# SUPPLEMENTARY MATERIAL TO "MARGINAL EMPIRICAL LIKELIHOOD AND SURE INDEPENDENCE FEATURE SCREENING"

By Jinyuan Chang* , Cheng Yong Tang and Yichao Wu†

*Peking University, University of Colorado Denver, North Carolina State University*

This Supplementary Material contains all techinical proofs.

## APPENDIX

Proof of Proposition 1: Note that

$$\mathrm{EL}(\mu)=\sup\bigg\{\prod_{i=1}^n w_i: w_i\geq 0,\ \sum_{i=1}^n w_i=1,\ \sum_{i=1}^n w_i(U_i-\mu)=0\bigg\}.$$

Define $V_i=U_i-\mu$ for any $i=1,\ldots,n$. Without lose of generality, we set $\mu_0>\mu$. Then,

$$\ell(\mu)=-2\log\{\mathrm{EL}(\mu)\}-2n\log n=2\sum_{i=1}^n\log(1+\lambda V_i)$$

where $\lambda$ satisfies $0=\frac{1}{n}\sum_{i=1}^n\frac{V_i}{1+\lambda V_i}$. Let $g(\lambda)=\frac{1}{n}\sum_{i=1}^n\frac{V_i}{1+\lambda V_i}$. Note that

$$g\bigg(\frac{2(\mu_0-\mu)}{S^2}\bigg)=\frac{1}{n}\sum_{i=1}^n V_i-\frac{2(\mu_0-\mu)}{S^2n}\sum_{i=1}^n\frac{V_i^2}{1+2(\mu_0-\mu)S^{-2}V_i},$$

*Supported in part by a National Natural Science Foundation of China Key Program Grant (No.11131002) and Center for Statistical Science at Peking University. This work was completed during Jinyuan Chang's PhD study at Peking University.

†Supported in part by NSF grants DMS-0905561 and DMS-1055210 and NIH/NCI grant R01-CA149569.

*AMS 2000 subject classifications:* Primary 62G09; secondary 62H99

*Keywords and phrases:* Empirical likelihood; high dimensional data analysis; independence sure screening; large deviation.

where $S^2 = \frac{1}{n}\sum_{i=1}^n V_i^2$. By Markov inequality, $\sup_{1\le i\le n}|V_i| = O_p(n^{\frac{1}{\nu}})$. Then, $\sup_{1\le i\le n}|(\mu_0-\mu)V_i| = o_p(1)$. Hence,

$$g\biggl(\frac{2(\mu_0-\mu)}{S^2}\biggr) = \frac{1}{n}\sum_{i=1}^n(U_i-\mu_0) - (\mu_0-\mu)\{1+o_p(1)\}.$$

Using the same argument of [1], we know that $\lambda = O_p\{|\mu_0-\mu|\}$.

By Taylor expansion,

$$\ell(\mu) = 2\sum_{i=1}^n\biggl\{\lambda V_i - \frac{1}{2}\lambda^2V_i^2 + \frac{\lambda^3V_i^3}{3(1+c_{i1}\lambda V_i)^3}\biggr\},$$

where $|c_{i1}|<1$ for all $i=1,\ldots,n$. On the other hand,

$$\lambda = \biggl(\frac{1}{n}\sum_{i=1}^n V_i^2\biggr)^{-1}\biggl(\frac{1}{n}\sum_{i=1}^n V_i\biggr) + \biggl(\frac{1}{n}\sum_{i=1}^n V_i^2\biggr)^{-1}\biggl\{\frac{1}{n}\sum_{i=1}^n\frac{\lambda^2V_i^3}{(1+c_{i2}\lambda V_i)^3}\biggr\},$$

where $|c_{i2}|<1$ for all $i=1,\ldots,n$. Then,

$$\ell(\mu) = n\biggl(\frac{1}{n}\sum_{i=1}^n V_i^2\biggr)^{-1}\biggl(\frac{1}{n}\sum_{i=1}^n V_i\biggr)^2 - n\biggl(\frac{1}{n}\sum_{i=1}^n V_i^2\biggr)^{-1}\biggl\{\frac{1}{n}\sum_{i=1}^n\frac{\lambda^2V_i^3}{(1+c_{i2}\lambda V_i)^3}\biggr\}^2 + \frac{2}{3}\sum_{i=1}^n\frac{\lambda^3V_i^3}{(1+c_{i1}\lambda V_i)^3}.$$

