Evolution of Controllability in Interbank Networks

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Supplementary information for the article "Evolution of Controllability in Interbank Networks"

I. The network of interbank money market

I. A Data set

Our data set is composed of 2750 daily snapshots of the Italian interbank money market, from January 4th, 1999 to September 30th, 2009. This data has been collected by “e-MID”, the only electronic market for Interbank Deposits in the Euro Area and US [1]. For the most part, the transactions correspond to overnight exchanges of deposits among banks. The amounts of transactions between parties, as well as their directions are recorded individually on a daily basis. Generally, any two banks i and j exchange liquidity in multiple tranches in both directions.

All these daily transactions form an oriented graph $G^{(1)}(V_t, E_t)$, with $t = 1, 2, \ldots 2750$, whose vertices are the banks taking part in the deals and the links have a weight given by the corresponding cash flow. We name V_t and E_t the sets of nodes and edges respectively, their cardinalities $|V_t|, |E_t|$ giving the number of nodes and edges of the graph. All these quantities depend on time because the number of banks entering and exiting the market as well as the amounts exchanged change considerably. In particular, we denote $w_t(i, j)$ the aggregate lending of bank i to j on day t. It is worth mentioning that we may well have also j lending to i, corresponding to $w_t(j, i) \neq 0$, that is flows are generally bidirectional. Fig. S1 illustrates the evolution of $G^{(1)}$ in terms of number of nodes, number of edges and total lending volumes (the sum of all link weights, indicated as v_t).

[1] e-MID Sim S.p.A, Milan, Italy.

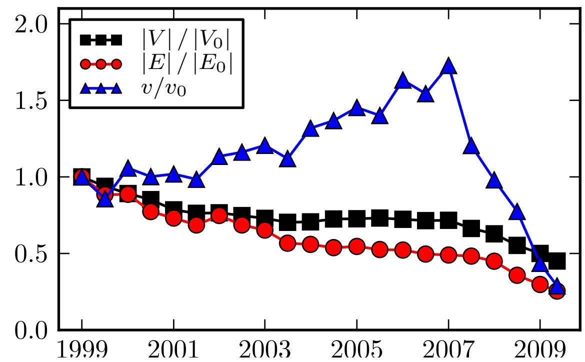

Figure S1: Evolution of the daily interbank lending network $G^{(1)}(V, E)$. The system experimented a sensible shrinking in both the number of agents ($|V|$) and the number of links ($|E|$); a credit crunch process is also clear starting in 2007, with a volume (v) drop of roughly 80% with respect to 1999. Curves are rescaled by the values of the first point of the corresponding quantity $|V_0|$, $|E_0|$, v_0 respectively.

I. B Network aggregation

As we can see from Fig. S1, the size of the network shrinks steadily over time, losing about half of the nodes and links. In contrast, the lending volume increases by 50% from 1999 to 2007. Afterwards, it experiences a fast drop between 2007 and 2009, decreasing by about 80%. So, we see that the network under examination undergoes important changes in terms of size and volumes flowing through the agents. However, some topological properties related to the ”fitness” of vertices seem to be preserved despite the shrinking process. For instance, the network remains scale free throughout the whole period, as shown in[1]. Remarkably, the density of links remains also around similar values (2.8% in 1999, 2.3 % in 2005, again 2.8% in 2009).

On a daily basis, the set of transactions E_t is highly volatile. However we can aggregate the daily loans over a number of trading days Δ, and define links and weights at this time scale as $w_{u,\Delta}(i,j) = \sum_{s=0}^{\Delta-1} w_{u+s}(i,j)$, where $u = 1, 1+\Delta, 1+2\Delta, \ldots$. In this way, we obtain the network $G^{(\Delta)}(V_{u,\Delta}, E_{u,\Delta})$, which is representative of the banking system during the time span $[u, u+\Delta)$. In our empirical study we compare results for five different time scales $\Delta = 1, 5, 21, 63, 126$ days, roughly corresponding to daily, weekly, monthly, three and six months time scales. It is worth noting that the recent Basel Committee proposals (Basel III) includes a 30-day liquidity coverage ratio requirement whose purpose is to ensure banks maintain adequate levels of unencumbered high quality assets against net cash outflows under stress.

