Freezing period strongly impacts the emergence of a global consensus in the voter model

Supplementary Information

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The presence of optimal freezing period H_{opt}: A simple proof.

According to the theoretical framework studying the dynamics of naming games [1, 2], we first analyze the scenario of $H = 1$. Let us consider a single interface separating two neighboring opinion clusters: in each one, all voters possess the same opinion (assuming $+1$ in the left-hand cluster and -1 in the right-hand cluster). For simplicity, we denote a voter with the opinion $+1$ and persistence time $\tau \geq H$ as the individual A, and regard a voter with the opinion -1 and persistence time $\tau \geq H$ as the individual B. Similarly, a voter having the opinion $+1$ and persistence time $\tau < H$ is denoted by the individual C, and a voter holding the opinion -1 and persistence time $\tau < H$ is represented by the individual D. The cluster in which all voters share the same opinion $A(B, C, D)$ is named $A(B, C, D)$-type cluster. We tag an interface with C_mD_n, if it has a C-type cluster of length m and a D-type cluster of length n (i.e., $...AAAC...CD...DBBB...$) in left- and right-hand sides, respectively. In the same way, an interface separating A- and

B-type clusters (i.e., ...$AAABBB$...) is registered as C_0D_0; an interface separating a C-type cluster of length m and a B-type cluster (i.e., ...$AAAC$...$CBBB$...) is denoted by C_mD_0; and an interface separating an A-type cluster and a D-type cluster of length n (i.e., ...$AAAD$...$DBBB$...) is represented by C_0D_n. These four cases cover the situations studied here.

With these definitions, we deduce the stationary probability that two neighboring clusters are separated by a C_mD_n interface. In a one-dimensional lattice composed of N sites, which are initially partitioned into two adjacent clusters of A and B (due to $H = 1$, we neglect the fact that at the first time step the opinion of each voter is either C or D), the probability of selecting the unique C_0D_0 interface is $1/N$. The dynamical rules result in the fact that a C_1D_0 or C_0D_1 interface will be generated with the same probability. Therefore, in a single time step, a C_0D_0 interface becomes a $C_1D_0(C_0D_1)$ interface with the probability $p_{00,10} = 1/2N(p_{00,01} = 1/2N)$; otherwise, it stays in C_0D_0. With respect to C_1D_0, the interface may evolve into C_0D_0, C_2D_0, or C_0D_1 with the probabilities $p_{10,00} = (1/2N) + (1-q)/2N$, $p_{10,20} = 1/2N$, and $p_{10,01} = q/2N$, respectively. Accordingly, the interface C_0D_1 may evolve into C_0D_0, C_0D_2, or C_1D_0. Subsequently, the C_2D_0 interface may evolve into C_0D_0, C_1D_0, or C_1D_1 with the probabilities $p_{20,00} = (1/2N)+(1-q)/2N$, $p_{20,10} = 1/N$, and $p_{20,11} = q/2N$, respectively. Note that we neglect the possible transition that the C_2D_0 interface evolves into C_3D_0, since the precondition $H = 1$ limits that the width of interface is small, i.e., $m+n \leq 2$. The transition rate of C_0D_2 or C_1D_1 interface is derived in the same way. We acquire the stationary probabilities of the Markovian chain defined by the transition matrix

$$U = \begin{pmatrix} 1-1/N & 1/2N & 1/2N & 0 & 0 & 0 \\ (2-q)/2N & 1-3/2N & q/N & 1/2N & 0 & 0 \\ (2-q)/2N & q/2N & 1-3/2N & 0 & 1/2N & 0 \\ (2-q)/2N & 1/N & 0 & 1-2/N & 0 & q/2N \\ (2-q)/2N & 0 & 1/N & 0 & 1-2/N & q/2N \\ 0 & (2-q)/2N & (2-q)/2N & q/2N & q/2N & 1-2/N \end{pmatrix}, \tag{S.1}$$

where the basis is $\{C_0D_0, C_1D_0, C_0D_1, C_2D_0, C_0D_2, C_1D_1\}$. The stationary probability vector $\hat{\mathrm{P}}= P_0, P_1, P_2, ..., P_5$ is obtained by solving $\hat{\mathrm{P}}(t+1)$-$\hat{\mathrm{P}}(t) = 0$, i.e., $(U^T - I)\hat{\mathrm{P}}= 0$, which results in

$$\begin{cases} P_0 = \dfrac{q^3 - 2q^2 - 10q + 20}{q^3 - 4q^2 - 9q + 40}, \\ P_1 = \dfrac{8 - q^2}{q^3 - 4q^2 - 9q + 40}, \\ P_2 = \dfrac{8 - q^2}{q^3 - 4q^2 - 9q + 40}, \\ P_3 = \dfrac{2}{q^3 - 4q^2 - 9q + 40}, \\ P_4 = \dfrac{2}{q^3 - 4q^2 - 9q + 40}, \\ P_5 = \dfrac{q}{q^3 - 4q^2 - 9q + 40}. \end{cases} \tag{S.2}$$