Hence,

$$\ell(\mu) = n\biggl[\frac{1}{n}\sum_{i=1}^n(U_i-\mu)^2\biggr]^{-1}\biggl[\frac{1}{n}\sum_{i=1}^n(U_i-\mu)\biggr]^2 + O_p\{n|\mu_0-\mu|^3\}.$$

As

$$n\biggl[\frac{1}{n}\sum_{i=1}^n(U_i-\mu)\biggr]^2 = n\biggl[\frac{1}{n}\sum_{i=1}^n(U_i-\mu_0)\biggr]^2 + n(\mu_0-\mu)^2 + 2n(\mu_0-\mu)\cdot\frac{1}{n}\sum_{i=1}^n(U_i-\mu_0),$$

noting $n(\mu_0-\mu)^2\to\infty$ and $\mu_0-\mu\to 0$, then

$$\frac{\ell(\mu)}{n(\mu_0-\mu)^2\sigma^{-2}} \xrightarrow{p} 1.$$

We complete the proof of this proposition. □

In order to establish the Theorem 1, we need the following lemma which is a large derivation result widely used in our following proofs.

LEMMA 1. *For independent and identically distributed random variables $\xi_1,\ldots,\xi_n$, suppose that there exist three positive constants $\widetilde{K}_1$, $\widetilde{K}_2$ and $\gamma$ such that $\mathbb{P}\{|\xi_i|>u\}\le \widetilde{K}_1\exp(-\widetilde{K}_2u^\gamma)$ for all $u>0$. Define $\mu_0=\mathbb{E}(\xi_i)$, $\delta=\max\{\frac{2}{\gamma}-1,0\}$, $H=2^{1+\delta}$ and $\bar{\Delta}=\frac{n^{\frac{1}{2}}\sigma}{2K}$, where $\sigma^2=\mathbb{E}\{(\xi_i-\mu_0)^2\}$ and $K>\sigma$ is a sufficiently large positive constant depending only on $\widetilde{K}_1$, $\widetilde{K}_2$ , $\gamma$ and $\mu_0$, then*

$$\mathbb{P}\bigg\{\pm\frac{1}{n^{\frac{1}{2}}\sigma}\sum_{i=1}^n(\xi_i-\mu_0)\ge x\bigg\}
\le\begin{cases}\exp\Big\{-\dfrac{x^2}{4H}\Big\}, & \text{if } 0\le x\le (H^{1+\delta}\bar{\Delta})^{\frac{1}{1+2\delta}};\\ \exp\Big\{-\dfrac{1}{4}(x\bar{\Delta})^{\frac{1}{1+\delta}}\Big\}, & \text{if } x\ge (H^{1+\delta}\bar{\Delta})^{\frac{1}{1+2\delta}}.\end{cases}$$

**Proof**: The key step is to bound $|\mathbb{E}(\xi_i^k)|$ for each $k\ge 3$. Note that

$$|\mathbb{E}(\xi_i^k)|\le \mathbb{E}(|\xi_i|^k)\le\int_0^\infty \widetilde{K}_1\exp\{-\widetilde{K}_2u^{\frac{\gamma}{k}}\}du=\frac{\widetilde{K}_1k}{\gamma\widetilde{K}_2^{\frac{k}{\gamma}}}\Gamma\left(\frac{k}{\gamma}\right).$$

If we pick $\delta=\max\{\frac{2}{\gamma}-1,0\}$, then there exists a positive constant $l$ such that $k^{\frac{k}{\gamma}+\frac{1}{2}}\le (k!)^{1+\delta}l^k$. By Stirling formula,

$$\Gamma\left(\frac{k}{\gamma}\right)=\sqrt{\frac{2\pi\gamma}{k}}\left(\frac{k}{e\gamma}\right)^{\frac{k}{\gamma}}\{1+O(k^{-1})\}.$$

Hence, there exists a positive constant $\tilde{K}>\sigma$ such that

$$|\mathbb{E}(\xi_i^k)|\le (k!)^{1+\delta}\tilde{K}^{k-2}\sigma^2,\quad k=3,4,\ldots.$$