I. C Centrality measures

The systemic relevance of banking institutions is key when designing regulation and monetary policies[2,3]. A reasonable assumption is to consider the case of the "too connected to fail", even though the precise notion of systemic relevance has not been defined yet and different measures have been proposed[4–6]. The degree $k(i)$ of a vertex provides information about the centrality in terms of number of connections to other nodes. This feature determines the properties of resilience of the network against link failure and epidemic spreading[7,8]. Another measure of importance is the *closeness centrality*. We introduce a notion of distance as follows. First we define the average lending $\bar{w}_u$ of the graph as the average weight of its edges. This represents the average amount of money which is exchanged between two banks on that period and provides a convenient measure unit. Given two connected nodes $i \to j$, we define their distance as the inverse of the (normalized) edge weight $d(i,j) = 1/(w(i,j)/\bar{w})$, so that the more i is lending to j, the closer is i to j. In this respect, this distance is a proxy of how much j is exposed towards i. For an oriented shortest path $(k_1 \to k_2 \to \cdots \to k_{n-1} \to k_n)$ connecting k_1 to k_n we define the distance as $d(k_1, k_n) = \sum_{s=1}^{n-1} d(k_s, k_{s+1})$. The closeness centrality $c_t(i)$ of a vertex is defined as the inverse of the average distance from the other nodes [2]. It measures the strengths of the credit relations among a bank i and its direct and indirect borrowers, also providing an indicator of potential spillover due to inefficient liquidity provision from bank i. In principle (but also in practice) we may have banks with very small degree (few connections) but with very tight links (large loans) or hub nodes which exchange very small amounts with many banks.

II. Network controllability

II. A General framework

It is possible to gain control of a directed network by identifying the "driver" nodes through the *Minimum Input Theorem*[9]: the minimum number of nodes to be controlled (*drivers*) corresponds to the number of *uncovered nodes* in a *maximum matching* of the corresponding graph.

For directed graphs a matching $M \subseteq E$ is a subset of directed edges such that any two edges do not share the same start or end vertex (note that for two edges in M, the end vertex of one may well be the start vertex of the other). The maximum matching M^{max} is the one with the largest possible cardinality among all the graph's matchings. A vertex is uncovered or *unmatched* when

[2]To account for changes in the network shape, we normalize closeness to $(|V|-1)/(|E|-1)$.

no edge in M^{max} reach it.

It has to be noted that, in general, a large multiplicity of maximum matchings exists. In the scope of the analysis presented in the main text, we always select the matching for which the sum of its edges' weights is maximal. Indeed, a maximum matching provides a maximal set of paths where some nodes are controlled indirectly (i.e. without external inputs) by their predecessors. Through these paths, the effect of external controllers on the drivers is propagated to other nodes. More precisely, if node i controls node j, the time variation of the matched node's state x_j gets a contribution from i that is proportional to the edge weight w_{ij}, see Eq. (1). In this sense, the larger the weight, the larger the impact of i upon j and, intuitively, maximising the sum of weights (all weights are positive in our case) would mean maximising the sum of the impacts.

The above theorem builds on a linear dynamics of the form

$$\frac{d\mathbf{x}(t)}{dt} = I\mathbf{x}(t) + C\mathbf{u}(t) \tag{1}$$

where $x_{i=1,\ldots,N}$ is a state variable for the N nodes, I is an $N \times N$ "influence" matrix representing the way every node is influenced by its predecessors. The components of the vector $\mathbf{u}$ correspond to external functions that are applied to a subset of N_D nodes and C is an $N \times N_D$ "control" matrix of external weights. The structure of I corresponds to that of the transposed weighted adjacency matrix (see[9] for more details), so that the time variation of x_i is determined, through I, by the state of the nodes that points to i. Generally, the exact values of the entries in I are unobservable and only the matrix structure is required in order to apply the theorem.

For the interbank lending network, we identify x_i with the aggregate lending of bank i, i.e. the sum of its outgoing lending flows, and we assume that the arrows of the mutual influence relationships among banks correspond to the observed flows. Indeed, it is reasonable to assume that variations of the amount of lending from any bank depend on the lending of the others [10–12]. Moreover, the information a given bank is able to access concerns only the level of lending it receives from its neighbors while information about all other deals is usually confidential. In particular, during a credit crunch, a bank gets less credit from its counterparts; as a reaction, that bank is likely to lower its lending to the others generating dangerous cascade effects. Even though this dependence may have a quite complicated formalization, as a first order approximation we will take it linear, so that we can postulate a dynamics of the form (1) for $\mathbf{x}(t)$, the functions $\mathbf{u}(t)$ representing external actions from regulators.

