# Supplementary information for: Observing fermionic statistics with photons in arbitrary processes

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**Operator for continuous, constant nearest neighbour coupling.** Evanescently coupled uniform waveguide arrays are modelled with the coupled oscillator Hamiltonian

$$H = \sum_j \beta a_j^\dagger a_j + C a_{j-1}^\dagger a_j + C a_{j+1}^\dagger a_j \tag{1}$$

for waveguide propagation constant (site potential) $\beta$ and constant coupling coefficient $C$, where $\beta, C \in \mathbb{R}$. In the basis of single position (or mode) states $\{|1\rangle_1, |1\rangle_2, ... |1\rangle_n\}$, the Hamiltonian is represented by a tri-diagonal $M \times M$ matrix

$$H_{i,j} = \begin{cases} \beta, & i = j \\ C, & i = j \pm 1 \end{cases} . \tag{2}$$

Unitary evolution of this Hamiltonian after time $T = cz$ is given by $U = e^{iHT}$, where $z$ is propagation distance in the waveguide and $c$ is the effective propagation speed which is equivalent to a continuous time quantum walk on a one dimensional graph[1].

The device reported is a uniform array of $M = 21$ waveguides. The propagation length $z$ is such that the linear ballistic propagation resulting from launching light into the central waveguides[1] does not extend beyond ten central waveguides which are accessed in the experiment. For computing similarity between experiment and theory the matrix $A$ is modelled by extracting the central $10 \times 10$ submatrix of a $21 \times 21$ unitary matrix $U = e^{iTH}$, computed using Eq. 1 with parameters $T = 13.9$, $\beta = 0$ and $C = 0.15$; the theoretical model assumes the array to be uniform and to have zero loss. Both assumptions can give rise to $S < 100\%$ in our analysis.

**2-particle correlation functions.** Quantum interference between two indistinguishable particles—modelled by creation operators $a_j^\dagger$ and annihilation operators $a_j$ acting on labeled modes $j$ in a finite set of modes associated to system $a$—is quantified by the correlation function $\Gamma_{r,q}^{\phi}$ and has been derived for photons (bosons)[2] and fermions[3] in multi-port networks. Here we review derivation and adapt for fractional statistics that allow continuous deformation between Bose-Einstein and Fermi-Dirac statistics.

Consider two indistinguishable particles that are governed by the relations

$$a_j^\dagger a_k^\dagger - e^{i\phi} a_k^\dagger a_j^\dagger = 0 \tag{3}$$

$$a_j a_k - e^{i\phi} a_k a_j = 0 \tag{4}$$

$$a_j a_k^\dagger - e^{i\phi} a_k^\dagger a_j = \delta_{j,k} \tag{5}$$

for Bose-Einstein ($\phi = 0$) and Fermi-Dirac ($\phi = \pi$) statistics, governing exchange according to the permutation group[4]. We can also define particles that undergo a fractional phase change $0 < \phi < \pi$ with $j < k$, on exchanging two such particles according to braiding operations defined as 'clockwise'. For $j > k$, anti-clockwise braids are defined with $-\pi < \phi < 0$.

A mode transformation $A$ on a state of two indistinguishable particles created in modes $j$ and $k$, maps the initial state $a_j^\dagger a_k^\dagger |0\rangle$ to a linear superposition of two particle states according to

$$a_j^\dagger a_k^\dagger |0\rangle \xrightarrow{A} \sum_{s,t} A_{s,j} A_{t,k} a_s^\dagger a_t^\dagger |0\rangle , \tag{6}$$

where $A$ is expressed as a matrix in the mode basis. Correlated detection at $r$ and $q$ will occur as a result of two possible events: (i.) the particle at $j$ is mapped to position $r$ and the particle at $k$ is mapped to $q$ with amplitude

$A_{r,j}A_{q,k}$; (ii.) the particle at $j$ is mapped to position $q$ and the particle at $k$ is mapped to $r$ with amplitude $A_{q,j}A_{r,k}$. Labelling modes such that $j < k$ and $r < q$ implies (ii.) is equivalent to at least one exchange of the position of the two particles. Here we define (i.) corresponds to no exchange while (ii.) corresponds to a single swap or braid operation according to (3-5).