Note that $\mathbb{E}(|\xi_i-\mu_0|^k)\le 2^{k-1}\{\mathbb{E}(|\xi_i|^k)+|\mu_0|^k\}$, then there exists a positive constant $K>\sigma$ depending only on $\widetilde{K}_1$, $\widetilde{K}_2$, $\gamma$ and $\mu_0$ such that

$$|\mathbb{E}\{(\xi_i-\mu_0)^k\}|\le (k!)^{1+\delta}K^{k-2}\sigma^2,\quad k=3,4,\ldots.$$

From Theorem 3.1 of [3], the result holds. □

PROOF OF THEOREM 1: Without lose of generality, let $\mu=0$. From [2], we can obtain that $\ell(0)=2\max_{\lambda\in\Lambda_n}\sum_{i=1}^n\log(1+\lambda U_i)$ where $\Lambda_n=\{\lambda:$

$1+\lambda U_i \geq n^{-1}$ for all $i=1,\ldots,n\}$. Pick $\lambda = (n^{\epsilon}\max_l |U_l|)^{-1}$ for some $\epsilon > 0$, then $\lambda \in \Lambda_n$ for sufficiently large $n$. We assume $\mu_0 = \mathbb{E}(U_i) > 0$. If $\mu_0 < 0$, define $\tilde{U}_i = -U_i$. We can find that

$$\begin{aligned}\mathrm{EL}(0) &= \sup\bigg\{\prod_{i=1}^n w_i : w_i \geq 0,\ \sum_{i=1}^n w_i = 1,\ \sum_{i=1}^n w_i U_i = 0\bigg\}\\ &= \sup\bigg\{\prod_{i=1}^n w_i : w_i \geq 0,\ \sum_{i=1}^n w_i = 1,\ \sum_{i=1}^n w_i \tilde{U}_i = 0\bigg\},\end{aligned}$$

then $\ell(0) = -2\log\{\mathrm{EL}(0)\} - 2n\log n$ does not depend on the sign of $\mu_0$. Pick $t > 0$,

$$\mathbb{P}\{\ell(0) < 2t\} \leq \mathbb{P}\bigg\{\sum_{i=1}^n \log\bigg[1 + \frac{U_i}{n^{\epsilon}\max_l |U_l|}\bigg] < t\bigg\}.$$

We will give an upper bound for the one on the right-hand side of above inequality. Note that

$$\log\bigg[1+\frac{U_i}{n^{\epsilon}\max_l |U_l|}\bigg] = \frac{U_i}{n^{\epsilon}\max_l |U_l|} - \frac{1}{2(1+c_i)^2}\frac{U_i^2}{n^{2\epsilon}\max_l |U_l|^2}$$

where $|c_i| \leq n^{-\epsilon}$, then

$$\sum_{i=1}^n \log\bigg[1+\frac{U_i}{n^{\epsilon}\max_l |U_l|}\bigg] = \sum_{i=1}^n \frac{U_i}{n^{\epsilon}\max_l |U_l|} + R_n,$$

where $|R_n| \leq n^{1-2\epsilon}$. Hence,

$$\mathbb{P}\{\ell(0) < 2t\} \leq \mathbb{P}\bigg\{\sum_{i=1}^n \frac{U_i}{n^{\epsilon}\max_l |U_l|} < t + n^{1-2\epsilon}\bigg\}.$$

It means that

$$\begin{aligned}\mathbb{P}\{\ell(0) < 2t\} &\leq \mathbb{P}\bigg\{\sum_{i=1}^n (U_i - \mu_0) < (tn^{\epsilon} + n^{1-\epsilon})\max_l |U_l| - n\mu_0\bigg\}\\ &\leq \mathbb{P}\bigg\{\frac{1}{n^{\frac{1}{2}}\sigma}\sum_{i=1}^n (U_i-\mu_0) < \frac{(tn^{\epsilon-\frac{1}{2}} + n^{\frac{1}{2}-\epsilon})M - n^{\frac{1}{2}}\mu_0}{\sigma}\bigg\}\\ &\quad + K_1\exp\{-K_2 M^{\gamma} + \log n\}.\end{aligned}$$

For $L \to \infty$, pick $\epsilon$ satisfies $n^{\epsilon} = \frac{L}{\mu_0}$. Choose $\eta \in (0, \frac{2}{3})$ and let $M = \eta L$ and $2t = \frac{n\mu_0^2}{L^2}$, then $\frac{tn^{\epsilon}M}{n\mu_0} = \frac{\eta}{2}$ and $\frac{n^{1-\epsilon}M}{n\mu_0} = \eta$. Hence, by Lemma 1, we can obtain the result. □

In the following, we consider the corresponding results in the procedure of variable screening. For given $j$, define $Z_{ij} = X_{ij}Y_i$.

