

## CALCULATING BAYES FACTORS FOR THE GENE EXPRESSION DATA

### 1. WHAT ARE BAYES FACTORS COMPUTING

Bayes factors are a Bayesian approach to model selection. Unlike hypothesis testing, which decides whether to reject a null hypothesis at some predetermined significance level, Bayes factors measure the evidence for one model  $M_1(\theta_1)$  against another model  $M_2(\theta_2)$ . If we denote the data by  $D$ , then the Bayes factor is given by

$$K = \frac{P(D|M_1)}{P(D|M_2)} = \frac{\int P(D|M_1, \theta_1) d\theta_1}{\int P(D|M_2, \theta_2) d\theta_2}$$

(we assume uniform priors). A ratio greater than 1 means that  $M_1$  has more support, given the data, than  $M_2$ . A ratio less than 1 means the opposite. To “normalize” the Bayes factor, a logarithm is often taken so that 0 indicates no preference between the models, and so that a positive number indicates the same strength of evidence for  $M_1$  as the negative equivalent indicates for  $M_2$ . A base of 10 is often chosen, and then  $\log_{10} K$  is measured in “decibans.”

We want to decide whether gene expression levels between two groups look like they are generated by one or two distributions. In other words, we want to decide whether or not the two groups exhibit differentiable gene expression levels. We assume the data are generated by a normal distribution. For our  $M_1$ , we assume the data for two groups are generated by different normal distributions (different mean and variance). For  $M_2$ , we assume the data for both groups are generated by the same normal distribution.

### 2. PROBABILITY OF DATA FROM THE SAME NORMAL DISTRIBUTION

Let  $N$  be the number of data points, and denote the data points by  $x_1, \dots, x_N$ . The probability of  $x_1, \dots, x_N$  all being generated from the same normal distribution  $N(\mu, \sigma)$  is:

$$P(x_1, \dots, x_N | \mu, \sigma) = \prod_i P(x_i | \mu, \sigma).$$

To marginalize over the distribution parameters, we need to integrate this product over the parameters  $\mu, \sigma$ . Rewriting the product to separate the parameters will allow us to calculate the integral. We start with

$$\begin{aligned} \prod_i P(x_i | \mu, \sigma) &= \prod_i \frac{1}{\sigma \sqrt{2\pi}} e^{\frac{(x_i - \mu)^2}{2\sigma^2}} \\ &= \frac{1}{(\sigma \sqrt{2\pi})^N} \exp \left\{ -\frac{1}{2\sigma^2} \sum_i (x_i - \mu)^2 \right\}. \end{aligned}$$

Let's just work on the exponential, so

$$\begin{aligned} \exp \left\{ -\frac{1}{2\sigma^2} \sum_i (x_i - \mu)^2 \right\} &= \exp \left\{ -\frac{1}{2\sigma^2} [\mu^2 - 2\mu x_1 + x_1^2 + \mu^2 - 2\mu x_2 + x_2^2 + \dots] \right\} \\ &= \exp \left\{ -\frac{1}{2\sigma^2} [N\mu^2 - 2\mu (\sum x_i) + \sum x_i^2] \right\} \\ &= \exp \left\{ -\frac{N}{2\sigma^2} \left[ \mu^2 - 2\mu \bar{x} + \sum \frac{x_i^2}{N} \right] \right\}, \end{aligned}$$

where  $\bar{x}$  is the average of the data points. This exponential almost looks like a Gaussian (which we know how to integrate), except that the term in the brackets isn't a square. So we complete the square:

$$\begin{aligned} &= \exp \left\{ -\frac{N}{2\sigma^2} \left[ \mu^2 - 2\mu \bar{x} + \bar{x}^2 + \left( \sum \frac{x_i^2}{N} \right) - \bar{x}^2 \right] \right\} \\ &= \exp \left\{ -\frac{N}{2\sigma^2} (\mu - \bar{x})^2 \right\} \exp \left\{ -\frac{N}{2\sigma^2} \left( \sum \frac{x_i^2}{N} - \bar{x}^2 \right) \right\}. \end{aligned}$$

Note that  $\left( \sum \frac{x_i^2}{N} - \bar{x}^2 \right)$  is a constant; let's abbreviate it as  $\kappa$ .

Now we just use  $u$ -substitution to compute the integral:

$$\int_{\mathbb{R}} P(x_1, \dots, x_N | \mu, \sigma) d\mu = \frac{1}{(\sigma \sqrt{2\pi})^N} \exp \left\{ -\frac{N\kappa}{2\sigma^2} \right\} \int_{\mathbb{R}} \exp \left\{ -\frac{N}{2\sigma^2} (\mu - \bar{x})^2 \right\} d\mu$$

let  $u = \sqrt{\frac{N}{2\sigma^2}}(\mu - \bar{x})$  so  $du = \sqrt{\frac{N}{2\sigma^2}} d\mu$ , then

$$\begin{aligned} &= \frac{1}{(\sigma \sqrt{2\pi})^N} \exp \left\{ -\frac{N\kappa}{2\sigma^2} \right\} \sqrt{\frac{2\sigma^2}{N}} \int_{\mathbb{R}} e^{-u^2} du \\ &= \frac{1}{(\sigma \sqrt{2\pi})^N} \exp \left\{ -\frac{N\kappa}{2\sigma^2} \right\} \sqrt{\frac{2\pi\sigma^2}{N}} \\ &= \frac{1}{\sqrt{N} (\sigma \sqrt{2\pi})^{N-1}} \exp \left\{ -\frac{N\kappa}{2\sigma^2} \right\}. \end{aligned}$$

