## SUPPLEMENTAL MATERIAL

### Single particle

At any time $t$, the position of the particle in the channel is described by the vector of probabilities $p_i(t)$ to be at a particular site $i$: $|p(t)\rangle = (p_1(t), ...p_i(t)...p_N(t))$, so that $\langle i|p(t)\rangle = p_i(t)$ The probabilities $p_i(t)$ obey the following equations [1–3]

$$\frac{d}{dt}p_i(t) = r(p_{i-1} + p_{i+1} - 2p_i) \text{ for } 1 < i < N \tag{1}$$

with the boundary conditions

$$\frac{d}{dt}p_1(t) = -(r_o + r)p_1 + rp_2) \quad \text{and} \quad \frac{d}{dt}p_N(t) = -(r_o + r)p_N + rp_{N-1}. \tag{2}$$

Equations (1,2) can be written in a matrix form as

$$\frac{d}{dt}|p(t)\rangle = \hat{M} \cdot |p(t)\rangle, \tag{3}$$

with

$$M_{i,i} = -2r \quad \text{and} \quad M_{i,i\pm1} = r \quad \text{for} \quad 1 < i < N, \tag{4}$$

and

$$M_{1,1} = -r - r_o; \quad M_{N,N} = -r - r_o; \quad M_{1,2} = r; \quad M_{N,N-1} = r. \tag{5}$$

### Explicit solution of single particle equations in terms of matrix elements

Here we re-derive the solutions obtained in the Letter, using the standard methods of linear algebra [4]. Assume an arbitrary Markov process that can be in $N$ states (such as defined by equation (3)). Time evolution of its probability distribution $\vec{p}(t) = (p_1, p_2, ..., p_N)$ and can be described by the following matrix equation (the equation (3) is an example):

$$\dot{\vec{p}} = \hat{U} \cdot \vec{p} \tag{6}$$

where $\vec{p}$ is an $N$-dimensional vector, and $\hat{U}$ is an $N \times N$ matrix. Let us denote the eigenvalues of the matrix $\hat{U}$ as $\omega_1...\omega_i...\omega_N$ and the corresponding eigenvectors as $\vec{v}^1, ..., \vec{v}^i, ...\vec{v}^N$. Then the general solution is

$$\vec{p}(t) = \sum_{j=1}^{N} a_j \vec{v}^j e^{\omega_j t}, \tag{7}$$

where $a_1...a_N$ is a set of numerical coefficients. In other words,

$$p_i(t) = \sum_{j=1}^{N} a_j v_i^j e^{\omega_j t}. \tag{8}$$

The coefficients $a_j$ can be determined from the initial condition:

$$p_i(0) = \sum_{j=1}^{N} a_j v_i^j, \tag{9}$$

which can be written as

$$\vec{p}(0) = \hat{V} \cdot \vec{a}, \tag{10}$$

where

$$\hat{V} = \begin{pmatrix} \vec{v}^1 \\ ... \\ \vec{v}^i \\ ... \\ \vec{v}^N \end{pmatrix}^T$$

so that $V_{ij} = v_i^j$. Finally

$$\vec{a} = \hat{V}^{-1} \cdot \vec{p}(0) \tag{11}$$

and

$$a_k = \sum_{j=1}^{N} (V^{-1})_{kj} p_j(0) \tag{12}$$

For the initial condition $p_i(0) = \delta_{i,1}$, we get $a_k = (V^{-1})_{k1}$. Now, the matric $U$ is diagonalized to its diagonal form

$$\hat{W} = \begin{pmatrix} w_1 & 0 & ... & 0 \\ ..... & ..... & .... & \\ 0 \; ... & w_i & ... & 0 \\ ..... & .... & .... & \\ 0 \; ... & 0 & & w_N \end{pmatrix}$$

by the transformation $W = V^{-1}UV$, or equivalently $U = VWV^{-1}$ [4].

