DebtRank-transparency: Controlling systemic risk in financial networks

Supplementary Information

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This document is the Supplementary Information (SI) for the manuscript *DebtRank-transparency: Controlling systemic risk in financial networks*.

S1. DETAILS OF THE MODEL

For simplicity every bank i has only a single commercial client, firm i. The number of banks (firms) is B, banks (firms) are indexed with $i = 1, 2, ..., B$. This simplification is justified by the fact that for example in Germany the number of large corporations and the number of relevant banks is of the same order of magnitude, for Germany, 4643[1] large companies versus 2093[2] banks.

Firms

Every time step each firm performs the following tasks in the following order

- repay loans to banks
- file a (random) loan request to their bank (for investments, paying wages, etc.)
- realize random profits or losses from previous investments
- receive the loan (or not) and make investments and paying salaries to households.

The firm's *investments*, are

$$I_i(t) = \sum_{t'=0}^{\tau} a_i(t - t'), \tag{1}$$

where $a_i(t)$ is the amount the firm invests at time t and pays to the households. These investments are tracked for τ time steps, because for simplicity we assume the firm i makes a random return (profit or loss) τ time steps later.

The firm's *cash* is deposited in the bank account of firm i. The size of the deposit at time t is $D_i(t)$. For these deposits the firms receive an interest rate of $r^{\text{f}-\text{deposit}}$.

The *liabilities* of a firm are loans received (and accumulated over time) from its bank, $L_i^{\text{f}}(t)$. For simplicity all loans have a the same maturity of τ time steps. The total amount of loans received by firm i (from bank i) at time t is

$$L_i^{\text{f}}(t) = \sum_{t'=0}^{\tau} l_i(t - t'), \tag{2}$$

where $l_i(t)$ is the size of the loan payed out at t. The *equity* of firm i is obtained by subtracting its liabilities from its cash and assets

$$C_i^{\text{f}}(t) = D_i(t) + I_i(t) - L_i(t), \tag{3}$$

At every time step all firms file new loan requests to their banks. The loan request of firm i follows a random process

$$l_i^{\text{req}}(t) = \zeta_t - \min[0, D_i(t)], \tag{4}$$

where ζ_t is the realization of an i.i.d. random number from a uniform distribution between zero and α, which is fixed at some constant (same for all i). The 'min' is present to ensure that the deposits of a firm is never negative. If the bank of firm i has enough cash reserves available the loan request will be immediately granted and payed out,

$$l_i(t) = \begin{cases} l_i^{\text{req}}(t) & \text{if } R_i(t) \geq l_i^{\text{req}}(t) \\ -\min[0, D_i(t)] & \text{if } l_i^{\text{req}}(t) > R_i(t) \geq -\min[0, D_i(t)] \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

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[1] `http://de.wikipedia.org/wiki/Unternehmen`
[2] `http://de.wikipedia.org/wiki/Deutsches_Bankwesen`

Here $R_i(t)$ is the reserves of bank i (see banks below). The firm pays and interest rate of $r^{\mathrm{f-loan}}$ for the loan. For simplicity, we fix the interest rate $r^{\mathrm{f-loan}}$ to be the same for all firms, see SI Table I. This is of course a somewhat unrealistic assumption. Interest rates for firm loans vary across banks and firms and reflect perceived risks of firms. More realistic heterogeneity of firm interest rates alone would favor profitable, and punish loss-making firms. Banks requiring low interest rates would have lower earnings. Since every bank has only one firm as a customer, banks offering low interest rates would not earn enough. A consistent natural extension to the model would therefore be to introduce an adaptive interest scheme such as in [1], and at the same time, a market for these loans, so that every bank could serve multiple firms. This is one of the aims of the FP7 CRISIS[3] model but beyond the scope of this work.

If the bank of firm i has not enough cash reserves to grant the loan request, the bank at least tries to grant a loan to balance the deposit account of the firm. If the reserves of the Bank and are not sufficient for this, the bank does not grant a loan.