Now we can integrate this expression with respect to  $\sigma$ . The integral has the same basic form as

$$\int_0^\infty \frac{1}{x^{N-1} e^{1/x^2}} dx = \frac{1}{2} \Gamma \left( \frac{N}{2} - 1 \right) \quad \text{for } N > 2,$$

(here  $\Gamma(x)$  is the gamma function). Using  $u$ -substitution with  $u = \sigma\sqrt{\frac{2}{N\kappa}}$ , so  $du = \sqrt{\frac{2}{N\kappa}} d\sigma$ , we have

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}} P(x_1, \dots, x_N | \mu, \sigma) d\mu d\sigma \\ &= \frac{1}{\sqrt{N}(\sqrt{2\pi})^{N-1}} \int_0^\infty \frac{1}{\sigma^{N-1}} \cdot \exp\left\{-\frac{N\kappa}{2\sigma^2}\right\} d\sigma \\ &= \frac{1}{\sqrt{N}(\sqrt{2\pi})^{N-1}} \left(\sqrt{\frac{2}{N\kappa}}\right)^{N-2} \int_0^\infty \frac{1}{u^{N-1} e^{1/u^2}} du \\ &= \frac{1}{\sqrt{N}(\sqrt{2\pi})^{N-1}} \left(\sqrt{\frac{2}{N\kappa}}\right)^{N-2} \Gamma\left(\frac{N}{2} - 1\right). \end{aligned}$$

Note that this expression has no dependence on the distribution parameters, and depends solely on the data points, as we would expect.

### 3. PROBABILITY OF DATA FROM TWO NORMAL DISTRIBUTIONS

Now suppose we have data from two groups, say  $x_1, \dots, x_N$  from group  $A$ , and  $y_1, \dots, y_M$  from group  $B$ . The probability that the  $x_i$  are generated by a normal distribution  $N(\mu_A, \sigma_A)$ , and that the  $y_j$  are generated by a normal distribution  $N(\mu_B, \sigma_B)$  is:

$$\begin{aligned} & P(x_1, \dots, x_N, y_1, \dots, y_M | \mu_a, \sigma_A, \mu_B, \sigma_B) \\ &= P(x_1, \dots, x_N | \mu_A, \sigma_A) \cdot P(y_1, \dots, y_M | \mu_B, \sigma_B) \end{aligned}$$

(since two groups are independent of each other)

$$= \prod_i P(x_i | \mu_A, \sigma_A) \cdot \prod_j P(y_j | \mu_B, \sigma_B).$$

So marginalizing over the parameters is simply:

$$\begin{aligned} & \int P(x_1, \dots, x_N, y_1, \dots, y_M | \mu_a, \sigma_A, \mu_B, \sigma_B) d\mu_A d\sigma_A d\mu_B d\sigma_B \\ &= \int \prod_i P(x_i | \mu_A, \sigma_A) \cdot \prod_j P(y_j | \mu_B, \sigma_B) d\mu_A d\sigma_A d\mu_B d\sigma_B \\ &= \int_0^\infty \int_{\mathbb{R}} \prod_i P(x_i | \mu_A, \sigma_A) d\mu_A d\sigma_A \cdot \int_0^\infty \int_{\mathbb{R}} \prod_j P(y_j | \mu_B, \sigma_B) d\mu_B d\sigma_B, \end{aligned}$$

and we can calculate each factor by simply using the formula in the previous section.

### 4. INTERPRETING THE BAYES FACTORS

If  $K$  is the Bayes factor, let  $B = \log_{10} K$ , with  $B$  measured in decibans. Positive  $B$  is evidence for  $M_1$ , that data for the two groups is generated by different normal distributions.

Important to note is that  $M_2$  (one normal distribution) is a special case of  $M_1$  (two normal distribution), where the means and variances are equal. Thus  $B$  will never be very negative, as evidence for  $M_2$  is not inconsistent with  $M_1$ . It is possible

for  $B$  to be negative, as Bayes factors have a slight preference for models with fewer parameters, which provides some protection from overfitting.

## AREA BETWEEN PIECEWISE-LINEAR CURVES

### 1. SETUP

A *piecewise-linear curve* is a curve specified by a sequence of points  $(P_1, \dots, P_n)$ , where neighboring points in the sequence are connected by a line segment. For our purposes, we will only be dealing with curves that are piecewise-linear, and henceforth “curve” will mean “piecewise-linear curve.”