Since the two particles are indistinguishable, quantum mechanics dictates outcomes (i.) and (ii.) are indistinguishable and hence the complex probability amplitudes $A_{r,j}A_{q,k}$ and $A_{q,j}A_{r,k}$ interfere in the correlation across $r$ and $q$. The correlated detection outcome $\Gamma^{\phi}_{r,q}$ is therefore computed by projecting the state $\langle 0| a_r a_q$ on the output state (6), using the inner product

$$\langle 0| a_r a_q a_j^\dagger a_k^\dagger |0\rangle = \delta_{q,j}\delta_{r,k} + e^{i\phi}\delta_{r,j}\delta_{q,k}, \tag{7}$$

which follows from the relations (3-5). Combining expressions (6) and (7) therefore yields

$$\Gamma^{\phi}_{r,q} = \left|A_{r,j}A_{q,k} + e^{i\phi}A_{q,j}A_{r,k}\right|^2 \tag{8}$$

for exchange statistics parameterised by $\phi$. Note that for fermions ($\phi = \pi$) the diagonal elements of the matrix $\Gamma^{\pi}_{r,r} = 0$ agree with the Pauli-exclusion principle.

**$N$-particle correlation functions.** Consider $N$ particles with exchange parameter $\phi$ and governed by the relations (3-5) which are launched into inputs labelled by tuple $\vec{\nu} = \{\nu_1, \nu_2, ..., \nu_N\}$ of mode transformation $A$. The $N$-fold correlation function for detection across output $N$-tuple $\vec{\mu} = \{\mu_1, \mu_2, ..., \mu_N\}$ is derived for multi-particle quantum walks with both Fermi-Dirac and Bose-Einstein statistics[5].

The input state is mapped by arbitrary $A$ to a linear superposition of $N$-particle states at the output of $A$ according to

$$\prod_{j=1}^{N} a^\dagger_{\nu_j} |0\rangle \xrightarrow{A} \prod_{j=1}^{N}\left(\sum_{p=1}^{N} A_{p,\nu_j} a^\dagger_p\right)|0\rangle . \tag{9}$$

Defining $\nu_1{<}\nu_2{<}...{<}\nu_N$ and $\mu_1{<}\mu_2{<}...{<}\mu_N$ we can consider the transformation $A$ maps the $N$ particles at $\vec{\nu}$ to the output $\vec{\mu}$ in $N!$ indistinguishable possibilities. One of these possibilities maps $\nu_j$ to $\mu_j$ for all $j$ which we define to not exchange a single pair of particles. The remaining $N! - 1$ map $\nu_j$ to $\sigma_{\mu_j}$ for all $j$ for some permutation $\sigma_\mu$ of $\mu$ and are defined to exchange particles with the minimum number of pairwise transpositions in clockwise braids that transform $\nu$ to $\sigma_\mu$. Each braid between any two particles makes the combined wave-function acquire a phase $\phi$; $m$ clockwise braids that are the minimum number required to map $\nu$ to $\sigma_\nu$ therefore cause the global wave-function to acquire phase $m\phi$. From this definition, we obtain an $N$-particle generalisation of expression (7):

$$\left(\langle 0|\prod_{s=1}^{N} a_{\mu_s}\right)\left(\prod_{j=1}^{N} a^\dagger{}_{\nu_j}|0\rangle\right) = \sum_{\sigma_\nu\in S_n} e^{i\tau(\sigma_\nu)\phi}\prod_{j=1}^{N}\delta_{\mu_j,\sigma_{\mu_j}}, \tag{10}$$

where $\tau(\sigma_\nu)$ is the minimum number of pairwise transpositions (neighbouring swaps) relating $\sigma_\nu$ to $\vec{\nu}$. Projection onto the output state (9) with $N$-particles at $\vec{\mu}$ uses relations (3-5) and (10) for a fractional parameter ($0 < \phi < \pi$) to yield

$$\Gamma^{\phi}_{\vec{\mu}} = \left|\sum_{\sigma_\nu\in S_n} e^{i\tau(\sigma_\nu)\phi}\prod_{j=1}^{n} A_{\mu_j,\sigma_{\nu_j}}\right|^2 . \tag{11}$$

Varying $\phi$ allows continuous deformation between Bose-Einstein ($\phi = 0$) and Fermi-Dirac ($\phi = \pi$) quantum interference.

