Supporting Information

Johnson 10.1073/pnas.1313476110

SI Text

This supplement contains two sections. The first section presents a comparison of Bayes factors obtained using uniformly most powerful Bayesian tests (UMPBTs) to Bayes factors obtained using standard Cauchy priors (1–3), intrinsic priors (4), and Bayesian information criterion (BIC)-based approximations to Bayes factors (5–7), all in the context of z tests. In the second, several lemmas are presented that describe the UMPBT(γ) in common testing scenarios. Finally, a table summarizing the results of these lemmas is provided.

Comparison of Bayes Factors

In this section, Bayes factors generated from UMPBT alternatives are compared with Bayes factors obtained from other default Bayesian testing procedures. Each Bayesian testing procedure was used to test whether the mean μ of a random sample of n normal observations with known variance $\sigma^2 = 1$ was equal to 0. Several default procedures were tested. The first, due to Jeffreys (1), is based on the assumption that the prior density for μ under the alternative hypothesis is a standard Cauchy distribution. The extension of this test for unknown σ^2 leads to the Zellner–Siow prior for linear models (2) and testing procedures advocated for psychological tests in ref. 3. The second default method was obtained by assuming an intrinsic prior for μ under the alternative hypothesis (4). The third default method was based on converting the BIC criterion (5) into an approximate Bayes factor, as suggested in refs. 6 and 7.

The prior densities that define the alternative hypothesis in the comparison group are based on the specification of local alternative prior densities, which means that the order at which they accumulate evidence in favor of a true null hypothesis is only $O_p(n^{1/2})$ (8). This slow rate of convergence occurs because local alternative prior densities are not zero at the parameter value the defines a point null hypothesis. Data that support the null hypothesis thereby also provide some support to the alternative, making it difficult to distinguish between the two hypotheses when the null is true. In contrast, the evidence achieved by the UMPBTs in favor of true null hypotheses is bounded by a function of the evidence threshold γ. This means that only a finite amount of evidence can be obtained in favor of a true null hypothesis if γ is held constant as the sample size grows.

All tests were considered to be two-sided. The prior densities for μ under the alternative hypotheses in the approximate UMPBT(γ) two-sided tests were defined by placing one-half of the prior mass corresponding to each of the one-sided UMPBT (2γ)s on μ.

The Bayes factors in favor of the alternative hypotheses under each testing procedure can be expressed as follows.

Cauchy.

$$BF_{10}^{C}(\mathbf{x}) = \exp\left(\frac{n\bar{x}^2}{2}\right)\int_{-\infty}^{\infty}\frac{\exp\left[-n(\bar{x}-\mu)^2/2\right]}{\pi(1+\mu^2)}\,d\mu.$$

Intrinsic.

$$BF_{10}^{I}(\mathbf{x}) = \frac{1}{\sqrt{2n+1}}\exp\left[\frac{(n\bar{x})^2}{2n+1}\right].$$

[Note that the intrinsic prior in this setting is $\mu \sim N(0,2)$.]

BIC.

$$BF_{10}^{B}(\mathbf{x}) = \exp\left\{0.5\left[n\bar{x}^2 - \log(n)\right]\right\}.$$

UMPBT.

$$BF_{10}^{U}(\mathbf{x}) = \exp\left(\frac{n\bar{x}^2}{2}\right)\left\{\frac{1}{2}\exp\left[-0.5n(\bar{x}-\mu_u)^2\right] + \frac{1}{2}\exp\left[-0.5n(\bar{x}+\mu_u)^2\right]\right\},$$

where

$$\mu_u = \sqrt{\frac{2\log(2\gamma)}{n}}.$$

To study the behavior of the Bayes factors obtained under each of the four procedures, the sample mean of the observed data was assumed to take one of the four values (0, 0.2, 0.4, 0.6). Note that the first value of 0 provides as much evidence in favor of the null hypothesis as can be obtained from the data. The remaining values represent standardized effect sizes of 0.2, 0.4, and 0.6, respectively, because the observational variance is 1. For each assumed value of the sample mean, the sample size was increased from 1 until either a sample size of 5,000 was reached or until the maximum of the Bayes factors exceeded 5,000. These maximum values were imposed to retain detail in the plots for values of the Bayes factors that are of practical interest. Finally, the evidence threshold γ for the UMPBT was determined by equating the rejection region for this test to the rejection region of a two-sided classical test of size 0.005. That is, γ was equal to $\exp(2.807^2/2)/2 = 25.7$.