If the firm receives the loan, the corresponding cash is 'invested' (think of paying workers, buying new machines, etc.).

$$a_i(t) = \max[0, l_i(t) - \min[0, D_i(t)]]. \tag{6}$$

Again, the min and max ensure that the firm does not invest more cash than it has. On these investments, firm i makes a random return (profit or loss) τ time steps later.

$$p_i(t) = \xi_i a_i(t - \tau), \tag{7}$$

where $\xi_i \in \mathcal{N}(\mu_i, \sigma^{\mathrm{return}})$, i.e. an i.i.d. random number from a normal distribution with mean μ_i and standard deviation σ^{return}. Each firms starts with an initial equity of $C_i^{\mathrm{f}}(0) = 1$. Their deposits evolve as

$$D_i(t) = (1 + r^{\mathrm{f-deposit}})D_i(t-1) - (1 + r^{\mathrm{f-loan}})l_i(t-\tau) + l_i(t) + p_i(t) - a_i(t). \tag{8}$$

A firm goes bankrupt in either of two cases: (i) its equity falls below a (negative) threshold $C_i^{\mathrm{default}} = -15$ or (ii) if its liquidity is below zero, $D_i(t) < 0$. Negative equity can be the result of a large loss or a series of losses on its investments. Liquidity problems of firms arise when banks do not receive loans from their main bank, i.e. when the bank does not have enough reserves or is unable to raise enough liquidity on the interbank market (see below). In both cases the firm is declared bankrupt and goes out of business.

If a firm i goes out of business it 'sells' its assets $I_i(t)$ to the households to repay as much as possible of the debt to the bank. This means that

$$D_j^{\mathrm{h}}(t) = D_j^{\mathrm{h}}(t) - w_j(t) I_i(t) \tag{9}$$

and

$$D_i^{\mathrm{f}}(t) = D_i^{\mathrm{f}}(t) + I_i(t). \tag{10}$$

Here $D_j^{\mathrm{h}}(t)$ are the deposits of households and $w_j(t)$ is the relative fraction of household deposits at bank j, see next section for details. The main bank of the firm gets the remaining funds and writes off the outstanding debt of the bankrupt firm. If a firm defaults at time t we set $I_i(t) = 0$, $D_i(t) = 0$ and $L_i^{\mathrm{f}}(t) = 0$.

Households

The role of households in the model is to provide a re-allocation and re-distribution mechanism of the economy. Households receive money for labour from their firms, they buy products from other firms, make deposits at banks (which are not necessarily the same as the main bank of their firm), etc. For simplicity, households are modeled as a single representative agent. They hold a bank account at every bank. We denote the deposits the households hold at bank i at time t by $D_i^{\mathrm{h}}(t)$. The relative fraction of household deposits at bank i, (i.e. the market share in total deposits of the bank), is $w_i(t) = D_i^{\mathrm{h}}(t)/\sum_i D_i^{\mathrm{h}}(t)$. These fractions change stochastically over time. We model these

[3] http://www.crisis-economics.eu

changes as a slowly mean-reverting random process, where the deposits mean revert around a market share of $1/B$ (B is the number of banks),

$$\tilde{w}_i(t) = \rho\tilde{w}_i(t-1) + \chi_i(t) + (1-\rho)\frac{1}{B}, \tag{11}$$

where χ_i is from an i.i.d. normal distribution $N(0,\sigma)$. With normalization after every realization, the market shares are $w_i(t) = \tilde{w}_i(t)/\sum_{i=1}\tilde{w}_i(t)$. For $\rho < 1$, $w_i(t)$ mean-reverts around $1/B$, and for appropriate choices of σ, the positivity of $w_i(t)$ can be ensured for all practical purposes. The deposits of the households at bank i evolve as

$$D_i^{\mathrm{h}}(t) = w_i(t)\sum_{i=1}^{B}\left[(1+r^{\mathrm{h}})D_i^{\mathrm{h}}(t-1) - p_i(t) + a_i(t)\right], \tag{12}$$

where r^{h} is the interest rate for these deposits. Deposits of the households are *demand deposits* which can be withdrawn at any time.

