Appendix B: Partial Variance Calculations in the θ-space

Based on eq. (6), $V^{(x_i)} = V_\theta(E_\theta(y\,|\,\theta_i))$. To calculate $V^{(x_i)}$, we first calculate

$$\begin{aligned} E_\theta(y\,|\,\theta_i) &= E(g(G(\theta_1),...,G(\theta_n)\,|\,\theta_i) \\ &= \int_0^{2\pi}\cdots\int_0^{2\pi} f(\theta_1,...,\theta_{i-1}\theta_{i+1},...,\theta_n\,|\,\theta_i)\, g(G(\theta_1),...,G(\theta_n))\,d\theta_1...d\theta_{i-1}d\theta_{i+1}\cdots d\theta_n. \end{aligned}$$

where $f(\theta_1,...,\theta_{i-1}\theta_{i+1},...,\theta_n\,|\,\theta_i)$ is the conditional probability density function of $(\theta_1,...,\theta_{i-1}\theta_{i+1},...,\theta_n)$ given θ_i. Since $\theta_1,...,\theta_n$ are independently and uniformly distributed between 0 and 2π, we have

$$E_\theta(y\,|\,\theta_i) = (\frac{1}{2\pi})^{n-1}\int_0^{2\pi}\cdots\int_0^{2\pi} g(G(\theta_1),...,G(\theta_n))\,d\theta_1...d\theta_{i-1}d\theta_{i+1}\cdots d\theta_n$$

Using the multiple Fourier transformation in eq. (12), we get

$$E_\theta(y\,|\,\theta_i) = (\frac{1}{2\pi})^{n-1}\int_0^{2\pi}\cdots\int_0^{2\pi}\sum_{r_1,...,r_n=-\infty}^{+\infty} C_{r_1,...,r_n} e^{i(r_1\theta_1+\cdots+r_n\theta_n)}\,d\theta_1...d\theta_{i-1}d\theta_{i+1}\cdots d\theta_n .$$

In view that

$$\int_0^{2\pi} e^{i(r_i\theta_i)}\,d\theta_i = 0 ,$$

we get

$$E_\theta(y\,|\,\theta_i) = \sum_{r_i=-\infty}^{+\infty} C_{0\cdots r_i\cdots 0}\, e^{\mathbf{i}(r_i\theta_i)} . \qquad \text{(B1)}$$

Based on Parseval's theorem

$$E_{\theta_i}(E_\theta(y\,|\,\theta_i))^2 = \sum_{r_i=-\infty}^{+\infty} |\,C_{0\cdots r_i\cdots 0}\,|^2 \qquad \text{(B2)}$$

In view that

$$\begin{aligned} E(C_{0\cdots r_i\cdots 0}\, e^{\mathbf{i}(r_i\theta_i)}) &= \frac{1}{2\pi}\int_0^{2\pi} C_{0\cdots r_i\cdots 0}\, e^{\mathbf{i}(r_i\theta_i)}\,d\theta_i \\ &= \begin{cases} 0, r_i \neq 0 \\ C_{0\cdots 0\cdots 0}, r_i = 0, \end{cases} \end{aligned}$$

we have

$$E_{\theta_i}(E_\theta(y\,|\,\theta_i)) = C_{0\cdots 0\cdots 0}. \qquad \text{(B3)}$$

Based on eq. (B2) and eq. (B3), we have

$$V^{(x_i)} = V_\theta(E_\theta(y\,|\,\theta_i)) = \sum_{|r_i|=1}^{+\infty} |\, C^{(\theta)}{}_{0\cdots r_i \cdots 0}\,|^2 \,. \qquad \text{(B4)}$$

For the partial variance contributed by uncertainties in parameters x_i and x_j, $V^{(x_i,x_j)} = V_\theta(E_\theta(y\,|\,\theta_i,\theta_j))$, we first calculate

$$\begin{aligned} &E_\theta(y\,|\,\theta_i,\theta_j) = E(g(G(\theta_1),...,G(\theta_n))\,|\,\theta_i,\theta_j) \\ &= \int_0^{2\pi}\cdots\int_0^{2\pi} f((\theta_1,..,\theta_n \setminus \theta_i,\theta_j)\,|\,\theta_i,\theta_j) g(G(\theta_1),...,G(\theta_n))(d\theta_1\cdots d\theta_n \setminus d\theta_i d\theta_j). \end{aligned}$$

where $(.\setminus.)$ represents all the elements in the first term excluding the elements in the second term and $f((\theta_1,..,\theta_n \setminus \theta_i,\theta_j)\,|\,\theta_i,\theta_j)$ is the conditional probability density function of $(\theta_1,..,\theta_n \setminus \theta_i,\theta_j)$ given (θ_i,θ_j). Since $\theta_1,...,\theta_n$ are independently and uniformly distributed, we have

$$E_\theta(y\,|\,\theta_i,\theta_j) = (\frac{1}{2\pi})^{n-2}\int_0^{2\pi}\cdots\int_0^{2\pi} g(G(\theta_1),...,G(\theta_n))(d\theta_1\cdots d\theta_n \setminus d\theta_i d\theta_j)\,.$$