The value of the Bayes factors obtained under these combinations of sample means and sample sizes is displayed in Fig. S1. This figure reveals a number of interesting features. Among these, this plot illustrates the consistency of the Bayes factors corresponding to the Cauchy, intrinsic, and BIC procedures. These procedures all produce Bayes factors that tend to 0 when $\bar{x} = 0$ and the sample size grows, even though this convergence is slow. In contrast, the UMPBT-based Bayes factor—based on a fixed evidence threshold γ—is constant and approximately equal to $1/2\gamma$ when $\bar{x} = 0$, independently of the sample size. In this respect, UMPBT tests with fixed evidence thresholds are similar to classical hypothesis tests: both maintain a constant "type I error" as the sample size is increased. Preliminary recommendations for increasing γ with sample size to achieve consistency are provided in ref. 9. Similarly, UMPBT-based Bayes factors eventually become smaller than the other three Bayes factors as n grows when γ is held constant, even though the UMPBT is consistent under a true alternative.

For sample sizes typically achieved in practice, the UMPBT-based Bayes factors appear to provide more useful summaries of the evidence in favor of either a true null or true alternative hypothesis than do the other Bayes factors. When $\bar{x} = 0$ for example, the Bayes factor in favor of the null hypothesis is ∼50 for all values of n, whereas the other Bayes factors do not achieve this level of support for the null hypothesis until n is greater than ∼1,250 (intrinsic), 1,700 (Cauchy), or 2,500 (BIC). For a standardized effect size of 0.2, none of the Bayes factors becomes much larger than 1 until sample sizes of about 50 are obtained, and then the UMPBT-based Bayes factors are larger than the other Bayes factors for all sample sizes for which the Bayes factors are all less than 5,000. Similar comments apply to observed effect sizes of 0.4 and 0.6, except that smaller sample sizes are needed for all of the Bayes factors to exceed 1. As stated in the main article, these observations demonstrate that UMPBT-based Bayes factors produce more extreme Bayes factors than other default Bayesian procedures for sample sizes and effect sizes of practical interest. This means that the false-positive rates that would be estimated from the other procedures for marginally significant P values would be higher than 17–25%, the range suggested by the use of UMPBTs.

The relative performance of the various Bayes factors for small values of n is also interesting. For all values of $\bar{x}$ considered, the UMPBT-based Bayes factors obtained for $n<5$ suggest more support for the null hypotheses than do the other hypothesis tests. This fact can be attributed to the fact that the UMPBTs are obtained using nonlocal alternative priors on μ, whereas the other tests are based on local priors. As demonstrated in ref. 8, this means that UMPBTs are able to more quickly obtain evidence in support of the null hypothesis. For instance, when $\bar{x}=0.2$ and $n=1$, the UMPBT-based Bayes factor suggests strong support for the null hypothesis, whereas the other Bayes factors assume noncommittal values near 1.0.

When viewed from a scientific perspective, the evidence provided by UMPBTs in favor of the null hypothesis for small values of n and values of $|\bar{x}|\leq 0.6$ seems quite reasonable. Clearly, most scientists would not design an experiment to test whether a normal mean was equal to 0 with fewer than five observations. Unless, of course, μ was assumed to be large relative to σ under the alternative hypothesis. Under such an assumption, the observation of a sample mean less than 0.6σ provides strong evidence in favor of the null hypothesis.

Along similar lines, most classical statisticians regard the sample size n as fixed and ancillary when they conduct hypothesis tests. Under this assumption, UMPBTs violate the likelihood principle because the alternative hypothesis depends on n. In actual practice, however, the sample size selected by a researcher to test an effect size is generally highly informative about the magnitude of that effect size. For instance, few researchers would collect 100,000 observations to detect a standardized effect size of 0.4. A scientist who collects this many observations obviously hopes to detect a much subtler departure from the standard theory. It is also worth noting that sample size calculations themselves require the specification of an alternative hypothesis.