LEMMA 2. *Under assumption A.2, then*

$$\mathbb{P}\{|Z_{ij}| > u\} \le 2K_1 \exp\{-K_2 u^{\gamma}\} \quad \textit{for any} \quad j = 1, \dots, p,$$

*where* $\gamma = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}$.

**Proof**: Pick $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}\{|Z_{ij}| > u\} &= \mathbb{P}\{|X_{ij}| > u^{\epsilon}, |X_{ij}Y_i| > u\} + \mathbb{P}\{|X_{ij}| \le u^{\epsilon}, |X_{ij}Y_i| > u\} \\ &\le \mathbb{P}\{|X_{ij}| > u^{\epsilon}\} + \mathbb{P}\{|Y_i| > u^{1-\epsilon}\} \\ &\le K_1 \exp\{-K_2 u^{\epsilon\gamma_1}\} + K_1 \exp\{-K_2 u^{(1-\epsilon)\gamma_2}\}.\end{aligned}$$

In order to get the best rate for the right-hand side of above inequality, we need $\epsilon\gamma_1 = (1-\epsilon)\gamma_2$. It means that $\epsilon = \frac{\gamma_2}{\gamma_1+\gamma_2}$. Hence,

$$\mathbb{P}\{|Z_{ij}| > u\} \le 2K_1 \exp\{-K_2 u^{\gamma}\}$$

where $\gamma = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}$. We complete the proof of this lemma. □

PROOF OF PROPOSITION 2: Note that $|\mu_j| \le \{\mathbb{E}(X_{ij}^2)\}^{\frac{1}{2}}\{\mathbb{E}(Y_i^2)\}^{\frac{1}{2}}$, then $|\mu_j|$ can be bounded by a uniform constant. From A.1, $n\mu_j^2 \ge c_1^2 n^{1-2\kappa}$ for any $j \in \mathcal{M}_*$. Then, $\mathbb{P}\{\ell_j(0) < \frac{c_1^2 n^{1-2\kappa}}{L^2}\} \le \mathbb{P}\{\ell_j(0) < \frac{n\mu_j^2}{L^2}\}$. Using Lemma 2 and Theorem 1, we can bound the one on the right-hand side of above inequality. The constant $C_1$ appeared in Proposition 2 depends only on $K_1$, $K_2$ and $\gamma$. Then, Proposition 2 holds. □

PROOF OF THEOREM 2: Note that

$$\begin{aligned}\mathbb{P}\{\mathcal{M}_* \subsetneq \widehat{\mathcal{M}}_{\gamma_n}\} &= \mathbb{P}\{\text{There exists } j \in \mathcal{M}_* \text{ such that } \ell_j(0) < c_1^2 n^{2\tau}\} \\ &\le s \max_{j\in\mathcal{M}_*} \mathbb{P}\{\ell_j(0) < c_1^2 n^{2\tau}\},\end{aligned}$$

then the result holds. □

Lemmas 3 and 4 are two key results for the proof of Proposition 3. These two lemmas together show that the $\lambda$ defined by $0 = \frac{1}{n}\sum_{i=1}^n \frac{U_i}{1+\lambda U_i}$ can be bounded by $(|\frac{4}{n}\sum_{i=1}^n U_i|)(\frac{3}{n}\sum_{i=1}^n U_i^2)^{-1}$ with high probability if $\mathbb{E}(U_i) =$

0. In these two lemmas, we both assume the random variables $U_1, \dots, U_n$ are independent and identically distributed and there exist three positive constants $\widetilde{K}_1$, $\widetilde{K}_2$ and $\gamma$ such that $\mathbb{P}\{|U_i| > u\} \le \widetilde{K}_1 \exp(-\widetilde{K}_2 u^{\gamma})$ for all $u > 0$. We also suppose $\sigma^2 = \mathbb{E}(U_i^2) \ge M$ for some positive constant $M$. Throughout the proofs of the two lemmas, $C$ will denote a generic positive constant depending only on $\widetilde{K}_1$, $\widetilde{K}_2$, $\gamma$ and $M$, that may be different in different uses.

