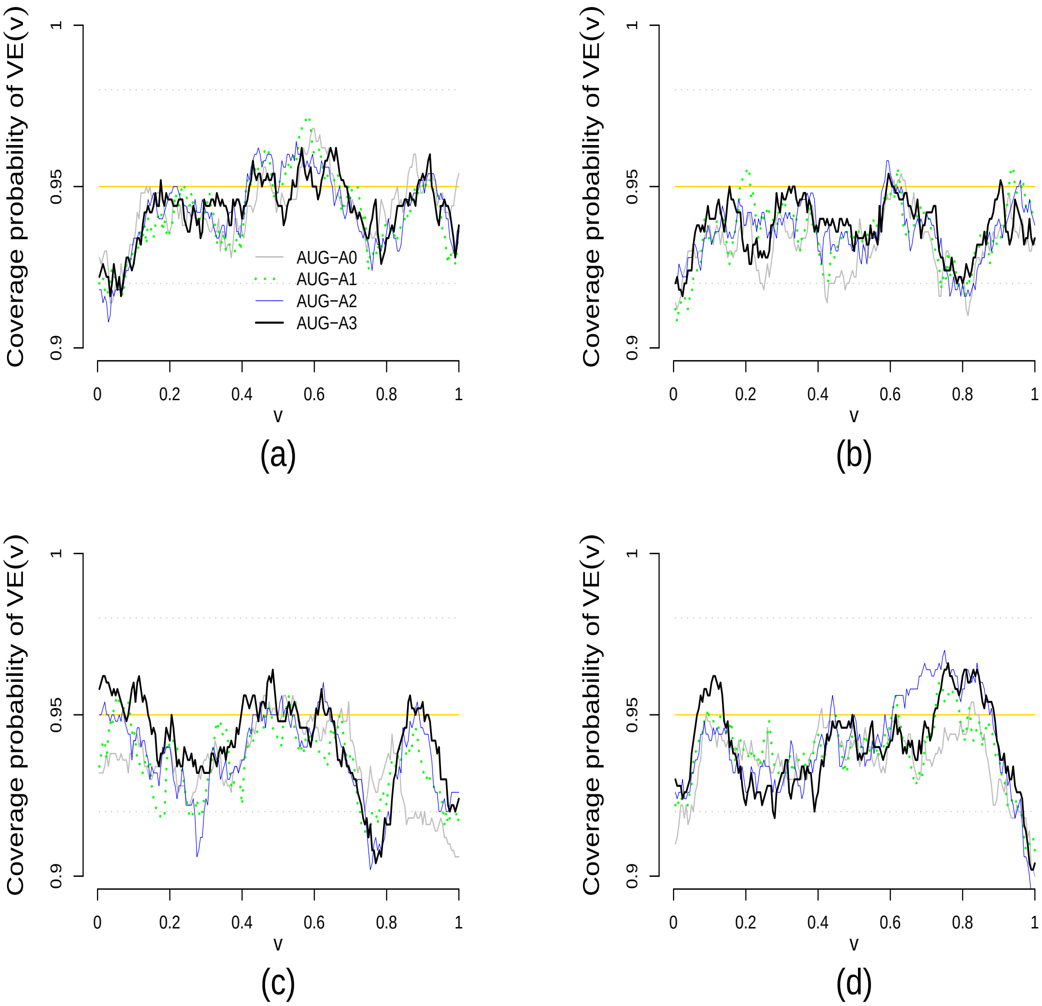


Figure 3: Estimated pointwise coverage probabilities of 95% confidence intervals for $\mathrm{VE}(v)$ constructed using the AUG estimator based on 500 simulations for $n = 500$, $b_1 = 0.1$ and $h = b_2 = 0.15$: (a) for model (M1), (b) for model (M2), (c) for model (M3) and (d) for model (M4). AUG-A0 is the AUG estimator corresponding to $\rho = 0$, AUG-A1 for $\rho \approx 0.78$, AUG-A2 for $\rho \approx 0.92$ and AUG-A3 for $\rho \approx 0.98$, where $\rho$ is the correlation coefficient between $A_{ki}$ and $V_{ki}$.