Appendix H: Estimation Errors of Sensitivity Indices in Stochastic Models

For a stochastic model, we can write the model as

$$y = g(x_1,..,x_n) + \xi \qquad \text{(H1)}$$

where ξ is a random error with $E(\xi)=0$ and $V(\xi)=\sigma_e{}^2$. Using search functions with eq. (35), we can transform $g(x_1,..,x_n)$ into a multiple periodic function of parameters in the θ -space [i.e., $(\theta_1,..,\theta_n)$]. Then the multiple Fourier transformation can be applied to $g(x_1,..,x_n)$ with eq. (12). However, the estimation of Fourier coefficient $\hat{C}^{(\theta)}_{r_1 \cdots\cdots r_n}$ will have an additional error term (C_e) due to the random noise (ξ). Namely,

$$\begin{aligned}
\hat{C}^{(\theta)}_{r_1 \cdots\cdots r_n} &= \frac{1}{N}\sum_{j=1}^{N}(g(G(\theta_1^{(j)}),...,G(\theta_n{}^{(j)}) + \xi^{(j)})e^{-\mathbf{i}(r_1\theta_1^{(j)}+\cdots+r_n\theta_n{}^{(j)})}) \\
&= \frac{1}{N}\sum_{j=1}^{N}[(g(G(\theta_1^{(j)}),...,G(\theta_n{}^{(j)})e^{-\mathbf{i}(r_1\theta_1^{(j)}+\cdots+r_n\theta_n{}^{(j)})}] + \frac{1}{N}\sum_{j=1}^{N}\xi^{(j)}e^{-\mathbf{i}(r_1\theta_1^{(j)}+\cdots+r_n\theta_n{}^{(j)})} \\
&= \hat{C}^{(\theta,g)}_{r_1 \cdots\cdots r_n} + \hat{C}_e,
\end{aligned}$$

where

$$\begin{aligned}
\hat{C}^{(\theta,g)}{}_{r_1 \cdots\cdots r_n} &= \frac{1}{N}\sum_{j=1}^{N}[(g(G(\theta_1^{(j)}),...,G(\theta_n{}^{(j)})e^{-\mathbf{i}(r_1\theta_1^{(j)}+\cdots+r_n\theta_n{}^{(j)})}] \\
\hat{C}_e &= \frac{1}{N}\sum_{j=1}^{N}\xi^{(j)}e^{-\mathbf{i}(r_1\theta_1^{(j)}+\cdots+r_n\theta_n{}^{(j)})}.
\end{aligned}$$

Finally, the partial variances contributed by main effects and interaction effects can be estimated based on eq. (30). Using the Central Limit Theorem, for $\hat{C}_e = \hat{a}_e - \mathbf{i}\hat{b}_e$, we have

$$\begin{aligned}
\hat{a}_e &\sim \ddot{N}(0, \frac{\sigma_e{}^2}{2N}), \\
\hat{b}_e &\sim \ddot{N}(0, \frac{\sigma_e{}^2}{2N}),
\end{aligned} \qquad \text{(H2)}$$

where $\ddot{N}(u,v)$ represents a normal distribution with mean *u* and variance *v* and

$$E\left[\left|\hat{C}^{(\theta)}_{r_1\cdots\cdots r_n}\right|^2\right]=E\left[\left|\hat{C}^{(\theta,g)}_{r_1\cdots\cdots r_n}\right|^2\right]+\frac{{\sigma_e}^2}{N}, \qquad \text{(H3)}$$

using the fact that

$$\begin{aligned}
&E\left[\left|\hat{C}^{(\theta)}_{r_1\cdots\cdots r_n}\right|^2\right]\\
&=E\left[\left|\hat{C}^{(\theta,g)}_{r_1\cdots\cdots r_n}+\hat{C}_e\right|^2\right]\\
&=E\left[\left(\hat{a}^{(\theta,g)}_{r_1\cdots\cdots r_n}+\hat{a}_e\right)^2+\left(\hat{b}^{(\theta,g)}_{r_1\cdots\cdots r_n}+\hat{b}_e\right)^2\right]\\
&=E\left({\hat{a}^{(\theta,g)}_{r_1\cdots\cdots r_n}}^2+{\hat{b}^{(\theta,g)}_{r_1\cdots\cdots r_n}}^2\right)+E\left({\hat{a}_e}^2\right)+E\left({\hat{b}_e}^2\right)+2E(\hat{a}^{(\theta,g)}_{r_1\cdots\cdots r_n}\hat{a}_e)+2E(\hat{b}^{(\theta,g)}_{r_1\cdots\cdots r_n}\hat{b}_e)\\
&=E\left[\left|\hat{C}^{(\theta,g)}_{r_1\cdots\cdots r_n}\right|^2\right]+\frac{{\sigma_e}^2}{N}\ \text{with}\ E(\hat{a}^{(\theta,g)}_{r_1\cdots\cdots r_n}\hat{a}_e)=0\ \text{and}\ E(\hat{b}^{(\theta,g)}_{r_1\cdots\cdots r_n}\hat{b}_e)=0.
\end{aligned}$$

This shows that there will be an additional bias term $\frac{{\sigma_e}^2}{N}$. If the random noise is very large, then a very large sample size is needed to identify the partial variances contributed by different model parameters. Otherwise, they would be difficult to identify the variance contributions by parameters.