Thus, the probability to be in state $i$ at time $t$ is

$$p_i(t) = \sum_{j=1}^{N} a_j v_i^j \exp w_j t = \sum_{j=1}^{N} (V^{-1})_{j1} V_{ij} e^{w_j t} = \left(V e^{Wt} V^{-1}\right)_{i1} = \left(e^{Ut}\right)_{i1} \tag{13}$$

and in particular,

$$p_N(t) = \left(e^{Ut}\right)_{N1} \tag{14}$$

The probability flow to exit to the right is $r_o p_N(t)$, and the total probability of exit to the right is

$$P_{\rightarrow} = r_o \int_0^{\infty} p_N(t)dt = -r_o \sum_{j=1}^{N} a_j v_N^j \frac{1}{w_j} = -r_o \sum_{j=1}^{N} (V^{-1})_{j1} V_{Nj} \frac{1}{w_j} \left(VW^{-1}V^{-1}\right)_{N1} = -r_o U_{N1}^{-1} \tag{15}$$

in agreement with equation (2) in the Letter. This result can be also obtained using the following reasoning. Instead of considering a single particle hopping through the states, starting at the position 1, let us consider the steady state where a flux $J$ enters to a position 1, with a steady state probability distribution $\vec{p}$. Then the probability to exit to the right is the ratio of the transmitted flux to the entrance flux: $r_o p_N/J$. We have

$$0 = \hat{U} \cdot \vec{p} + \vec{J} \tag{16}$$

and therefore $\vec{p} = -U^{-1} \cdot \vec{J}$ so that $p_N = -U_{N1}^{-1}$ because $J_i = J\delta_{i,1}$.

The probability distribution of exit times to the right is simply $r_o p_N(t)$ [3] and any moment of it can be calculated easily. For instance, the mean first passage time to exit to the right is:

$$\bar{T}_{\rightarrow} = r_o \int_0^{\infty} tP_N(t)dt = r_o \sum_{j=1}^{N} (V^{-1})_{j1} V_{Nj} \frac{1}{w_j^2} = r_o \left(V(W^2)^{-1}V^{-1}\right)_{N1} = r_o \left(U^2\right)_{N1}^{-1} \tag{17}$$

in agreement with the equation (3) in the Letter.

**Steady state**

In the case of current $J$ impinging on the channel entrance, one can describe the system in terms of average site occupancies $n_i$, whose kinetics is described in the mean field approximation by the following equations [5–7].

$$\frac{d}{dt}n_i = rn_{i-1}(1-\frac{n_i}{m})+rn_{i+1}(1-\frac{n_i}{m})-rn_i(1-\frac{n_{i-1}}{m})-rn_i(1-\frac{n_{i+1}}{m}) = r(n_{i-1}+n_{i+1}-2n_i) \quad \text{for} \quad 1 < i < N. \tag{18}$$

where $m$ is the maximal site occupancy. The boundary conditions at sites 1 and $N$ are

$$\frac{d}{dt}n_1 = -(r + r_o)n_1 + rn_2 + J(1 - \frac{n_1}{m})$$
$$\frac{d}{dt}n_N = -(r + r_o)n_N + rn_{N-1}. \tag{19}$$

In a matrix form:

$$\frac{d}{dt}|n(t)\rangle = \hat{M}^J \cdot |n(t)\rangle + \vec{J} \tag{20}$$

where the matrix $\hat{M}^J$ is the same as $\hat{M}$ with the only change $\hat{M}^J_{1,1} = -J/m - r - r_o$ and $\vec{J} = (J, 0, ...0)$. Note that for an internally uniform channel (as the one described in Fig. 1) the mean-field equations (Eqs. (18,20)) are exact [5, 6].

The steady state density profile can be obtained from Eq. (20) as $|n\rangle^{ss} = -\left(\hat{M}^J\right)^{-1} \cdot \vec{J}$, or more specifically as:

$$n_i^{ss} = \frac{J\left(1 + (N-i)\frac{r_o}{r}\right)}{r_o\left(2 + (N-1)\frac{r_o}{r}\right) + \frac{J}{m}\left(1 + (N-1)\frac{r_o}{r}\right)}. \tag{21}$$

The average exit flux to the right is $J_\rightarrow = r_o n_N^{ss}$. This together with Eq. (21) yield the probability of an individual particle within the flux to exit to the right:

$$P^{ss}_\rightarrow = \frac{J_\rightarrow}{J(1 - n_1/m)} = \frac{1}{2 + (N-1)r_o/r}. \tag{22}$$

As already established before, the exit probability of individual particles to exit to the right is the same as in the single-particle case (at least for uniform channels), even though they are interfering with each other's passage through the channel [7].