Lemma 3. *If $\mathbb{E}(U_i) = 0$, then*

$$\mathbb{P}\bigg\{\frac{|n^{-1}\sum_{i=1}^n U_i|}{n^{-1}\sum_{i=1}^n U_i^2}\max_l |U_l| \ge \frac{1}{4}\bigg\} \le \begin{cases} \exp(-Cn^{\frac{\gamma}{4}}), & \text{if } 0<\gamma<2; \\ \exp(-Cn^{\frac{\gamma}{\gamma+2}}), & \text{if } \gamma \ge 2. \end{cases}$$

**Proof**: Pick $\epsilon \in (0, \frac{1}{2})$, then

$$\begin{aligned} &\mathbb{P}\bigg\{\frac{|n^{-1}\sum_{i=1}^n U_i|}{n^{-1}\sum_{i=1}^n U_i^2}\max_l |U_l| \ge \frac{1}{4}\bigg\} \\ =\ &\mathbb{P}\bigg\{\frac{|n^{-1}\sum_{i=1}^n U_i|}{n^{-1}\sum_{i=1}^n U_i^2}\max_l |U_l| \ge \frac{1}{4}, \max_l |U_l| > n^{\epsilon}/4\bigg\} \\ &+ \mathbb{P}\bigg\{\frac{|n^{-1}\sum_{i=1}^n U_i|}{n^{-1}\sum_{i=1}^n U_i^2}\max_l |U_l| \ge \frac{1}{4}, \max_l |U_l| \le n^{\epsilon}/4\bigg\} \\ =:\ & I_1 + I_2. \end{aligned}$$

For $I_1$, we have $I_1 \le n\widetilde{K}_1 \exp(-4^{-\gamma}\widetilde{K}_2 n^{\epsilon\gamma})$. On the other hand, noting that

$$\mathbb{P}\bigg\{\bigg|\frac{1}{n}\sum_{i=1}^n U_i\bigg| \ge \frac{1}{n^{1+\epsilon}}\sum_{i=1}^n U_i^2\bigg\} \le \mathbb{P}\bigg\{\bigg|\frac{1}{n}\sum_{i=1}^n U_i\bigg| \ge \frac{\sigma^2}{2n^{\epsilon}}\bigg\} + \mathbb{P}\bigg\{\frac{1}{n}\sum_{i=1}^n U_i^2 < \frac{\sigma^2}{2}\bigg\},$$

then

$$I_2 \le \mathbb{P}\bigg\{\bigg|\frac{1}{n}\sum_{i=1}^n U_i\bigg| \ge \frac{\sigma^2}{2n^{\epsilon}}\bigg\} + \mathbb{P}\bigg\{\frac{1}{n}\sum_{i=1}^n U_i^2 < \frac{\sigma^2}{2}\bigg\} =: A_1 + A_2.$$

We will use large derivation results to bound the two parts on the right-hand side. For $A_1$, define $\delta = \max\{\frac{2}{\gamma} - 1, 0\}$, by Lemma 1,

$$A_1 \le \begin{cases} 2\exp(-Cn^{1-2\epsilon}), & \text{if } (1-2\epsilon)(1+2\delta) < 1; \\ 2\exp(-Cn^{\frac{1-\epsilon}{1+\delta}}), & \text{if } (1-2\epsilon)(1+2\delta) \ge 1. \end{cases}$$

For $A_2$, define $\tilde{\sigma}^2 = \text{var}(U_i^2)$ and $\tilde{\delta} = \max\{\frac{4}{\gamma} - 1, 0\}$, by Lemma 1,

$$A_2 = \mathbb{P}\left\{\frac{1}{n^{\frac{1}{2}}\tilde{\sigma}}\sum_{i=1}^{n}(U_i^2 - \sigma^2) < -\frac{n^{\frac{1}{2}}\sigma^2}{2\tilde{\sigma}}\right\} \leq \exp(-Cn^{\frac{1}{1+\tilde{\delta}}}).$$