Banks

Every bank i has only one firm i as a customer. At each time step banks perform the following tasks:

- collect deposits from households and pay interest at the rate r^{h} for last time step
- hold deposits of firms
- issue new loans to firms at the interest rate $r^{\mathrm{f-loan}}$
- borrow or lend in the interbank market at the interest rate r^{ib}

Banks collect deposits from firms and households and provide loans to firms. The *assets* of bank i are its cash reservers $R_i(t)$ plus the total loans provided to firms, $L_i^{\mathrm{f}}(t)$, and loans to other banks. For IB loans we assume the same time to maturity τ (same duration as firm loans), so that the total outstanding interbank loans of bank i are

$$L_i^{\mathrm{IB}}(t) = \sum_{t'=0}^{\tau}\sum_{j=1}^{B} l_{ji}(t-t'), \tag{13}$$

Note that the entries in $L_{ij}(t) = \sum_{t'=0}^{\tau} l_{ij}(t-t')$ are the liabilities bank i has towards bank j at time t. We use the convention to write liabilities in the rows (second index) of l. If the matrix is read column-wise (transpose of l) we get the assets or claims, banks hold with each other. The *liabilities* of bank i are the deposits of firms $D_i^{\mathrm{f}}(t)$, deposits of households $D_i^{h}(t)$, and IB loans from other banks $B_i^{\mathrm{IB}}(t)$,

$$B_i^{\mathrm{IB}}(t) = \sum_{t'=0}^{\tau}\sum_{j=1}^{B} l_{ij}(t-t'). \tag{14}$$

The equity of a bank is given by subtracting its liabilities from its assets,

$$C_i^{\mathrm{b}}(t) = R_i(t) + L_i^{\mathrm{IB}}(t) + L_i^{\mathrm{f}}(t) - B_i^{\mathrm{IB}}(t) - D_i(t) - D_i^{\mathrm{h}}(t). \tag{15}$$

Banks are the only agents in the model that keep cash reserves. All other agents always deposit their cash in bank accounts. The reserves of banks only change through interbank loans, or, if agents with bank accounts at different banks conduct mutual business, e.g. think of households buying products at different firms which then make a profit and deposit that cash at their bank accounts. However, if a firm receives a loan from its main bank this does not affect the cash reserves of the bank. Each bank starts with an initial reserve $R_i(0) = 1$, which evolves according to

$$R_i(t) = R_i(t-1) + \Delta L_i^{\mathrm{IB}} + \Delta B_i^{\mathrm{IB}} + \Delta D_i^{\mathrm{h}}(t), \tag{16}$$

with

$$\Delta L_i^{\rm IB} = (1+r^{\rm ib})\sum_j l_{ji}(t-\tau) - \sum_j l_{ji}(t),$$
$$\Delta B_i^{\rm IB} = \sum_j l_{ij}(t) - (1+r^{\rm ib})\sum_j l_{ij}(t-\tau), \quad (17)$$
$$\Delta D_i^{\rm h}(t) = D_i^{\rm h}(t) - D_i^{\rm h}(t-1).$$

The equity of a bank is not affected by a change in its reserves. It changes only as a consequence of interest payments or because of credit defaults of firms or other banks.

A bank goes bankrupt if either its equity $C_i(t)$ or reserves $R_i(t)$ become negative. Negative equity can be triggered by the bankruptcy of a firm. Liquidity problems can occur when banks are unable to raise enough liquidity on the interbank market. There is no recovery of IB loans considered, and lending banks write off the loans extended to the defaulted banks.