Because the value of the sample size selected for an experiment often reflects prior information regarding the magnitude of an effect size, it is the author's opinion that it is appropriate (and often desirable) to use the sample size chosen by an investigator to specify an alternative hypothesis.

Lemmas

The following lemmas describe the UMPBT(γ) for several common tests.

Lemma 1. *Suppose $X_1,\ldots,X_n$ are independent and identically distributed (iid) according to a normal distribution with mean μ and variance σ^2 (i.e., $N(\mu,\sigma^2)$). Then the one-sided UMPBT(γ) for testing $H_0:\mu=\mu_0$ against any alternative hypothesis that requires $\mu>\mu_0$ is obtained by taking $H_1:\mu=\mu_1$, where*

$$\mu_1=\mu_0+\sigma\sqrt{\frac{2\log(\gamma)}{n}}. \quad \textbf{[S1]}$$

Similarly, the UMPBT(γ) one-sided test for testing $\mu<\mu_0$ is obtained by taking

$$\mu_1=\mu_0-\sigma\sqrt{\frac{2\log(\gamma)}{n}}.$$

Proof: Provided in ref. 9.

Lemma 2. *Suppose $X_{1,1},\ldots,X_{1,n_1}$ are iid $N\left(\mu-\frac{n_2}{n_1+n_2}\delta,\sigma^2\right)$, and $X_{2,1},\ldots,X_{2,n_2}$ are iid $N\left(\mu+\frac{n_1}{n_1+n_2}\delta,\sigma^2\right)$, where σ^2 is known and the prior distribution for μ is assumed to be uniform on the real line. The one-sided UMPBT(γ) for testing $H_0:\delta=0$ against alternatives that require $\delta>0$ is obtained by taking*

$$H_1:\delta=\sigma\sqrt{\frac{2(n_1+n_2)\log(\gamma)}{n_1n_2}}. \quad \textbf{[S2]}$$

Proof. Consider first simple alternative hypotheses on $\delta>0$. Up to a constant factor that arises from the uniform distribution on μ, the marginal distribution of the data under the null hypothesis can be obtained by integrating out μ to obtain

$$m_0(\mathbf{x})=\left(2\pi\sigma^2\right)^{-(n_1+n_2-1)/2}(n_1+n_2)^{-1/2}\times\exp\left[-\frac{1}{2\sigma^2}\sum_{j=1}^{2}\sum_{i=1}^{n_j}\left(x_{j,i}-\bar{x}_j\right)^2\right]. \quad \textbf{[S3]}$$

Similarly, the marginal distribution of the data under the alternative that $\mu_2-\mu_1=\delta$ can be obtained by integrating out μ to obtain

$$m_1(\mathbf{x})=m_0(\mathbf{x})\exp\left\{-\frac{1}{2\sigma^2}\left[\frac{n_1n_2}{n_1+n_2}\delta^2-\frac{2n_1n_2}{n_1+n_2}(\bar{x}_2-\bar{x}_1)\delta\right]\right\}. \quad \textbf{[S4]}$$

It follows that

$$\mathbf{P}[\log(BF_{10})>log(\gamma)]=\mathbf{P}\left[\bar{x}_2-\bar{x}_1>\frac{(n_1+n_2)\sigma^2\log(\gamma)}{n_1n_2\delta}+\frac{\delta}{2}\right]. \quad \textbf{[S5]}$$

Regardless of the distribution of $(\bar{x}_2-\bar{x}_1)$, this probability can be maximized by minimizing the right-hand side of the last inequality with respect to δ. The UMPBT value for δ is thus

$$\delta^*=\sigma\sqrt{\frac{2(n_1+n_2)\log(\gamma)}{n_1n_2}}. \quad \textbf{[S6]}$$