However, crowding does influence transport and is manifested in obstruction of the entrance site. The transport *efficiency*, defined as the ratio of the exit flux to the right $J_\rightarrow$ to the total impinging flux $J$, $\mathrm{Eff}_\rightarrow = \frac{J_\rightarrow}{J}$, decreases with $J$ due to jamming at the entrance.

$$\mathrm{Eff}_\rightarrow = \frac{J_\rightarrow}{J} = \frac{r_o}{2r + (N-1)r_o + J(1 + (N-1)r_o/r)/m}. \tag{23}$$

**Derivation of the analytical expressions for the mean exit times in the jammed regime**

The boundary conditions of equation (8) of the Letter, describing the probability of the tagged particle are:

$$\frac{d}{dt}p_1 = -r_o p_1 - rp_1(1-n_2^{ss}/m) + rp_2(1-n_1^{ss}/m)$$
$$\frac{d}{dt}p_N = -r_o p_N - rp_N(1-n_{N-1}^{ss}/m) + rp_{N-1}(1-n_N^{ss}/m) \tag{24}$$

Using the matrix form of Eq. (9) of the paper, the elements of the matrix $M^{ss}$ are given by

$$M_{i,i}^{ss} = -r(2-n_{i-1}^{ss}/m-n_{i+1}^{ss}/m) = -2r(1-n_i^{ss}/m); \quad \text{and} \quad M_{i,i\pm1}^{ss} = r(1-n_i^{ss}/m) \quad \text{for} \quad 1<i<N, \tag{25}$$

and

$$M_{1,1}^{ss} = -r(1-\frac{n_1^{ss}}{m}) - r_o; \quad M_{N,N}^{ss} = -r(1-\frac{n_{N-1}^{ss}}{m}) - r_o; \quad M_{1,2}^{ss} = r(1-\frac{n_1^{ss}}{m}); \quad M_{N,N-1}^{ss} = r(1-\frac{n_N^{ss}}{m}) \tag{26}$$

The average (over particles actually exited to the left) time to exit to the left is

$$\overline{T}_{\leftarrow}^{ss} = r_o \left\langle 1\right| \left(\left(M^{ss}\right)^{-1}\right)^2 \left|1\right\rangle / P_{\leftarrow}, \tag{27}$$

where

$$P_{\leftarrow} = 1 - P_{\rightarrow} = \frac{1+(N-1)\,r_o/r}{2+(N-1)\,r_o/r}. \tag{28}$$

In order to obtain an explicit expression for $\overline{T}_{\leftarrow}^{ss}$ we define

$$\left|W\right\rangle = D\left(M^{ss}\right)^{-1}\left|1\right\rangle \tag{29}$$

and

$$\left\langle Q\right| = D\left\langle 1\right|\left(M^{ss}\right)^{-1}. \tag{30}$$

In the equations above we introduced the notation for the determinant of $M^{ss}$, $D \equiv det\left(M^{ss}\right)$. The elements of these vectors are given by

$$W_n = r^{N-2}\left(A_N r + nB_N r + (N-n)\,r_o\right)\prod_{k=2}^{N-1}\left(A_N + kB_N\right) \tag{31}$$

and

$$Q_n = r^{N-2}\left(A_N r + nB_N r + (N-n)\,r_o\right)\frac{\prod_{k=1}^{N-1}\left(A_N + kB_N\right)}{A_N + nB_N}. \tag{32}$$