IB market

Banks can borrow or lend from other banks on the interbank market at interest rate $r^{\rm ib}$. Constant IB interest for all banks is a simplification that is not fully realistic. It is especially invalid in times of turmoil. For feasibility however, we make this simplification and intend to study the potentially important systemic effects of heterogeneous IB interest rates by an extension of this model in a separate work within the FP7 CRISIS framework. Whether two banks consider lending to each other, is specified with the interbank network A. We refer to A as the (symmetric) *bank-relation network*. $A_{ij} = 1$ means that bank i would in general consider lending to- and borrowing from bank j, they have a business relation, and $A_{ij} = 0$ means that i and j have no business relations. If a bank needs additional liquidity (for servicing a firm-loan request) it contacts other banks with which it is connected in the IB network, and issues an IB loan request. It contacts its neighbors in the IB network until the liquidity requirements are satisfied. Note that the order in which these IB loan requests are made on the IB market is random in the normal mode, and follows the ordering according to Debt- or Katz rank in the transparent scheme (the least risky bank is approached first). The IB loan request of bank i is of a size such that an external firm-loan request can be serviced,

$$l_{\rm i}^{\rm IB-req}(t) = \max\left[0, l_{\rm i}^{\rm req}(t) - R_{\rm i}(t)\right]. \quad (18)$$

The contacted bank j will grant the requested loan if it has enough liquidity available. The amount bank i has finally borrowed from bank j at time t is

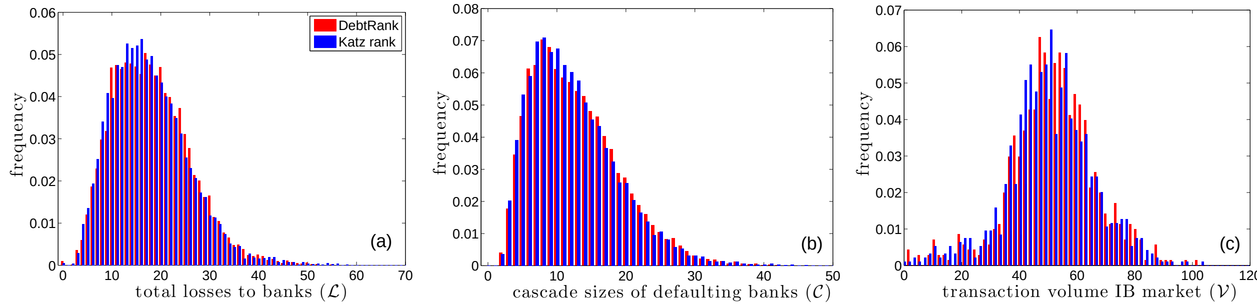
$$l_{ji}(t) = \begin{cases} 0 & \text{if } R_j(t) - l_j^{\rm req}(t) \le 0 \\ l_{\rm i}^{\rm IB-req}(t) & \text{if } R_j(t) - l_j^{\rm req}(t) \ge l_{\rm i}^{\rm IB-req}(t) \\ R_j(t) - l_j^{\rm IB-req}(t) & \text{otherwise.} \end{cases} \quad (19)$$

Note that the loan size depends on the firm-loan extended to firm j in that time step. In case the first contacted bank j can not provide the full requested liquidity the requesting bank i takes a smaller amount ($l_{ji}(t) \le l_{\rm i}^{\rm IB-req}(t)$) and continues by asking the set other banks it has connections to, $\{j \mid A_{ij} = 1\}$, for the remaining funds.

