Appendix G: Variance Decomposition for the Test Model

For the test model in eq. (45), we have the variance of the model output y decomposed as follows

$$V(y) = \sum_{i=1}^{3} i^2 \, V(x_i{}^2) + V(x_1 x_2) + 4V(x_2 x_3) + 9V(x_1 x_3) + V(x_1 x_2 x_3) \tag{G1}$$

In view that

$$\begin{aligned} Cov(x_i x_j, x_i{}^2) &= E(x_i{}^3 x_j) - E(x_i x_j) E(x_i{}^2) \\ &= E(x_i{}^3) E(x_j) - E(x_i) E(x_j) E(x_i{}^2) \\ &= 0. \end{aligned} \tag{G2}$$

Since x_i follows a standard normal distribution, $x_i{}^2$ follows a chi-squared distribution with degree of freedom equal to one. Thus, $V(x_i{}^2) = 2$ and

$$V(y) = \sum_{i=1}^{3} 2\, i^2 \; + 1 + 4 + 9 + 1 = 43. \tag{G3}$$

For the expected value of y given a specific parameter x_1, we have

$$\begin{aligned} E(y|x_1) &= x_1{}^2 + x_1 E(x_2) + 3x_1 E(x_3) + 2E(x_2{}^2) + 3E(x_3{}^2) \\ &= x_1{}^2 + 5. \end{aligned}$$

The variance of $E(y|x_1)$ then can be calculated as follows,

$$\begin{aligned} V(E(y|x_1)) &= V(x_1{}^2) \\ &= V(x_1{}^2) \\ &= 2. \end{aligned} \tag{G4}$$

Similarly, we can show that

$$\begin{aligned} V(E(y|x_2)) &= 4V(x_2{}^2) = 8 \\ V(E(y|x_3)) &= 9V(x_3{}^2) = 18. \end{aligned} \tag{G5}$$

For the expected value of y given two parameters x_1 and x_2, we have

$$\begin{aligned} E(y|x_1, x_2) &= x_1{}^2 + x_1 x_2 + 2E(x_2{}^2) + 3E(x_3{}^2) \\ &= x_1{}^2 + x_1 x_2 + 5. \end{aligned}$$

Thus, the variance of the expected value can be calculated as follows,

$$V(E(y|x_1, x_2)) = V(x_1^{\ 2}) + V(x_1 x_2) = 2 + 1 = 3. \qquad \text{(G6)}$$

Similarly, we can show that,

$$\begin{aligned} &V(E(y|x_1, x_3)) = 9V(x_3^{\ 2}) + 9V(x_1 x_3) = 27, \\ &V(E(y|x_2, x_3)) = 4V(x_2^{\ 2}) + 4V(x_2 x_3) = 12, \\ &V(E(y|x_1, x_2, x_3)) = V(y) = 43. \end{aligned} \qquad \text{(G7)}$$

Based on eq. (8), we can calculate the partial variances contributed by the main effects, the second-order interaction effects, and the third-order interaction effects (see Table 1 for detailed results).