Now consider composite alternative hypotheses, and let $BF_{10}(\delta)$ denote the value of the Bayes factor when evaluated at a particular value of δ and fixed $\mathbf{x}$. Define an indicator function s according to

$$s(\mathbf{x},\delta)=Ind(BF_{10}(\delta)>\gamma). \quad \textbf{[S7]}$$

Then it follows from Eq. **S5** that

$$s(\mathbf{x},\delta)\leq s(\mathbf{x},\delta^*)\quad\text{for all }\mathbf{x}. \quad \textbf{[S8]}$$

This implies that

$$\int_0^\infty s(\mathbf{x},\delta)\pi(\delta)\leq s(\mathbf{x},\delta^*) \quad \textbf{[S9]}$$

for all probability densities $\pi(\delta)$. It follows that

$$\mathbf{P}_{\delta_t}(BF_{10} > \gamma) = \int_{\mathcal{X}} s(\mathbf{x}, \delta) f(\mathbf{x}|\delta_t) d\delta_t \qquad \textbf{[S10]}$$

is maximized by a prior density that concentrates its mass δ^*. Here $f(\mathbf{x}|\delta_t)$ is the sampling density of $\mathbf{x}$ for $\delta = \delta_t$,

Lemma 3. *Suppose that X is distributed according to a χ^2 distribution on 1 degree of freedom and noncentrality parameter λ; that is, $X \sim \chi_1^2(\lambda)$. The UMPBT(γ) for testing $H_0 : \lambda = 0$ is obtained by taking $H_1 : \lambda = \lambda_1$, where λ_1 is the value of λ that minimizes*

$$\frac{1}{\sqrt{\lambda}} \log\left(e^{\lambda/2}\gamma + \sqrt{e^{\lambda}\gamma^2 - 1}\right). \qquad \textbf{[S11]}$$

Proof. As in *Lemma 2*, consider first simple alternative hypotheses on $\lambda > 0$. By taking the ratio of a noncentral χ^2 density on 1 degree of freedom to the central χ^2 density on 1 degree of freedom, it follows that the Bayes factor in favor of the alternative can be expressed as

$$\sum_{i=0}^{\infty} \frac{e^{\lambda/2}\Gamma\left(\frac{1}{2}\right)\left(\frac{\lambda x}{2}\right)^i}{i!2^i\Gamma\left(\frac{1}{2}+i\right)}. \qquad \textbf{[S12]}$$

Noting that

$$\Gamma\left(\frac{1}{2}+i\right) = \frac{(2i)!\Gamma(1/2)}{4^i i!}, \qquad \textbf{[S13]}$$

and that

$$\cosh\left(\sqrt{\lambda x}\right) = \sum_{i=0}^{\infty} \frac{(\lambda x)^i}{(2i)!}, \qquad \textbf{[S14]}$$

it follows that

$$BF_{10}(\lambda) = e^{-\lambda/2}\cosh\left(\sqrt{\lambda x}\right). \qquad \textbf{[S15]}$$

The probability that the Bayes factor exceeds the evidence threshold is given by

$$\begin{aligned}\mathbf{P}_{\lambda_t}[BF_{10} > \gamma] &= \mathbf{P}_{\lambda_t}\left[\cosh\left(\sqrt{\lambda x}\right) > e^{\lambda/2}\gamma\right] \\ &= \mathbf{P}_{\lambda_t}\left[\sqrt{x} > \lambda^{-1/2} \log\left(e^{\lambda/2}\gamma + \sqrt{e^{\lambda}\gamma^2 - 1}\right)\right].\end{aligned} \qquad \textbf{[S16]}$$

Minimizing the right-hand side of the inequality maximizes the probability, regardless of the value of λ_t. The extension to composite hypotheses follows along from the same logic used in Eqs. **S7–S10**.

Lemma 4. *Suppose that X has a binomial distribution with success probability p and denominator n. The UMPBT(γ) for testing $H_0 : p = p_0$ against alternatives that require $p > p_0$ is obtained by taking $H_1 : p = p_1$, where p_1 is the value of p that minimizes*

$$\frac{\log(\gamma) - n[\log(1-p) - \log(1-p_0)]}{\log[p/(1-p)] - \log[p_0/(1-p_0)]}. \qquad \textbf{[S17]}$$

The UMPBT(γ) for alternatives that require $p < p_0$ is obtained by taking p_1 to be the value of p that maximizes Eq. **S17**.
Proof. Provided in ref. 9.