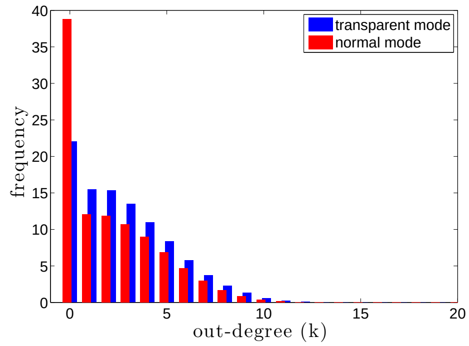
IB market topology

For the bank-relation network A we use two types of networks in our simulations, random graphs of Erdös-Renyi type (ER), and for comparison the more realistic, scale free (SF) networks [2]. For the ER networks we specify the linking probability [3], for any pair of the B banks by $0 < \gamma \le 1$. The SF networks are produced with the Barabási-Albert preferential attachment algorithm [4], using $m = 6$ links, with which any new node is linked to the existing network. If simulations are carried out repeatedly with the same parameters, we generate for each simulation a new bank-relation network.

In our simulations we find the distributions of IB loans ($l_{ji}(t)$), within the range from approximately 0 to 10. The distribution itself is an approximate power law with a slope of about -2, which it is not incompatible with empirical values [2] (not shown).



Supplementary Figure 1. (a) Distributions of losses $\mathcal{L}$, (b) cascade sizes $\mathcal{C}$, and (c) transaction volume in the IB market $\mathcal{V}$, for the Katz rank (blue) and the DebtRank (red) method.



Supplementary Figure 2. Distributions of out-degrees k of the IB liability network $\mathrm{sgn}(L_{ij})$ for the normal (red) and transparent mode (blue) at time $t = 100$. Same parameter settings as in the corresponding figure in the main text.

In SI Fig. 2 the distributions of out-degrees k of the IB liability network $\mathrm{sgn}(L_{ij})$ for the normal (red) and transparent mode (blue) are shown. The simulation parameters are as in the main text. The out-degree distribution (total number of different banks a bank has received loans from) is mainly influenced by the cash needs of a bank. Therefore the out-degree distribution of the transparent mode is similar to the approximate random network distributions also seen in the in-degree of the normal mode. The different number of nodes with a degree of zero is a result of the slightly higher transaction volume in the transparent mode.

Model parameters

All parameters of the model are collected in SI Table I. The simulations for the comparisons of Katz vs. DebtRank, SF vs. ER, and different connectivities, were run with different initial random number seeds. For any realistic situation we assume that $r^{\mathrm{f-loan}} > r^{\mathrm{h}} > r^{\mathrm{ib}} > r^{\mathrm{f-deposit}}$.

Fits to distribution functions

The following fits to the curves have been attempted. The heavy tails in the losses and cascade size in the normal mode, are fitted by a power law. The exponents κ for the various cases are found in SI Table II. The loss and cascade curves in the transparent and fast mode were fitted (least squares) by $\exp[-a\log(x)^2 + b\log(x) - c]$. Values for a, b and c are in SI Table II. For the transaction volume we present the first four moments, mean, variance, skewness, and kurtosis also in SI Table II. Fit ranges were 10-70 (losses) and 10-50 (cascades) for the power laws, 2-35 (losses) and 2-25 (cascades) for the (approximate) log-exponentials.

Supplementary Table I. List of the parameters and initial values as used in the model.

number of banks and firms	$B = F = 100$
interest rate for IB market	$r^{\mathrm{ib}} = 0.0008$
interest rate for deposits of households	$r^{\mathrm{h}} = 0.00085$
interest rate for deposits of firms	$r^{\mathrm{f-deposit}} = 0.0001$
interest rate for loans to firms	$r^{\mathrm{f-loan}} = 0.001$
mean reversion parameter in re-allocation process	$\rho = 0.99$
standard deviation for re-allocation process	$\sigma = 0.0001$
upper limit for loan requests of firms	$\alpha = 3$
mean return of firms on loans	$\mu_{\mathrm{i}} = 1.01$
standard deviation of firm returns	$\sigma^{\mathrm{return}} = 0.3$
initial deposits of households	$D_i^{\mathrm{h}}(0) = 3$
initial equity of banks	$C_i^{\mathrm{b}}(0) = 1$
initial equity of firms	$C_i^{\mathrm{f}}(0) = 1$
ER linking probability if IB network	$\gamma = [0.115, 0.25, 1]$
SF network parameter (number of links per new node)	$m = 6$

