Appendix E: Estimation Error of Fourier Coefficients for Simple Random Sampling

For the Fourier coefficient defined in eq. (13), we define

$$C^{(\theta)}{}_{r_1..r_n} = a^{(\theta)}{}_{r_1..r_n} - \mathbf{i}b^{(\theta)}{}_{r_1..r_n} \qquad \text{(E1)}$$

where $a^{(\theta)}{}_{r_1..r_n}$ is the cosine Fourier coefficient and $b^{(\theta)}{}_{r_1..r_n}$ is the sine Fourier coefficient with

$$\begin{aligned} a^{(\theta)}{}_{r_1..r_n} &= (\frac{1}{2\pi})^n \int_0^{2\pi} \cdots \int_0^{2\pi} g(G(\theta_1),...,G(\theta_n))Cos(r_1\theta_1 + \cdots + r_n\theta_n)d\theta_1 \cdots d\theta_n \\ &= E(g(G(\theta_1),...,G(\theta_n))Cos(r_1\theta_1 + \cdots + r_n\theta_n)), \\ b^{(\theta)}{}_{r_1..r_n} &= (\frac{1}{2\pi})^n \int_0^{2\pi} \cdots \int_0^{2\pi} g(G(\theta_1),...,G(\theta_n))Sin(r_1\theta_1 + \cdots + r_n\theta_n)d\theta_1 \cdots d\theta_n \\ &= E(g(G(\theta_1),...,G(\theta_n))Sin(r_1\theta_1 + \cdots + r_n\theta_n)). \end{aligned} \qquad \text{(E2)}$$

For the estimation of Fourier coefficient $C^{(\theta)}{}_{00\cdots r_i\cdots\cdots 00}$ in eq.(15), we have

$$\hat{C}^{(\theta)}_{r_1..r_n} = \hat{a}^{(\theta)}_{r_1..r_n} - \mathbf{i}\hat{b}^{(\theta)}_{r_1..r_n} \qquad \text{(E3)}$$

where

$$\begin{aligned} \hat{a}^{(\theta)}_{r_1..r_n} &= \frac{1}{N}\sum_{j=1}^{N}[g(G(\theta_1^{(j)}),...,G(\theta_n^{(j)}))Cos(r_1\theta_1^{(j)} + \cdots + r_n\theta_n^{(j)})], \\ \hat{b}^{(\theta)}_{r_1..r_n} &= \frac{1}{N}\sum_{j=1}^{N}[g(G(\theta_1^{(j)}),...,G(\theta_n^{(j)}))Sin(r_1\theta_1^{(j)} + \cdots + r_n\theta_n^{(j)})]. \end{aligned} \qquad \text{(E4)}$$

Clearly, both $\hat{a}^{(\theta)}_{r_1..r_n}$ and $\hat{b}^{(\theta)}_{r_1..r_n}$ are unbiased estimator in view that

$$\begin{aligned} E(\hat{a}^{(\theta)}_{r_1..r_n}) &= a^{(\theta)}{}_{r_1..r_n}, \\ E(\hat{b}^{(\theta)}_{r_1..r_n}) &= b^{(\theta)}{}_{r_1..r_n}. \end{aligned} \qquad \text{(E5)}$$

In view that $\{\theta_1^{(j)},...,\theta_n^{(j)}, j = 1,...,N\}$ are random sample drawn in the θ-space, the variance of $\hat{a}^{(\theta)}_{r_1..r_n}$ [i.e., $V(\hat{a}^{(\theta)}_{r_1..r_n})$] can be estimated as follows,

$$\begin{aligned} V(\hat{a}^{(\theta)}_{r_1..r_n}) &= \frac{1}{N}V[g(G(\theta_1),...,G(\theta_n))Cos(r_1\theta_1 + \cdots + r_n\theta_n)] \\ &= \frac{1}{N}\{E[g^2(G(\theta_1),...,G(\theta_n))Cos^2(r_1\theta_1 + \cdots + r_n\theta_n)] - (a^{(\theta)}{}_{r_1..r_n})^2\}. \end{aligned} \qquad \text{(E6)}$$