Lemma 5. *Assume that the conditions of Lemma 1 apply, except that σ^2 is not known. Suppose that the prior distribution for σ^2 is an inverse gamma distribution with parameters α and λ, and define*

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad W = \sum_{i=1}^{n}(x_i - \bar{x})^2 + 2\lambda, \quad \textit{and} \quad \gamma^* = \gamma^{\frac{2}{n+\alpha}}. \qquad \textbf{[S18]}$$

Then the value of μ_1 that minimizes a_n in Eq. **S4** *is*

$$\mu_1 = \mu_0 + \sqrt{\frac{W(\gamma^* - 1)}{n}}. \qquad \textbf{[S19]}$$

If a noninformative prior is assumed for σ^2 (i.e., $\alpha = \lambda = 0$), then the UMPBT(γ) alternative is obtained by taking

$$\mu_1 = \mu_0 + s\sqrt{(\gamma^* - 1)\frac{(n-1)}{n}},$$

where

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2.$$

Proof. As in the previous proofs, consider first the case of simple alternative hypotheses. By integrating out the variance parameter, it follows that the Bayes factor in favor of the alternative hypothesis can be expressed as

$$BF_{10}(\mu_1) = \left[\frac{W + n(\bar{x} - \mu_0)^2}{W + n(\bar{x} - \mu_1)^2}\right]^{n/2+\alpha}. \qquad \textbf{[S20]}$$

After some algebra, this expression leads to the following equation:

$$\mathbf{P}_{\mu_t}[BF_{10}(\mu_1) > \gamma] = \mathbf{P}_{\mu_t}[a_n < \bar{x} < b_n], \qquad \textbf{[S21]}$$

where

$$a_n = \frac{\gamma^*\mu_1 - \mu_0}{\gamma^* - 1} - \sqrt{\frac{\gamma^*(\mu_1 - \mu_0)^2}{(\gamma^* - 1)^2} - \frac{W}{n}} \qquad \textbf{[S22]}$$

and

$$b_n = \frac{\gamma^*\mu_1 - \mu_0}{\gamma^* - 1} + \sqrt{\frac{\gamma^*(\mu_1 - \mu_0)^2}{(\gamma^* - 1)^2} - \frac{W}{n}}. \qquad \textbf{[S23]}$$

Minimizing a_n as a function of μ_1 leads to the stated result.

Lemma 6. *Assume that the conditions of Lemma 2 apply, except that the variance σ^2 is unknown. Suppose the prior distribution for σ^2 is an inverse gamma distribution with parameters α and λ, and define*

$$\bar{x}_j = \frac{1}{n_j}\sum_{i=1}^{n_j} x_{j,i}, \quad W = \sum_{j=1}^{2}\sum_{i=1}^{n_j}\left(x_{j,i} - \bar{x}_j\right)^2 + 2\lambda, \quad \textit{and} \quad \gamma^* = \gamma^{\frac{2}{n_1+n_2+2\alpha-1}}. \qquad \textbf{[S24]}$$

Then the value of δ than minimizes a_n in Eq. **S5** *is*

$$\delta = \sqrt{\frac{W(\gamma^* - 1)(n_1 + n_2)}{n_1 n_2}}. \qquad \textbf{[S25]}$$

Taking $\alpha = \lambda = 0$ *and*

$$s^2 = \frac{1}{n_1 + n_2 - 2} \sum_{j=1}^{2} \sum_{i=1}^{n_j} \left(x_{j,i} - \bar{x}_j\right)^2,$$

the UMPBT(γ) *alternative is defined by taking*

$$\delta = s\sqrt{\frac{(\gamma^* - 1)\nu(n_1 + n_2)}{n_1 n_2}}.$$

Proof. Similar to the proofs of *Lemmas 2* and *5*. Using the expressions for the marginal distributions obtained in the case of a known variance in *Lemma 2*, it can be shown that the Bayes factor takes the form of the ratio of t densities. Solving for the difference in means $\mu_2 - \mu_1$ leads to an inequality similar to Eq. **S21**, and the result follows.