Since the width of interface is small ($m + 2n \leq 2$) in this case, we assume that these interfaces are localized around their central position x. The central position is defined as $x = (x_{left} + x_{right})/2$, where x_{left} is the position of the rightmost side of the left-hand A-type cluster, and x_{right} is the position of the leftmost site of right-hand B-type cluster. With the transition between interfaces, i.e., $C_m D_n \rightarrow C_{m'} D_{n'}$, the central position x will have a set of possible movements. If we define $W(x \rightarrow x \pm \delta)$ as the transition probability for an interface centered at x moving to the position $x \pm \delta$, the symmetric items characterizing the movements of interfaces is obtained by enumeration of all possible cases:

$$W(x \rightarrow x + \frac{1}{2}) = \frac{1}{2N} P_1 + \frac{3-q}{2N} P_2 + 0 * P_3 + \frac{1}{N} P_4 + 0 * P_5 + \frac{2-q}{2N} P_6, \tag{S.3}$$

$$W(x \rightarrow x - \frac{1}{2}) = \frac{1}{2N} P_1 + 0 * P_2 + \frac{3-q}{2N} P_3 + 0 * P_4 + \frac{1}{N} P_5 + \frac{2-q}{2N} P_6, \tag{S.4}$$

$$W(x \rightarrow x + 1) = \frac{2-q}{2N} P_4, \tag{S.5}$$

$$W(x \rightarrow x - 1) = \frac{2-q}{2N} P_5. \tag{S.6}$$

With Eq.S.2, we have the specific values of the transition probabilities in the above equations.

Denote $\mathcal{P}(x, t)$ as the probability that the interface is located in the position x at time t. With $q = 0.01$, we present the numerical results of the evolution of the position of a typical interface ...$AAABBB$... in Figure S1. The probability $\mathcal{P}(x, t)$ shows a Gaussian distribution around the initial position, while the mean-square displacement reached by the interface at time t accords with a diffusion law $< x^2 >= 2D_{exp}^{H=1} t/N$ with the diffusion coefficient $D_{exp}^{H=1} \simeq 0.1997$.

In the standard voter model, there exists only one kind of interface (i.e., ...$AAABBB$...). Under the above theoretical framework, it is evident that at each time step the central position x can only move to the position $x \pm \frac{1}{2}$, and thus the transition probability only has one symmetric contribution:

$$W(x \rightarrow x \pm \frac{1}{2}) = \frac{1}{2N}. \tag{S.7}$$

The numerical results show that the diffusion coefficient $D_{exp}^{H=0} = 0.125$.
The dynamics of the voter model on a one-dimensional lattice can be sketched as follows: at the initial time steps, local interactions lead to the formation of a great many of small opinion clusters. As time proceeds, the interfaces begin to diffuse. Once two interfaces encounter each other, the cluster located between the interfaces annihilates, which means that these two interfaces will vanish and the adjacent clusters merge together to form a larger cluster. Since the diffusion coefficient of the scenario $H = 1$ is larger than that of the standard voter model ($H = 0$), the introduction of the freezing period $H = 1$ reduces the consensus time T_c.
When $H > 1$, the theoretical framework is also available in similar analysis. Due to the fact that the number of possible cluster interfaces becomes very huge (the procedure will be utmostly complicated), we can not present the details in the present work. But if H increases to an extremely large value, e.g., $H = 4000$, it is obvious that the freezing period itself will sustain the existence of opinion clusters for a long time. Meanwhile, a long freezing period seriously impairs the effect of biased random walks of the interfaces (as shown in Figure 3 of the text), which dramatically defers the formation of global consensus. Since the variance of T_c experiences a valley as the freezing period H increases, we ensure that an optimal value of H (H_{opt}) producing the shortest T_c definitely exists.

References

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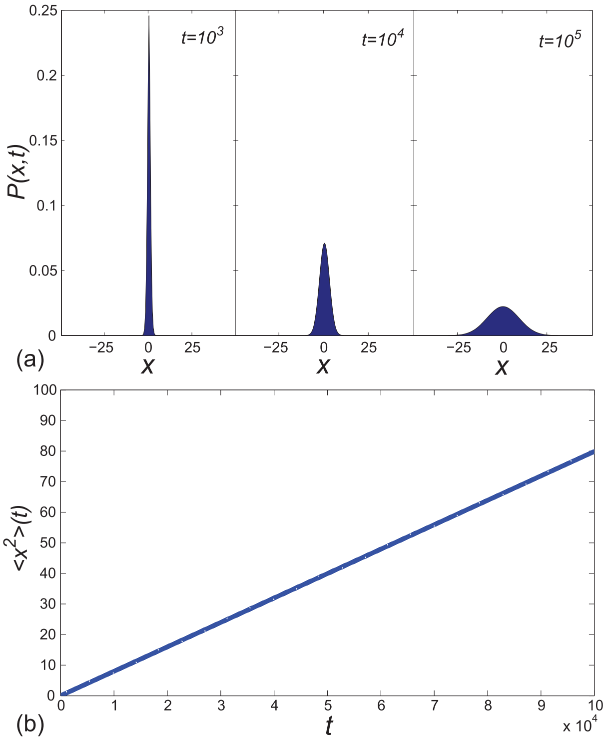

Figure S1. Evolution of the position of a given interface $...AAABBB...$ **with** $H = 1$**,** $q = 0.01$**.** (a) Evolution of the distribution $\mathcal{P}(x, t)$. (b) Evolution of the mean-square displacement, which shows that there presents a diffusion law $< x^2 >= 2D_{exp}t/\mathcal{N}$ with a coefficient $D_{exp}^{H=1} \simeq 0.1997$.