**Generating $N-$partite, $N-$level entanglement.** A quantum algorithm for anti-symmetrising arbitrary products of $n$ wave functions $|\rho_1, \rho_2, ..., \rho_n\rangle_A$ is presented[6] that would use three registers of $n$, $m-$digit quantum words or "qu-words" (strings of $n\log_2 m$ qubits). This was then simplified[7] to use two registers of qubits $A$ and $B$; the first encodes the wavefunctions $|\rho_1, \rho_2, ..., \rho_n\rangle$. The second register is mapped to a symmetrized state with $O(n(\ln m)^2)$

local operations followed by $O(n^2 \ln m)$ controlled operations leaving the second register in the state

$$\frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} |\sigma(1, ..., N)\rangle_B \tag{12}$$

where $\sigma(1, .., N)$ is a permutation $\sigma \in S_N$ of the $N-$tuple $\{1, 2, ..., N\}$. The state Eq. 12 is then antisymmetrised by a sorting algorithm using $O(n \ln n)$ operations, applying the same swaps (via controlled operations) and phase shifts to the first register to antisymmetrize $|\rho_1, \rho_2, ..., \rho_n\rangle_A$. For our purpose, we require only the one register $B$, since the state we wish to generate is known and of the form

$$\begin{aligned} |\psi_N(\phi)\rangle &= \frac{1}{\sqrt{N!}} \sum_{\sigma_\nu \in S_N} e^{i\tau(\sigma_\nu)\phi} \prod_{j=1}^{N} a^{(j)\dagger}_{\sigma_{\nu j}} |0\rangle \\ &= \frac{1}{\sqrt{N!}} \sum_{\sigma_\nu \in S_N} e^{i\tau(\sigma_\nu)\phi} |\sigma_\nu(1, 2, ..., N)\rangle \end{aligned} \tag{13}$$

where $\sigma(1, .., N)$ can be decomposed into $\tau(\sigma)$ adjacent transpositions.

The approach shown in Fig. 1 represents each $i^{th}$ qu-word (denoted $B[i]$) in the register as an $N-$level qudit, for the examples $N = 2, 3$, and generates the state $|\psi_N(\phi)\rangle$ in an iterative process, using $\sum_{i=1}^{N-1} i \sum_{j=1}^{i} j < N^3$ controlled operations. Here each controlled operation is counted as one swap operation of two modes in a given qudit, conditional on an excitation in one of the $N$ levels of the control qudit. Realisation with linear optics for example, could use a heralded KLM CNOT gate[8,9], each requiring two extra (ancilla) photons, and an interferometric structure of seven beamsplitters.

As with Refs. 6,7, the $N$ qudits are initialised in superposition using local operations (mode splitters; $MS$)

$$\begin{aligned} & |1\rangle_{B[N]} \otimes ... \otimes |1\rangle_{B[1]} \\ & \xrightarrow{MS} \frac{1}{\sqrt{N!}} \left( \sum_{k=1}^{N} |k\rangle_{B[N]} \right) \otimes \left( \sum_{k=1}^{N-1} |k\rangle_{B[N-1]} \right) \otimes ... \otimes |1\rangle_{B[1]} \end{aligned} \tag{14}$$

Note that this step puts the state in the correct number of $N!$ terms required for the eventual entangled state. Using linear optics, this would use one photon for each qudit and $\sum_{j=1}^{N} (j-1) = O(N^2)$ two-mode beam splitters (with appropriate reflectivities).

The next step in generating $|\psi_N(\phi)\rangle$ is to introduce the phase relationship $\phi$ according to

$$\begin{aligned} \frac{1}{\sqrt{N!}} \left( \sum_{k=1}^{N} e^{i(k-1)\phi} |k\rangle_{B[N]} \right) \otimes \left( \sum_{k=1}^{N-1} e^{i(k-1)\phi} |k\rangle_{B[N-1]} \right) \otimes ... \\ ... \otimes |1\rangle_{B[1]} \end{aligned} \tag{15}$$

which uses $\sum_{j=1}^{N} (j-1) = O(N^2)$ local phase shift operations. A set of $\sum_{i=1}^{N-1} i \sum_{j=1}^{i} j = O(N^4)$ controlled operations are then applied iteratively to generate $|\psi_N(\phi)\rangle$; this process can be thought of as performing a series of controlled operations on the intermediate states

$$\frac{1}{\sqrt{q+1}} \sum_{j=1}^{q+1} e^{i(j-1)\phi} |j\rangle_{B[q+1]} \otimes |\psi_q(\phi)\rangle_{B[q,...,1]} \tag{16}$$