In the above expressions

$$A_N = \frac{r_o\left(2r + (N-1)\,r_o - J/m\right)}{(J/m)\left(r + (N-1)\,r_o\right) + r_o\left(2r + (N-1)\,r_o\right)} \tag{33}$$

and

$$B_N = \frac{Jr_o/m}{(J/m)\left(r + (N-1)\,r_o\right) + r_o\left(2r + (N-1)\,r_o\right)}. \tag{34}$$

The mean escape time to the left is then

$$\overline{T}^{ss}_{\leftarrow} P_{\leftarrow} = \frac{r_o}{D^2}\sum_{n=1}^{N} W_n Q_n = \frac{r_o}{D^2}\sum_{n=1}^{N} r^{2N-4}\left(A_N r + nB_N r + (N-n)\,r_o\right)^2 \frac{(A_N + B_N)}{A_N + nB_N}\prod_{k=2}^{N-1}\left(A_N + kB_N\right)^2. \tag{35}$$

Using the notations above we can express $D$ as:

$$D = r^{N-2} r_o\left(2A_N r + (N+1)\,B_N r + (N-1)\,r_o\right)\prod_{k=2}^{N-1}\left(A_N + kB_N\right). \tag{36}$$

Substituting $D$ into the expression for $\overline{T}^{ss}_{\leftarrow}$ one obtains

$$\overline{T}^{ss}_{\leftarrow} P_{\leftarrow} = \frac{(A_N + B_N)\sum_{n=1}^{N}\frac{(A_N r + nB_N r + (N-n)r_o)^2}{A_N + nB_N}}{r_o\left(2A_N r + (N+1)\,B_N r + (N-1)\,r_o\right)^2}. \tag{37}$$

Performing the summation, we get for the average time to exit the channel to the left

$$\begin{aligned}\frac{\overline{T}^{ss}_{\leftarrow} P_{\leftarrow}}{(A_N + B_N)} &= \frac{\left(N(B_N r - r_o)(2A_N(B_N r + r_o) + B_N(B_N(N+1)r + (3N-1)r_o))\right)}{2B_N^2 r_o\left(2A_N r + (N+1)\,B_N r + (N-1)\,r_o\right)^2} \\ &+ \frac{r_o(A_N + NB_N)^2\left(\psi\left(\frac{A_N}{B_N} + N + 1\right) - \psi\left(\frac{A_N + B_N}{B_N}\right)\right)}{B_N^3\left(2A_N r + (N+1)\,B_N r + (N-1)\,r_o\right)^2}.\end{aligned} \tag{38}$$

Here $\psi(x) = \frac{d\ln(\Gamma(z))}{dz}|_{z=x}$, where $\Gamma(x)$ is the $\gamma$-function.

Substituting the expressions for $A_N$ and $B_N$, we get the explicit expression for the average time as

$$\begin{aligned}\overline{T}^{ss}_{\leftarrow} &= -\frac{N\left(2r_o\left(2r + r_o(N-1)\right) + \frac{J}{m}\left(4r + 3r_o(N-1)\right)\right)}{2\,(J/m)^2\left(r + r_o(N-1)\right)} \\ &+ \frac{(J/m)\left(r + r_o(N-1)\right) + r_o\left(2r + r_o(N-1)\right)}{(J/m)^3 r_o\left(r + r_o(N-1)\right))}\left(\psi\left(N + \frac{2r + (N-1)\,r_o}{J/m}\right) - \psi\left(\frac{2r + (N-1)r_o}{J/m}\right)\right).\end{aligned} \tag{39}$$

In the Letter we use the definitions of the probabilities to exit to the right/left in order to simplify this cumbersome expression. In the single particle limit, $J \to 0$, the expression for

$\overline{T}_{\leftarrow}^{ss}$ reduces to the previously obtained single particle expression (Eq. (6) in the paper). At the other extreme when the input flux $J \to \infty$, the mean escape time is

$$\lim_{J\to\infty} \overline{T}_{\leftarrow}^{ss} = \frac{r + (N-1)r_o}{r_o(2r + (N-1)r_o)} = \frac{P_{\leftarrow}}{r_o}. \qquad (40)$$

The mean time to exit to the right and the mean trapping time can be obtained in a similar fashion.


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