Supplementary Table II. Fits to distribution functions

γ/m	NW	Mode	Losses (κ/a b c)	Cascades (κ/a b c)	Volume (mean std kurt skew)
1	ER	normal	1.15	1.37	46.83 15.57 3.90 0.39
1	ER	transparent DR	1.60 8.08 13.20	1.38 5.86 8.88	51.88 15.05 4.91 0.04
1	ER	transparent KR	1.57 7.81 12.62	1.52 6.35 9.22	52.39 14.73 4.18 0.21
1	ER	fast KR	2.02 9.28 13.27	1.82 7.11 9.29	50.28 14.27 4.03 0.13
6	SF	normal	1.42	1.47	47.49 16.06 3.50 0.15
6	SF	transparent	1.81 7.07 9.19	1.74 5.87 7.06	43.63 13.50 3.60 0.22
0.25	ER	normal	1.27	1.39	46.57 15.43 3.57 0.24
0.115	ER	normal	1.66	1.75	47.25 15.46 3.55 0.26
0.25	ER	transparent	2.25 9.87 13.26	2.17 8.12 9.73	46.33 13.61 3.69 0.10
0.115	ER	transparent	2.30 8.85 10.60	2.24 7.57 8.33	42.55 12.77 3.49 0.21

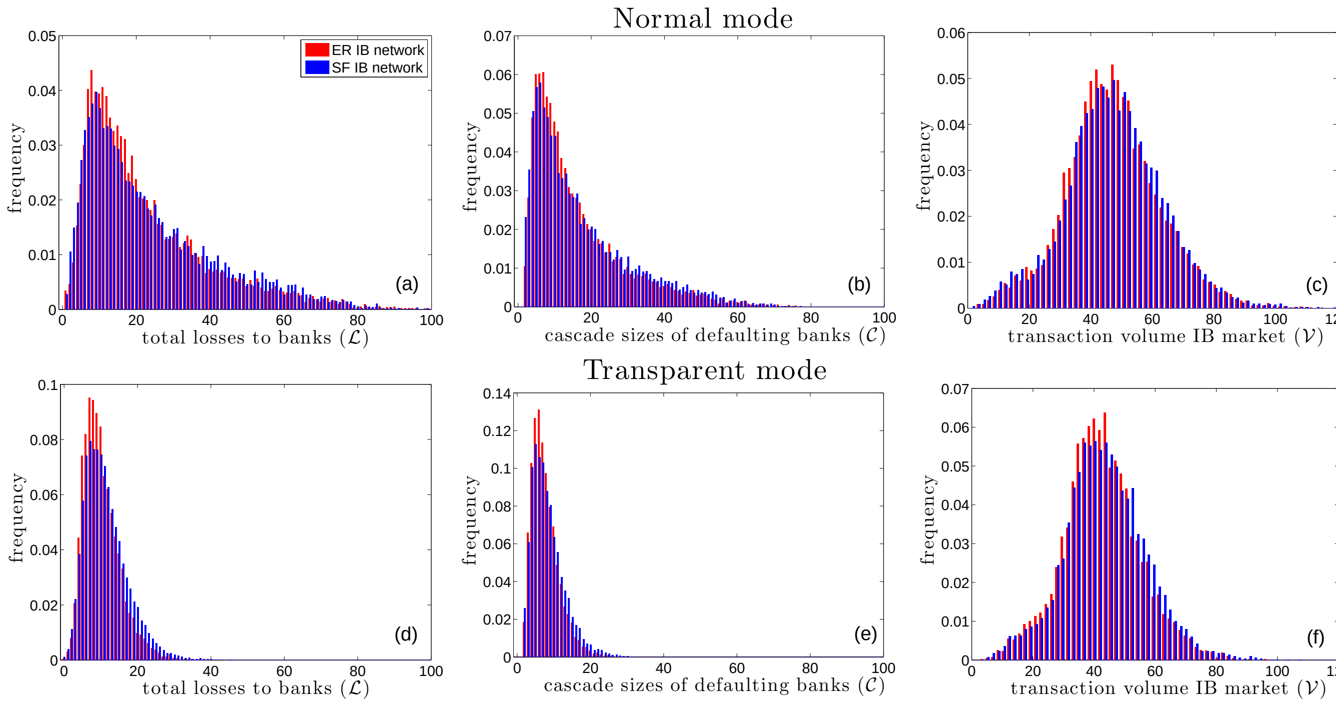
S2. COMPARISON OF DEBTRANK AND KATZ RANK

In SI Fig. 1 we show the distribution functions of the three measures for (a) losses $\mathcal{L}$, (b) cascade sizes $\mathcal{C}$, and (c) transaction volume in the IB market $\mathcal{V}$, for the simulation performed with the DebtRank (red) algorithm and the Katz rank (blue). It is clearly seen, that in all measures there is practically no difference between the methods. Since the Katz rank is easier to implement this might practically favor the Katz rank, especially when it is intended to implement the fast mode.