for each step of the iteration $q = 2, ..., N-1$. This is verified for $q = 2$ (see Fig. 1(a) and Fig. 1(b)). Next we assume inductively the process works for some $q = r$ to generate $|\psi_q(\phi)\rangle$. For $q = r + 1$, the process would begin with the state

$$\frac{1}{\sqrt{r+1}}\sum_{j=1}^{r+1}e^{i(j-1)\phi}|j\rangle_{B[r+1]}\otimes|\psi_r(\phi)\rangle_{B[r,..,1]} \tag{17}$$
$$=\frac{1}{\sqrt{r+1}}\sum_{j=1}^{r+1}e^{i(j-1)\phi}|j\rangle_{B[r+1]}\otimes\frac{1}{\sqrt{r!}}\sum_{\sigma\in S_r}|\sigma(1,..,r)\rangle_{B[r,..,1]}$$

A series of controlled qudit shift operations are then applied, such that an excitation of mode $j$ of the $B[r+1]$ qudit shifts each of the labeled modes $\{j,...,r\}$ to $\{j+1,...,r+1\}$ in the target qudits labelled $\{B[r],\ldots B[1]\}$, such that the overall state evolves to

$$\frac{1}{\sqrt{(r+1)!}}\sum_{\sigma\in S_r}e^{i\tau_\nu(\sigma)\phi}\Big\{|1\rangle_{B[r+1]}|\sigma(2,...,r+1)\rangle_{B[r,...,1]}+e^{i\phi}|2\rangle_{B[r+1]}|\sigma(1,3,4,...,r+1)\rangle_{B[r,...,1]}$$
$$\ldots+e^{i(j-1)\phi}|j\rangle_{B[r+1]}|\sigma(1,2,3,...,j-1,j+1,...,r+1)\rangle_{B[r,...,1]}+\ldots$$
$$\ldots+e^{ir\phi}|r+1\rangle_{B[r+1]}|\sigma(1,2,...,r)\rangle_{B[r,...,1]}\Big\} \tag{18}$$

The tuple $\{j,1,2,...,j-1,j+1,...,r+1\}$ is $(j-1)$ adjacent transpositions from the tuple $\{1,2,...,r+1\}$. Since the permutation $\sigma(1,2,..,j-1,j+1,...,r+1)$ is $\chi(\sigma)$ adjacent transpositions away from the ordered set $\{1,2,...,j-1,j+1,...,r+1\}$, it follows that the tuple $\{j,\sigma(1,2,...,j-1,j+1,...,r+1)\}$ is $(j-1+\chi(\sigma))$ adjacent transpositions different from $\{1,2,..,r+1\}$. Hence the Eq. 18 is equivalent to the state

$$|\psi_{r+1}(\phi)\rangle=\frac{1}{\sqrt{(r+1)!}}\sum_{\gamma\in S_r}e^{i\tau_\nu(\gamma)\phi}|\gamma(1,..,r+1)\rangle_{B[r+1,..,1]} \tag{19}$$

as required. Therefore, by induction we have an iterative process that generates $|\psi_N(\phi)\rangle$ for arbitrary size $q=N$, requiring $O(N^2)$ local operations, and $O(N^4)$ controlled swap operations.

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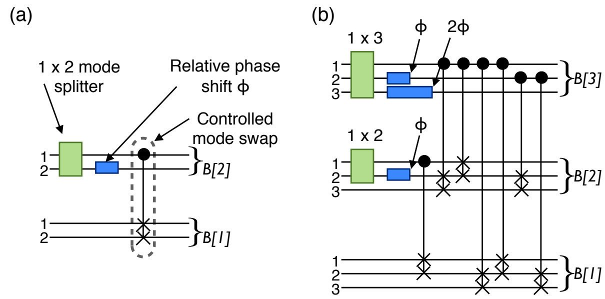


FIG. 1: Qudit circuits for geneating $|\psi_N(\phi)\rangle$ for (a) $N = 2$ and (b) $N = 3$. Qudits are initialised in the state $|1\rangle_{B[N]} \otimes ... \otimes |1\rangle_{B[1]}$, where $B[i]$ denotes the $i^{\mathrm{th}}$ qudit in the register. The green squares represent $1 \times k$ mode splitters that can be decomposed into $k-1$ two-mode splitters. Blue rectangles represent a relative phase shift with labelled phase $j\phi$. Each controlled swap between two modes denoted '$\times$' are conditional on an excitation in the corresponding mode denoted denoted by a '$\bullet$'.