Aim to obtain the optimal rate, we pick $\epsilon = \frac{1}{4}\mathbb{1}\{0 < \gamma < 2\} + \frac{1}{\gamma+2}\mathbb{1}\{\gamma \geq 2\}$. Then, we complete the proof. □

LEMMA 4. *If* $\mathbb{E}(U_i) = 0$*, then*

$$\mathbb{P}\left\{|\lambda| \geq \frac{4|n^{-1}\sum_{i=1}^{n}U_i|}{3n^{-1}\sum_{i=1}^{n}U_i^2}\right\} \leq \begin{cases} \exp(-Cn^{\frac{\gamma}{4}}), & \text{if } 0 < \gamma < 2; \\ \exp(-Cn^{\frac{\gamma}{\gamma+2}}), & \text{if } \gamma \geq 2; \end{cases}$$

*where* $\lambda$ *is defined by* $0 = \frac{1}{n}\sum_{i=1}^{n}\frac{U_i}{1+\lambda U_i}$.

**Proof**: Note that

$$\lambda = \left(\frac{1}{n}\sum_{i=1}^{n}U_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{U_i^2}{1+\lambda U_i}\right)^{-1}.$$

If $\frac{1}{n}\sum_{i=1}^{n}U_i > 0$, then $\lambda > 0$. Note that $\frac{1}{n} < 1 + \lambda\min_l U_l < 1 + \lambda\max_l U_l$, then

$$0 < \lambda \leq \frac{n^{-1}\sum_{i=1}^{n}U_i}{n^{-1}\sum_{i=1}^{n}U_i^2}\left(1 + \lambda\max_l U_l\right).$$

On the other hand, if $\frac{1}{n}\sum_{i=1}^{n}U_i < 0$, then $\lambda < 0$, by the same argument,

$$0 > \lambda \geq \frac{n^{-1}\sum_{i=1}^{n}U_i}{n^{-1}\sum_{i=1}^{n}U_i^2}\left(1 + \lambda\max_l U_l\right).$$

Then

$$|\lambda| \leq \frac{|n^{-1}\sum_{i=1}^{n}U_i|}{n^{-1}\sum_{i=1}^{n}U_i^2}\left(1 + |\lambda|\max_l |U_l|\right).$$

Hence,

$$\mathbb{P}\left\{|\lambda| < \frac{4|n^{-1}\sum_{i=1}^{n}U_i|}{3n^{-1}\sum_{i=1}^{n}U_i^2}\right\} \geq \mathbb{P}\left\{\frac{|n^{-1}\sum_{i=1}^{n}U_i|}{n^{-1}\sum_{i=1}^{n}U_i^2}\max_l |U_l| < \frac{1}{4}\right\}.$$

From Lemma 3, we complete the proof. □

Proof of Proposition 3: For any $j \notin \mathcal{M}_*$, $\mu_j = 0$, then

$$\begin{aligned}\ell_j(0) &= n\bigg(\frac{1}{n}\sum_{i=1}^n Z_{ij}^2\bigg)^{-1}\bigg(\frac{1}{n}\sum_{i=1}^n Z_{ij}\bigg)^2 - n\bigg(\frac{1}{n}\sum_{i=1}^n Z_{ij}^2\bigg)^{-1}\bigg\{\frac{1}{n}\sum_{i=1}^n \frac{\lambda^2 Z_{ij}^3}{(1+c_{i2}\lambda Z_{ij})^3}\bigg\}^2\\ &\quad + \frac{2}{3}\sum_{i=1}^n \frac{\lambda^3 Z_{ij}^3}{(1+c_{i1}\lambda Z_{ij})^3}\\ &=: I_1 + I_2 + I_3.\end{aligned}$$

If $|\lambda| < |\frac{4}{n}\sum_{i=1}^n Z_{ij}|(\frac{3}{n}\sum_{i=1}^n Z_{ij}^2)^{-1}$, then

$$\max_l |\lambda Z_{lj}| < \frac{|4n^{-1}\sum_{i=1}^n Z_{ij}|}{3n^{-1}\sum_{i=1}^n Z_{ij}^2}\max_l |Z_{lj}|.$$