S3. COMPARISON OF RANDOM AND SCALE-FREE IB NETWORK TOPOLOGIES

For the comparison of the scale free and ER networks we chose an average connectivity of $\langle k \rangle = 11.5$. Again, we compare the normal mode, for which random selection of transaction partners among the neighbors in the IB network is performed (shown in top panels), with the transparent mode, where the selection of transaction partners in the IB networks follows the DebtRank matching (lower panels).

In SI Fig. 3 we show the distribution of the losses to banks $\mathcal{L}$ in (a) and (d) for the normal and transparent mode respectively, the cascade sizes $\mathcal{C}$ in these modes are given in (b) and (e), and the transaction volume in the IB market $\mathcal{V}$ in (c) and (f). The results for the ER IB networks (red) are obtained with $\gamma = 0.115$ and the scale-free networks (blue) have the same average connectivity.



Supplementary Figure 3. Distributions of the losses $\mathcal{L}$ for the normal mode (a) and the transparent mode (d), for an ER (red) and a SF network (blue), both with the same average connectivity $\langle k \rangle = 11.5$. Cascade sizes $\mathcal{C}$ for the normal mode are given in (b) and (e) for the transparent mode, the transaction volume in the IB market $\mathcal{V}$ is seen for the normal in (c), and the transparent mode in (f).