A summary of the results of *Lemmas 1–6* appears in Table S1. Also provided in this table are expressions for the Bayes factors (expressed in terms of standard test statistics), rejection regions, and the relation between evidence threshold γ and the size of the corresponding classical test.

1. Jeffreys H (1961) *Theory of Probability* (Oxford Univ Press, Oxford), 3rd Ed.
2. Zellner A, et al., eds (1980) Posterior odds ratios for selected regression hypotheses. *Bayesian Statistics: Proceedings of the First International Meeting Held in Valencia (Spain)*, eds Bernardo JM, DeGroot MH, Lindley DV, Smith AFM (University Press, Valencia, Spain), pp 585–603.
3. Rouder JN, Speckman PL, Sun D, Morey RD, Iverson G (2009) Bayesian t tests for accepting and rejecting the null hypothesis. *Psychon Bull Rev* 16(2):225–237.
4. Berger JO, Pericchi LR (1996) The intrinsic Bayes factor for model selection and prediction. *J Am Stat Assoc* 91(433):109–122.
5. Schwarz G (1978) Estimating the dimension of a model. *Ann Stat* 6(2):461–464.
6. Kass RE, Raftery AE (1995) Bayes factors. *J Am Stat Assoc* 90(430):773–795.
7. Wagenmakers E-J (2007) A practical solution to the pervasive problems of p values. *Psychon Bull Rev* 14(5):779–804.
8. Johnson VE, Rossell D (2010) On the use of non-local prior densities in Bayesian hypothesis tests. *J Roy Stat Soc Ser B Method* 72(2):143–170.
9. Johnson VE (2013) Uniformly most powerful Bayesian tests. *Ann Stat* 41(4):1716–1741.

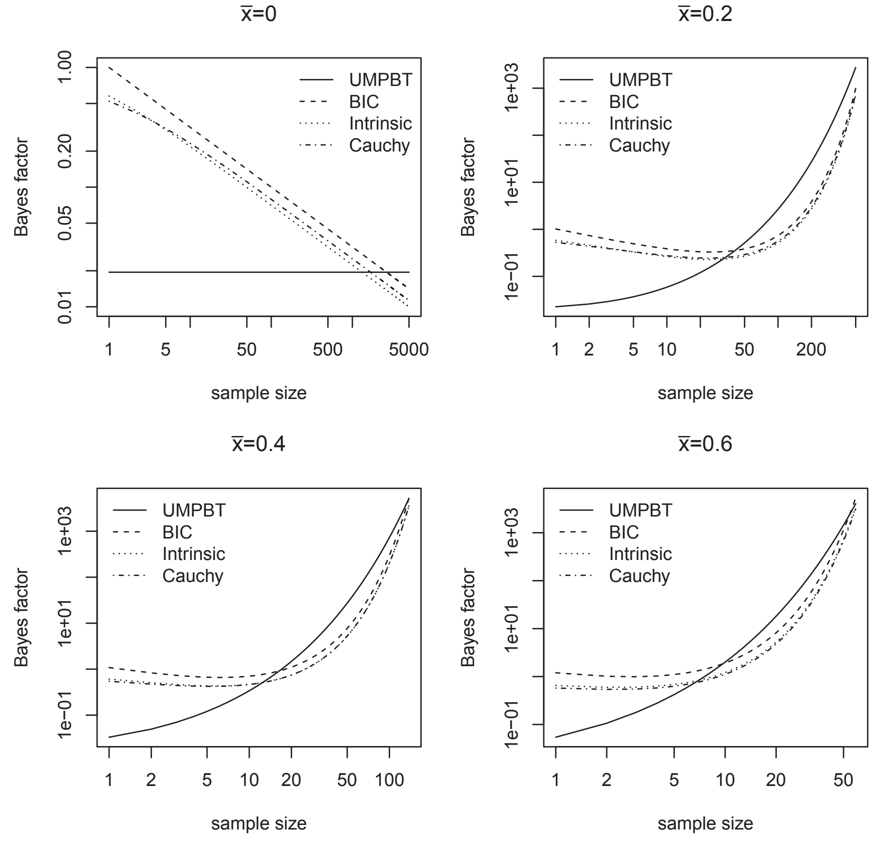

Fig. S1. Comparison of default Bayesian procedures for testing a null hypothesis that the mean of n $N(\mu,1)$ random variables is 0.