Thus, we can calculate the expected value of $(\hat{a}^{(\theta)}_{r_1..r_n})^2$ as follows,

$$E(\hat{a}^{(\theta)}_{r_1..r_n})^2 = V(\hat{a}^{(\theta)}_{r_1..r_n}) + E^2(\hat{a}^{(\theta)}_{r_1..r_n})\,. \tag{E7}$$

Based on eq. (E5) and (E6), $E(\hat{a}^{(\theta)}_{r_1..r_n})^2$ in eq. (E7) can be estimated as follows,

$$\begin{aligned} E(\hat{a}^{(\theta)}_{r_1..r_n})^2 = &\frac{1}{N}\{E[g^2(G(\theta_1),...,G(\theta_n))Cos^2(r_1\theta_1+\cdots+r_n\theta_n)]-(a^{(\theta)}{}_{r_1..r_n})^2\} \\ &+(a^{(\theta)}{}_{r_1..r_n})^2. \end{aligned}\,. \tag{E8}$$

Similarly, we can show that the expected value of $\hat{b}^{(\theta)}_{r_1..r_n}$

$$\begin{aligned} E(\hat{b}^{(\theta)}_{r_1..r_n})^2 = &\frac{1}{N}\{E[g^2(G(\theta_1),...,G(\theta_n))Sin^2(r_1\theta_1+\cdots+r_n\theta_n)]-(b^{(\theta)}{}_{r_1..r_n})^2\} \\ &+(b^{(\theta)}{}_{r_1..r_n})^2. \end{aligned} \tag{E9}$$

Based on eq. (E3), (E8) and (E9), the expected value of amplitude $|\hat{C}^{(\theta)}_{r_1..r_n}|^2$ can be estimated as follows,

$$\begin{aligned} E(\left|\hat{C}^{(\theta)}_{r_1..r_n}\right|^2) &= E[(\hat{a}^{(\theta)}_{r_1..r_n})^2+(\hat{b}^{(\theta)}_{r_1..r_n})^2] \\ &= E[(\hat{a}^{(\theta)}_{r_1..r_n})^2]+E[(\hat{b}^{(\theta)}_{r_1..r_n})^2] \\ &= (a^{(\theta)}{}_{r_1..r_n})^2+(b^{(\theta)}{}_{r_1..r_n})^2+\delta_e \\ &= \left|C^{(\theta)}{}_{r_1..r_n}\right|^2+\delta_e \end{aligned} \tag{E10}$$

where

$$\begin{aligned} \delta_e &= \frac{1}{N}\{E[g(G(\theta_1),...,G(\theta_n))]^2-[(a^{(\theta)}{}_{r_1..r_n})^2+(b^{(\theta)}{}_{r_1..r_n})^2]\} \\ &= \frac{1}{N}\{E[g(G(\theta_1),...,G(\theta_n))]^2-\left|C^{(\theta)}{}_{r_1..r_n}\right|^2\}. \end{aligned} \tag{E11}$$

Based on eq.(E4) and the Central Limit Theorem, for a large sample size *N*, the estimation error of Fourier coefficient $\hat{a}^{(\theta)}_{r_1..r_n}$ and $\hat{b}^{(\theta)}_{r_1..r_n}$ will follow a normal distribution. Statistically, we have

$$\begin{aligned} \hat{a}^{(\theta)}_{r_1..r_n} &\sim \ddot{N}(a^{(\theta)}{}_{r_1..r_n}, V(\hat{a}^{(\theta)}_{r_1..r_n})), \\ \hat{b}^{(\theta)}_{r_1..r_n} &\sim \ddot{N}(b^{(\theta)}{}_{r_1..r_n}, V(\hat{b}^{(\theta)}_{r_1..r_n})), \end{aligned} \tag{E12}$$

where $\ddot{N}(u,v)$ represents a normal distribution with mean *u* and variance *v*. Based on eq. (E6), this suggests that the estimation error for $C^{(\theta)}{}_{r_1..r_n}$ decays at a rate of

$\frac{1}{\sqrt{N}}$ with an increasing sample size *N*.