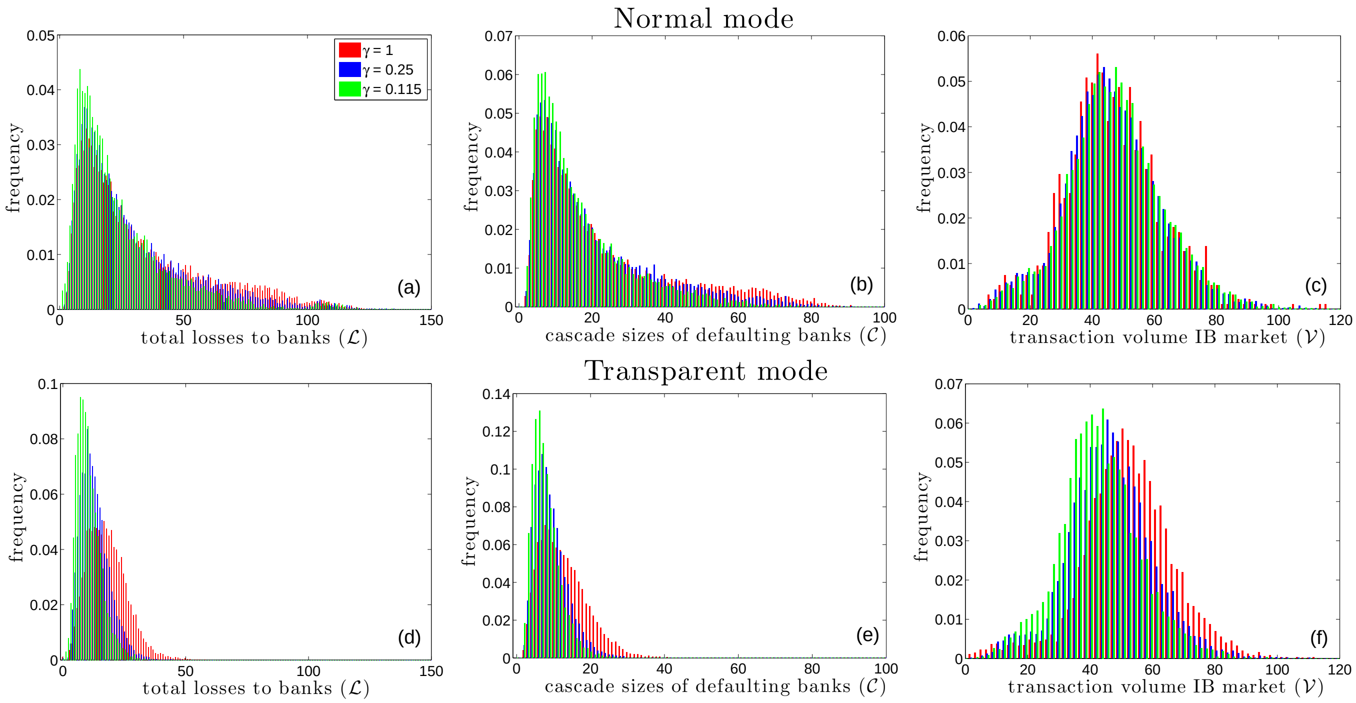
In both cases the SF IB network leads to a slightly less favorable situation in terms of losses and cascades. This is easily understandable in terms vulnerability of SF networks, meaning that if a hub defaults it affects usually more banks than if a node in a random network defaults (with a degree much lower than a hub), see e.g. [5].

In the transaction volume an interesting effect occurs. For the transparent mode the volume decreases a little bit, see SI Table II. This is the case for the lower connectivity, $\langle k \rangle = 11.5$. In the main text (fully connected network) the volume increases in the transparent mode. The situation is made more clear in the next section where different connectivities are discussed.

S4. COMPARISON OF DIFFERENT CONNECTIVITIES OF IB NETWORKS

To compare the effect of connectivity in the IB networks we use a ER network with $\gamma = 1$ (fully connected – as in main text), $\gamma = 0.25$, and $\gamma = 0.115$. In SI Fig. 4 we show the three measures, losses $\mathcal{L}$ in (a) for the normal mode and (d) for the transparent mode for the connectivities $\gamma = 1$ (red), $\gamma = 0.25$ (blue), and $\gamma = 0.115$ (green). The cascade sizes $\mathcal{C}$ in the same modes are given in (b) and (e), and the transaction volume in the IB market $\mathcal{V}$ in (c) and (f), respectively.

In both modes, higher connectivity means higher losses and larger cascades. The transaction volume in the normal mode is almost insensitive to connectivity, but lowers in the transparent mode with decreasing connectivity. When comparing the transaction volumes for the normal and transparent modes we see that in the highly connected case the volume does not change. For the less connected cases, the transaction volume decreases a little in the transparent mode.



Supplementary Figure 4. Distribution of losses $\mathcal{L}$ in the normal (a) and transparent mode (d), for a ER network with $\gamma = 1$ (red), $\gamma = 0.25$ (blue), and $\gamma = 0.115$ (green). The cascade sizes $\mathcal{C}$ for the normal mode are given in (b) and (e) for the transparent mode, the transaction volume $\mathcal{V}$ is seen for the normal in (c), and the transparent mode in (f).

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