Table S1. Properties of UMPBTs in common testing situations

Test	Variables	H_1	Bayes factor	Reject region	$\gamma = f(\alpha)$
One-sample z	$z = \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma}$	$\mu_1 = \mu_0 + \sigma\sqrt{\frac{2\log(\gamma)}{n}}$	$\exp\left[z\sqrt{2\log(\gamma)} - \log(\gamma)\right]$	$z > \sqrt{2\log(\gamma)}$	$\gamma = \exp\left(\frac{z_\alpha^2}{2}\right)$
Two-sample z	$z = \sqrt{\frac{n_1 n_2}{n_1+n_2}}\ \frac{\bar{x}_2-\bar{x}_1}{\sigma}$	$\delta = \sigma\sqrt{\frac{2(n_1+n_2)\log(\gamma)}{n_1 n_2}}$	$\exp\left[z\sqrt{2\log(\gamma)} - \log(\gamma)\right]$	$z > \sqrt{2\log(\gamma)}$	$\gamma = \exp\left(\frac{z_\alpha^2}{2}\right)$
One-sample t	$t = \frac{\sqrt{n}(\bar{x}-\mu_0)}{s}$				
	$s^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}$	$\mu_1 = \mu_0 + s\sqrt{\nu\gamma^*/n}$	$\left(\frac{\nu+t^2}{\nu+\left[t-\sqrt{\nu\gamma^*}\right]^2}\right)^m$	$t > \sqrt{\nu\gamma^*}$	$\gamma = \left(\frac{t_\alpha^2}{\nu}+1\right)^m$
	$\nu = n-1$				
	$\gamma^* = \gamma^{2/n} - 1$				
	$m = n/2$				
Two-sample t	$t = \sqrt{\frac{n_1 n_2}{n_1+n_2}}\ \frac{\bar{x}_2-\bar{x}_1}{s}$				
	$s^2 = \frac{\sum\sum(x_{i,j}-\bar{x}_j)^2}{n_1+n_2-2}$	$\delta = s\sqrt{\frac{2m\gamma^*\nu}{n_1 n_2}}$	$\left(\frac{\nu+t^2}{\nu+\left[t-\sqrt{\nu\gamma^*}\right]^2}\right)^m$	$t > \sqrt{\nu\gamma^*}$	$\gamma = \left(\frac{t_\alpha^2}{\nu}+1\right)^m$
	$\nu = n_1+n_2-2$				
	$\gamma^* = \gamma^{2/(n_1+n_2-1)} - 1$				
	$m = (n_1+n_2)/2$				
χ_1^2	x	$\lambda_1 = \arg\min_\lambda \frac{\log\left(a+\sqrt{a^2-1}\right)}{\sqrt{\lambda}}$	$\exp\left(-\frac{\lambda_1}{2}\right)\cosh\left(\sqrt{\lambda_1 x}\right)$		
	$a = \gamma e^{\lambda/2}$				
Proportion	(x,n)	$p_1 = \arg\min_p \frac{\log(\gamma) - n\Delta(p,p_0)}{\text{logit}(p)-\text{logit}(p_0)}$	$\left(\frac{p_1}{p_0}\right)^x\left(\frac{1-p_1}{1-p_0}\right)^{n-x}$		
p_0	$\Delta(p,p_0) = \log\left(\frac{1-p}{1-p_0}\right)$				

Note that the Bayes factors listed for the one- and two-sample t tests should only be used for $t < \sqrt{\nu\gamma^*} + \sqrt{\nu\gamma^* + 4\nu}$. Values for quantities in empty cells must be determined using numerical techniques.