Using the multiple Fourier transformation in eq. (6), we get

$$\begin{aligned} E_\theta(y\,|\,\theta_i,\theta_j) &= (\frac{1}{2\pi})^{n-2}\int_0^{2\pi}\cdots\int_0^{2\pi} \sum_{r_1,...,r_n=-\infty}^{+\infty} C^{(\theta)}{}_{r_1,...,r_n} e^{i(r_1\theta_1+\cdots+r_n\theta_n)} (d\theta_1\cdots d\theta_n \setminus d\theta_i d\theta_j) \\ &= \sum_{r_i,r_j=-\infty}^{+\infty} C^{(\theta)}{}_{0\cdot\cdot r_i \cdot\cdot r_j \cdot\cdot 0}\, e^{\mathbf{i}(r_i\theta_i+r_j\theta_j)}. \end{aligned}$$

Similar to the Parseval's theorem, we can show that

$$\begin{aligned} &E_{\theta_i,\theta_j}(E_\theta(y\,|\,\theta_i,\theta_j))^2 \\ &= (\frac{1}{2\pi})^2\int_0^{2\pi}\int_0^{2\pi}(\sum_{r_i,r_j=-\infty}^{+\infty} C_{0\cdot\cdot r_i\cdot\cdot r_j\cdot\cdot 0}\, e^{\mathbf{i}(r_i\theta_i+r_j\theta_j)})(\sum_{r_i,r_j=-\infty}^{+\infty} \overline{C}_{0\cdot\cdot r_i\cdot\cdot r_j\cdot\cdot 0}\, e^{-\mathbf{i}(r_i\theta_i+r_j\theta_j)}) d\theta_i d\theta_j \\ &= (\frac{1}{2\pi})^2\{\int_0^{2\pi}\int_0^{2\pi}\sum_{r_i,r_j=-\infty}^{+\infty} |\, C_{0\cdot\cdot r_i\cdot\cdot r_j\cdot\cdot 0}\,|^2 d\theta_i d\theta_j \\ &\quad + \sum_{l_i,l_j,r_i,r_j=-\infty,\, l_i\neq r_i \text{ or } l_j\neq r_j}^{+\infty} \int_0^{2\pi}\int_0^{2\pi} C_{0\cdot\cdot r_i\cdot\cdot r_j\cdot\cdot 0}\, e^{\mathbf{i}(r_i\theta_i+r_j\theta_j)}\, \overline{C}_{0\cdot\cdot l_i\cdot\cdot l_j\cdot\cdot 0}\, e^{-\mathbf{i}(l_i\theta_i+l_j\theta_j)} d\theta_i d\theta_j\}, \end{aligned}$$

where $\overline{C}_{0\cdot\cdot r_i\cdot\cdot r_j\cdot\cdot 0}$ indicates the conjugate of $C_{0\cdot\cdot r_i\cdot\cdot r_j\cdot\cdot 0}$. Using the fact that

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} C_{0\cdots r_i\cdots r_j\cdots 0} e^{\mathbf{i}(r_i\theta_i + r_j\theta_j)} \overline{C}_{0\cdots l_i\cdots l_j\cdots 0} e^{-\mathbf{i}(l_i\theta_i + l_j\theta_j)} d\theta_i d\theta_j, l_i \neq r_i \ or \ l_j \neq r_j \\
&= C_{0\cdots r_i\cdots r_j\cdots 0} \overline{C}_{0\cdots l_i\cdots l_j\cdots 0} \int_0^{2\pi} e^{\mathbf{i}(r_i - l_i)\theta_i} d\theta_i \int_0^{2\pi} e^{\mathbf{i}(r_j - l_j)\theta_j} d\theta_j \\
&= 0,
\end{aligned}$$

we have

$$E_{\theta_i,\theta_j}(E_\theta(y \mid \theta_i, \theta_j))^2 = \sum_{r_i, r_j = -\infty}^{+\infty} |C^{(\theta)}{}_{0\cdots r_i\cdots r_j\cdots 0}|^2. \tag{B5}$$

Based on eq. (B3) and eq. (B5), we have

$$\begin{aligned}
V^{(x_i,x_j)} &= V_\theta(E_\theta(y \mid \theta_i, \theta_j)) \\
&= \sum_{|r_i|,|r_j|=1}^{+\infty} |C^{(\theta)}{}_{0\cdots r_i\cdots r_j\cdots 0}|^2 + \sum_{|r_i|=1}^{+\infty} |C^{(\theta)}{}_{0\cdots r_i\cdots\cdots 0}|^2 + \sum_{|r_j|=1}^{+\infty} |C^{(\theta)}{}_{0\cdots\cdots r_j\cdots 0}|^2
\end{aligned} \tag{B6}$$

Using eq. (8), eq. (B4) and eq. (B6), it is clear that

$$\begin{aligned}
V_{x_i x_j} &= V^{(x_i,x_j)} - V_{x_i} - V_{x_j} \\
&= \sum_{|r_i|,|r_j|=1}^{+\infty} |C^{(\theta)}{}_{0\cdots r_i\cdots r_j\cdots 0}|^2.
\end{aligned} \tag{B7}$$

Using similar equations as eq. (B5) and eq. (B6), it can also be shown that

$$\begin{aligned}
V_{x_i x_j x_k} &= \sum_{|r_i|,|r_j|,|r_k|=1}^{+\infty} |C^{(\theta)}{}_{00\cdots r_i\cdots r_j\cdots r_k\cdots 00}|^2 \\
&\cdots \\
V_{x_1\cdots x_n} &= \sum_{|r_1|,\cdots,|r_n|=1}^{+\infty} |C^{(\theta)}{}_{r_1\cdots\cdots\cdots r_n}|^2.
\end{aligned} \tag{B8}$$