Now let us focus on the fraction of drivers $n_D = |D|/N$, where $D = \{i_{k_1}, i_{k_2}, \ldots, i_{k_{|D_t|}}\}$ is the set of the drivers given M^{max}. A large value of n_D would indicate that the interbank system is problematic from a control perspective, since many banks would be responsible for variations of the state of whole network. Conversely, regulators and policy makers would appreciate small values of n_D. Indeed, in this case it is enough to intervene only in a restricted number of banks.

II. B Effect of network aggregation

For temporal networks, the fraction of drivers increases, in general, in the process of aggregating the edges of Δ consecutive instances of the graph. However, the exact scaling of n_D *vs* Δ is difficult to predict. For the interbank lending system, our analysis detects a neat power-law scaling (see Fig. 1 (c) in the main text), and we may wonder if this law is valid any times we add graph instances and independently on the specific dynamics underlying the graph evolution. General arguments, as developed below, show that this is not the case, at least when graph instances are independent.

First, consider a sequence of independent Erdös-Renyi graphs with N nodes and linking probability p and the graph $G^{(\Delta)}$ obtained by aggregating the links of Δ instances (for the sake of clarity, we limit the discussion to unweighted graphs.). It can ben shown that the average degree of $G^{(\Delta)}$ would be $\langle k \rangle^{(\Delta)} = 2p(\Delta)(N-1)$, with $p(\Delta) = [1-(1-p)^{\Delta}]$, and the graph rapidly becomes complete with n_D approaching its lower limit $1/N$. In [9] it has been shown that n_D is approximately an exponential function of $\langle k \rangle$, both for random and scale-free graphs. In our example we would then expect $n_D \approx e^{-c(1-p)^{\Delta}}$, with c constant, rather than a power-law decay.

In the case of identical, independent scale-free graphs, we haven't got a closed-form expression for $\langle k \rangle^{(\Delta)}$. For large $\langle k \rangle$, we would expect $\log n_D \approx -c \sum_{k=0}^{k_{max}} k(F_k^{\Delta} - F_{k-1}^{\Delta})$, where F_k is the cumulative distribution function of the nodes' degree evaluated in k. To get an insight of what scaling should be expected, we simulate $H = 100$ instances of a scale-free graph with $N = 200$ nodes and average degree approximately 6. These numbers correspond (approximately) to the average number of active nodes and their average degree for the interbank lending system. Graphs are generated from the static model [13] with in-degree and out-degree tail exponents equal to 2.7 and 2.15 respectively, as observed empirically in [1]. In Fig. S2 we plot the scaling of n_D *vs* Δ.

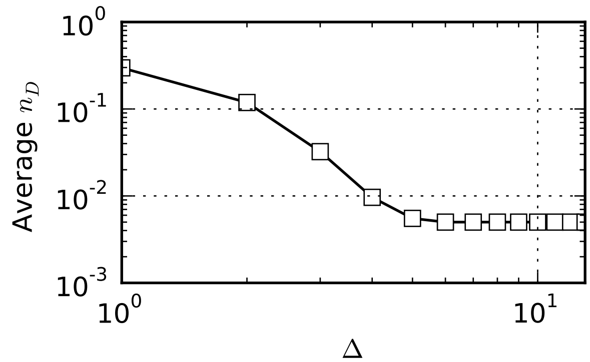

Figure S2: Average fraction of drivers for a network obtained aggregating Δ independent instances of a scale-free graph. Instances are generated from the static model with $N = 200$ nodes, instance's average degree $\langle k \rangle = 6$, in-degree and out-degree tail exponents $\gamma_{in} = 2.7$ and $\gamma_{out} = 2.15$ respectively.

Comparing with Fig. 1 (c) in the main text, here we see a rather different scaling; in particular, when adding independent scale-free graphs, the aggregated network saturates at a much higher speed than for the interbank system.

We conclude that the scaling we observe empirically is a genuine feature of the dynamics and topology of the interbank lending network. In particular, the process of aggregation involves instances that can not be considered realizations of the same graph and, furthermore, the underlying financial dynamics is likely to generated correlations among those instances.

II. C Driver persistence

For a temporal network, like the interbank lending one, a critical aspect is represented by the stability of the set of drivers with respect to time. In order to investigate this issue, we define the *drivers' resilience* as the average fraction of drivers which persists in the network control set after a time lag τ (in days)

$$r_{D,\tau} = \left\langle \frac{D_{u+\tau} \cap D_u}{\mid D_u \mid} \right\rangle \tag{2}$$

where the average is taken with respect to the time index u.

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