Supplementary Material: Palmer et al. An inverse power law distribution of crosslink lifetimes predicts fractional derivative viscoelasticity in biological tissue

In this Supplement, we will evaluate the integrals of Equation 22 reiterated here:

$$\tilde{Y}(\omega) = \xi - \left(\frac{\xi}{\bar{t}_{att}\left(e^{-(k+2)} + k + 1\right)} \right) \left[\int_0^{t_{crit}} \left(e^{-(k+2)} + (k+1) e^{-(k+2)(\tau / t_{crit})} \right) e^{-i\omega\tau} \, d\tau + \right.$$

$$\left. + \int_{t_{crit}}^{\infty} (k+2) e^{-(k+2)} (\tau / t_{crit})^{-(k+1)} e^{-i\omega\tau} \, d\tau \right] \qquad \text{(Eq S.1)}$$

In the main text, we have used a method suggested by Newman and colleagues (1, 2) to provide a mathematical description for $T_{att}(\tau)$ and $P(\tau)$ over the range $0 < \tau < \infty$. For the purposes of this paper, the most important feature of this description of $P(\tau)$ is the inverse power law term for $\tau > t_{crit}$. There are many other descriptions that could have been used for $\tau < t_{crit}$ and not change the most important outcome of the analysis below, i.e., the result that describes a fractional derivative complex modulus. The first integral term is evaluated according to Gradshteyn and Ryzhik (3), page 92, section 2.311:

$$\int_0^{t_{crit}} \left(e^{-(k+2)} + (k+1) e^{-(k+2)(\tau / t_{crit})} \right) e^{-i\omega\tau} \, d\tau =$$

$$\frac{e^{-(k+2)} t_{crit}}{i\omega t_{crit}} \left(1 - e^{-i\omega t_{crit}} \right) + \frac{(k+1) t_{crit}}{(k+2) + i\omega t_{crit}} \left(1 - e^{-[(k+2) + i\omega t_{crit}]} \right) \qquad \text{(Eq S.2)}$$

We can use the series expansion of the (1-exp(arg))/arg terms to provide the following form:

$$\int_0^{t_{crit}} \left(e^{-(k+2)} + (k+1) e^{-(k+2)(\tau / t_{crit})} \right) e^{-i\omega\tau} \, d\tau =$$

$$t_{crit} \left[e^{-(k+2)} \sum_{n=1}^{\infty} \frac{(-i\omega t_{crit})^{n-1}}{n!} + (k+1) \sum_{n=1}^{\infty} \frac{\left[-\left((k+2) + i\omega t_{crit} \right) \right]^{n-1}}{n!} \right] \qquad \text{(Eq S.3)}$$

In evaluating the second integral term, we apply t_{crit} as a scaling factor to τ.

$$\int_{t_{crit}}^{\infty} (k+2) e^{-(k+2)} (\tau / t_{crit})^{-(k+1)} e^{-i\omega\tau} \, d\tau =$$

$$t_{crit} (k+2) e^{-(k+2)} \int_1^{\infty} (\tau / t_{crit})^{-(k+1)} e^{-i\omega t_{crit} (\tau / t_{crit})} \, d\left(\tau / t_{crit} \right) \qquad \text{(Eq S.4)}$$

The integral is then evaluated according to Gradshteyn and Ryzhik (3), page 317, section 3.381.

$$\int_{t_{crit}}^{\infty} (k+2) e^{-(k+2)} (\tau / t_{crit})^{-(k+1)} e^{-i\omega\tau} \, d\tau =$$

$$t_{crit} (k+2) e^{-(k+2)} (i\omega t_{crit})^{k} \, \Gamma(-k, i\omega t_{crit}) \qquad \text{(Eq S.5)}$$

The incomplete upper gamma function, $\Gamma(-k, i\omega t_{crit})$ can be written as a series expansion according to Gradshteyn and Ryzhik (3) page 941, section 8.354, equation #2:

$$\Gamma(-k, i\omega t_{crit}) = \Gamma(-k) - (i\omega t_{crit})^{-k} \sum_{n=0}^{\infty} \frac{(-i\omega t_{crit})^n}{(n-k)n!} \qquad \text{(Eq S.6)}$$

Noting again the identity $\Gamma(-k) = -\Gamma(1-k)/k$ and allowing the zeroth-ordered term of the series to stand from the summation, we can write the evaluation of the second integral as follows:

$$\int_{t_{crit}}^{\infty} (k+2)e^{-(k+2)}(\tau / t_{crit})^{-(k+1)} e^{-i\omega\tau} d\tau =$$

$$- t_{crit}(k+2)e^{-(k+2)}\left[(i\omega t_{crit})^k \frac{\Gamma(1-k)}{k} - \frac{1}{k} + \sum_{n=1}^{\infty} \frac{(-i\omega t_{crit})^n}{(n-k)n!} \right] \qquad \text{(Eq S.7)}$$