We assume that for a finite number of  $x$ -values, we are given two  $y$ -values:  $a_i$  and  $b_i$ . For every  $x_i$  we have the points  $(x_i, a_i)$  and  $(x_i, b_i)$ , and the two respective curves  $A$  and  $B$ . It suffices to choose two  $x$ -values  $x_i$  and  $x_{i+1}$ , and compute the area between  $A$  and  $B$  over this interval, as the total area will be the sum of the areas over such intervals. The curves over the interval  $[x_i, x_{i+1}]$  are completely determined by the points  $(x_i, a_i)$ ,  $(x_i, b_i)$ ,  $(x_{i+1}, a_{i+1})$ , and  $(x_{i+1}, b_{i+1})$ , and we will express the area in terms of these values. There are two cases:  $A$  and  $B$  intersect over the interval  $[x_i, x_{i+1}]$ , or not.

### 2. $A$ AND $B$ DO NOT INTERSECT

To simplify notation, we assume  $i = 1$ , and without loss of generality, we assume  $a_1 > b_1$  and  $a_2 > b_2$ . To find the area between  $A$  and  $B$  we will compute an integral, but we need equations for  $A$  and  $B$  over  $[x_1, x_2]$  to do that. The slope of  $A$  is  $\frac{a_2 - a_1}{x_2 - x_1}$ , thus the equation for  $A$  over  $[x_1, x_2]$  is

$$\begin{aligned} y - a_1 &= \left( \frac{a_2 - a_1}{x_2 - x_1} \right) (x - x_1) \\ y &= \left( \frac{a_2 - a_1}{x_2 - x_1} \right) x - \left( \frac{a_2 - a_1}{x_2 - x_1} \right) x_1 + a_1. \end{aligned}$$

Using an integral to compute the area between  $A$  and  $B$  over  $[x_1, x_2]$ ,

$$\begin{aligned} &\int_{x_1}^{x_2} \left( \frac{a_2 - a_1 - b_2 + b_1}{x_2 - x_1} \right) x - x_1 \left( \frac{a_2 - a_1 - b_2 + b_1}{x_2 - x_1} \right) + (a_1 - b_1) dx \\ &= \left[ \frac{1}{2} \left( \frac{a_2 - a_1 - b_2 + b_1}{x_2 - x_1} \right) x^2 - x_1 \left( \frac{a_2 - a_1 - b_2 + b_1}{x_2 - x_1} \right) x + (a_1 - b_1)x \right]_{x_1}^{x_2} \\ &= \frac{1}{2}(a_2 - a_1 - b_2 + b_1)(x_2 + x_1) - x_1(a_2 - a_1 - b_2 + b_1) + (a_1 - b_1)(x_2 - x_1) \\ &= \frac{1}{2}(a_2 + a_1 - b_2 - b_1)(x_2 - x_1). \end{aligned}$$

This expression can be written as:

$$\frac{1}{2} \cdot \frac{-(a_1 - b_1)^2 + (a_2 - b_2)^2}{a_2 - a_1 - b_2 + b_1} (x_2 - x_1).$$

Why one would want to make the expression more complex will become clear in the next section.

3. *A* AND *B* DO INTERSECT

We use the same simplifying assumptions as above, except we assume  $a_1 > b_1$  and  $a_2 < b_2$ . In this case, the area between *A* and *B* is the sum of the areas of two triangles. We can use the ordinary area formula for a triangle after we find the intersection point of *A* and *B*. At the intersection point  $(x_{\text{int}}, y_{\text{int}})$ , the *y*-values are equal, so:

$$\begin{aligned} \left( \frac{a_2 - a_1}{x_2 - x_1} \right) x_{\text{int}} - x_1 \left( \frac{a_2 - a_1}{x_2 - x_1} \right) + a_1 &= \left( \frac{b_2 - b_1}{x_2 - x_1} \right) x_{\text{int}} - x_1 \left( \frac{b_2 - b_1}{x_2 - x_1} \right) + b_1 \\ \left( \frac{a_2 - a_1 - b_2 + b_1}{x_2 - x_1} \right) x_{\text{int}} &= x_1 \left( \frac{a_2 - a_1 - b_2 + b_1}{x_2 - x_1} \right) + (b_1 - a_1) \\ x_{\text{int}} &= x_1 + \left( \frac{x_2 - x_1}{a_2 - a_1 - b_2 + b_1} \right) (b_1 - a_1). \end{aligned}$$

Now the sum of the areas of the triangles is:

$$\begin{aligned} \frac{1}{2}(a_1 - b_1)(x_{\text{int}} - x_1) + \frac{1}{2}(b_2 - a_2)(x_2 - x_{\text{int}}) &= \frac{1}{2} \cdot \frac{(a_1 - b_1)(b_1 - a_1)}{a_2 - a_1 - b_2 + b_1} (x_2 - x_1) + \frac{1}{2} \cdot \frac{(b_2 - a_2)(a_2 - b_2)}{a_2 - a_1 - b_2 + b_1} (x_2 - x_1) \\ &= \frac{1}{2} \cdot \frac{-(a_1 - b_1)^2 - (a_2 - b_2)^2}{a_2 - a_1 - b_2 + b_1} (x_2 - x_1). \end{aligned}$$

This expression is nearly identical to the one above, differing by a change of sign in the numerator.







