Appendix C: Estimation Errors of Fourier Coefficients over the Auxiliary Variable *s*

The rectangle rule is a numerical approach to calculate the integral of a function $\psi(s)$. For the calculation of Fourier coefficients over the auxiliary variable *s* in eq. (23), we have $\psi(s) = g(G(s))\cos(ks)$ or $\psi(s) = g(G(s))\sin(ks)$. The rectangle rule first divides the range of s (in this paper, s is between 0 and 2π) into *N* small equal intervals. Each interval will form a rectangle with height determined by *f*(s) at the mid points of the interval (see Figure C1 for a better understanding). Then the integral ($\int_0^{2\pi} \psi(s)ds$) is approximated by the sum of areas of rectangles. Mathematically, the integral can be calculated as follows,

$$\int_0^{2\pi} \psi(s)ds = \frac{2\pi}{N}\sum_{j=1}^{N}\psi(s^{(j)}) \tag{C1}$$

where $\frac{2\pi}{N}$ is the rectangle width and $s^{(j)}$ is the mid-point at the *jth* interval defined by the grid samples in eq. (19). Using eq. (C1), it can be easily shown that the Fourier coefficients over the auxiliary variable *s* in eq. (23) can be calculated using the rectangle rule as follows

$$\frac{1}{2\pi}\int_0^{2\pi} g(G(\varphi(\omega_1 s)),...,G(\varphi(\omega_n s)))\cos(ks)ds = \frac{1}{N}\sum_{j=1}^{N} g(G(\varphi(\omega_1 s^{(j)})),...,G(\varphi(\omega_n s^{(j)})))\cos(ks^{(j)})$$
$$\frac{1}{2\pi}\int_0^{2\pi} g(G(\varphi(\omega_1 s)),...,G(\varphi(\omega_n s)))\sin(ks)ds = \frac{1}{N}\sum_{j=1}^{N} g(G(\varphi(\omega_1 s^{(j)})),...,G(\varphi(\omega_n s^{(j)})))\sin(ks^{(j)}).$$

Thus, the sample mean based on grid samples in eq. eq. (26) is equivalent to the numerical integral using the rectangle rule. If ψ (s) is differentiable to the second-order, the estimation error (*e*) for the numerical integral using the rectangle rule decays as the square of interval length (Davis and Rabinowitz 1984). Mathematically, we have

$$e \leq \frac{\Delta^2 (b-a)}{24} \max_{s \in [0,2\pi]} \psi''(s)$$

where Δ is the interval length; a is the lower range of integral; and b is the upper range of integral; and $\psi''(s)$ is the second derivative. For *s* between 0 and 2π, we have

$$e \leq \frac{\pi^3}{3N^2} \max_{s \in [0,2\pi]} \psi''(s) \qquad \text{(C2)}$$

which suggests that the estimation error decays at a rate of $1/N^2$ with an increasing sample size.

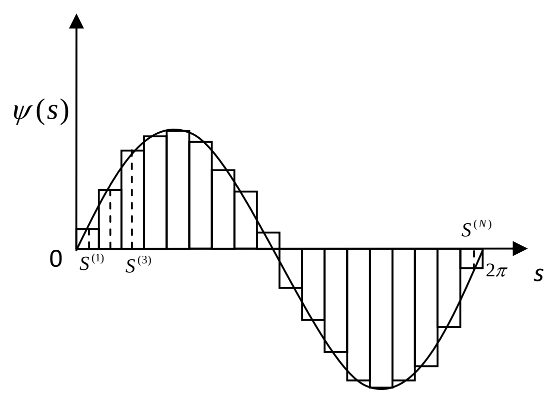

Figure C1 Illustration of the numerical integral using the rectangle rule.

References

Davis, P. J., and P. Rabinowitz. 1984. Methods of numerical integration, 2nd edition. Academic Press, Orlando.