Assessments of the first and second integrals of Equation S.1 have now been provided in Equations S.3 and S.7. Using the description for $\bar{t}_{att}$ provided in Equation 19, which cancels t_{crit} in the numerators, Equation S.1 can be rewritten as follows:

$$\begin{aligned}
\tilde{Y}(\omega) = {} & \xi \frac{k\left(e^{-(k+2)} + k + 1\right) + (k+2)(k+2)e^{-(k+2)}}{k\left(e^{-(k+2)} + k + 1\right) + (k+2)(k+2)e^{-(k+2)}} \\
& - \xi\left(\frac{k(k+2)}{k\left(e^{-(k+2)} + k + 1\right) + (k+2)(k+2)e^{-(k+2)}} \right) e^{-(k+2)} \sum_{n=1}^{\infty} \frac{(-i\omega t_{crit})^{n-1}}{n!} \\
& - \xi\left(\frac{k(k+2)}{k\left(e^{-(k+2)} + k + 1\right) + (k+2)(k+2)e^{-(k+2)}} \right) (k+1) \sum_{n=1}^{\infty} \frac{\left[-\left((k+2) + i\omega t_{crit}\right)\right]^{n-1}}{n!} \\
& + \xi\left(\frac{k(k+2)(k+2)e^{-(k+2)}}{k\left(e^{-(k+2)} + k + 1\right) + (k+2)(k+2)e^{-(k+2)}} \right) \left[(i\omega t_{crit})^k \frac{\Gamma(1-k)}{k} - \frac{1}{k} + \sum_{n=1}^{\infty} \frac{(-i\omega t_{crit})^n}{(n-k)n!} \right]
\end{aligned} \qquad \text{(Eq S.8)}$$

After rearranging the $1/k$ term, we arrive with the following result:

$$\begin{aligned}
\tilde{Y}(\omega) = {} & \xi \frac{k\left(e^{-(k+2)} + k + 1\right) + (k+2)(k+2)e^{-(k+2)}}{k\left(e^{-(k+2)} + k + 1\right) + (k+2)(k+2)e^{-(k+2)}} - \xi\left(\frac{(k+2)(k+2)e^{-(k+2)}}{k\left(e^{-(k+2)} + k + 1\right) + (k+2)(k+2)e^{-(k+2)}} \right) \\
& - \xi\left(\frac{k(k+2)}{k\left(e^{-(k+2)} + k + 1\right) + (k+2)(k+2)e^{-(k+2)}} \right) e^{-(k+2)} \sum_{n=1}^{\infty} \frac{(-i\omega t_{crit})^{n-1}}{n!} \\
& - \xi\left(\frac{k(k+2)}{k\left(e^{-(k+2)} + k + 1\right) + (k+2)(k+2)e^{-(k+2)}} \right) (k+1) \sum_{n=1}^{\infty} \frac{\left[-\left((k+2) + i\omega t_{crit}\right)\right]^{n-1}}{n!} \\
& + \xi\left(\frac{k(k+2)(k+2)e^{-(k+2)}}{k\left(e^{-(k+2)} + k + 1\right) + (k+2)(k+2)e^{-(k+2)}} \right) \left[(i\omega t_{crit})^k \frac{\Gamma(1-k)}{k} + \sum_{n=1}^{\infty} \frac{(-i\omega t_{crit})^n}{(n-k)n!} \right]
\end{aligned} \qquad \text{(Eq S.9)}$$

Now we rearrange terms in accordance to their influence on the measured complex modulus predicted to arise from the approximate inverse power law distribution described in Equation 17 of the main text:

$$\begin{aligned}\tilde{Y}(\omega) = {} & \xi\left(\frac{(k+2)(k+2)e^{-(k+2)}}{k\left(e^{-(k+2)}+k+1\right)+(k+2)(k+2)e^{-(k+2)}}\right)(i\omega t_{crit})^{k}\,\Gamma(1-k) \\ & +\xi\left(\frac{k\left(e^{-(k+2)}+k+1\right)}{k\left(e^{-(k+2)}+k+1\right)+(k+2)(k+2)e^{-(k+2)}}\right) \\ & -\xi\left(\frac{k(k+2)}{k\left(e^{-(k+2)}+k+1\right)+(k+2)(k+2)e^{-(k+2)}}\right)e^{-(k+2)}\sum_{n=1}^{\infty}\frac{(-i\omega t_{crit})^{n-1}}{n!} \\ & -\xi\left(\frac{k(k+2)}{k\left(e^{-(k+2)}+k+1\right)+(k+2)(k+2)e^{-(k+2)}}\right)(k+1)\sum_{n=1}^{\infty}\frac{\left[-\left((k+2)+i\omega t_{crit}\right)\right]^{n-1}}{n!} \\ & +\xi\left(\frac{k(k+2)}{k\left(e^{-(k+2)}+k+1\right)+(k+2)(k+2)e^{-(k+2)}}\right)(k+2)e^{-(k+2)}\sum_{n=1}^{\infty}\frac{(-i\omega t_{crit})^{n}}{(n-k)n!}\end{aligned}$$

(Eq S.10)

References

1. Newman, M. E. J. 2005. Power laws, Pareto distributions and Zipf's law. Contemporary Physics 46:323-351.

2. Clauset, A., C. R. Shalizi, and M. E. J. Newman. 2009. Power-law distributions in empirical data. SIAM Review 51:661-703.

3. Gradshteyn, I. S., and I. M. Ryzhik, editors. 1980. Tables of Integrals, Series, and Products. Academic Press.