In addition, if $|\frac{1}{n}\sum_{i=1}^n Z_{ij}|\max_l |Z_{lj}| < \frac{1}{4n}\sum_{i=1}^n Z_{ij}^2$, then $\max_l |\lambda Z_{lj}| < \frac{1}{3}$. Define

$$\mathcal{A} = \bigg\{|\lambda| < \frac{|4n^{-1}\sum_{i=1}^n Z_{ij}|}{3n^{-1}\sum_{i=1}^n Z_{ij}^2} \text{ and } \bigg|\frac{1}{n}\sum_{i=1}^n Z_{ij}\bigg|\max_l |Z_{lj}| < \frac{1}{4n}\sum_{i=1}^n Z_{ij}^2\bigg\},$$

then by Lemma 3 and Lemma 4,

$$\mathbb{P}(\mathcal{A}^c) \le \begin{cases} \exp(-Cn^{\frac{\gamma}{4}}), \text{ if } 0 < \gamma < 2;\\ \exp(-Cn^{\frac{\gamma}{\gamma+2}}), \text{ if } \gamma \ge 2.\end{cases}$$

If $\mathcal{A}$ holds,

$$|I_3| \le C\bigg(\sum_{i=1}^n |Z_{ij}|^3\bigg)\bigg|\frac{1}{n}\sum_{i=1}^n Z_{ij}\bigg|^3\bigg(\frac{1}{n}\sum_{i=1}^n Z_{ij}^2\bigg)^{-3}.$$

Note that

$$\mathbb{P}\{\ell_j(0) \ge c_1^2 n^{2\tau}\} \le \mathbb{P}\bigg\{I_1 \ge \frac{c_1^2 n^{2\tau}}{2}\bigg\} + \mathbb{P}\bigg\{I_3 \ge \frac{c_1^2 n^{2\tau}}{2}, \mathcal{A} \text{ holds}\bigg\} + \mathbb{P}(\mathcal{A}^c),$$

by the same way in the proofs of Lemma 3 and noting Lemma 1, we can get

$$\mathbb{P}\bigg\{I_1 \ge \frac{c_1^2 n^{2\tau}}{2}\bigg\} \le \begin{cases} \exp(-Cn^{1\wedge\frac{\gamma}{4}}) + \exp(-Cn^{2\tau}), \text{ if } 2\tau < \frac{1}{1+2\delta};\\ \exp(-Cn^{1\wedge\frac{\gamma}{4}}) + \exp(-Cn^{\frac{2\tau+1}{2+2\delta}}), \text{ if } 2\tau \ge \frac{1}{1+2\delta};\end{cases}$$

and

$$\mathbb{P}\bigg\{I_3 \ge \frac{c_1^2 n^{2\tau}}{2}, \mathcal{A} \text{ holds}\bigg\} \le \begin{cases} \exp(-Cn^{1\wedge\frac{\gamma}{6}}) + \exp(-Cn^{\frac{4\tau+1}{3}}), \text{ if } \frac{4\tau+1}{3} < \frac{1}{1+2\delta};\\ \exp(-Cn^{1\wedge\frac{\gamma}{6}}) + \exp(-Cn^{\frac{2\tau+2}{3+3\delta}}), \text{ if } \frac{4\tau+1}{3} \ge \frac{1}{1+2\delta};\end{cases}$$

where $\delta = \max\{\frac{2}{\gamma} - 1, 0\}$. Then,

$$\mathbb{P}\Big\{I_1 \geq \frac{c_1^2 n^{2\tau}}{2}\Big\} + \mathbb{P}\Big\{I_3 \geq \frac{c_1^2 n^{2\tau}}{2}, \mathcal{A} \text{ holds}\Big\} \leq \begin{cases} \exp(-Cn^{2\tau}), \text{ if } \tau \leq \frac{\gamma}{12}; \\ \exp(-Cn^{\frac{\gamma}{6}}), \text{ if } \tau > \frac{\gamma}{12}. \end{cases}$$

Combining with the upper bound of $\mathbb{P}(\mathcal{A}^c)$, we can obtain the result. □

Similar to the proof of Proposition 3, the following two lemmas are needed for Proposition 4, which can be viewed as the extension of the results in Lemma 3 and Lemma 4. We still assume that $U_1, \ldots, U_n$ are independent and identically distributed random variables and there exist three positive constants $\widetilde{K}_1$, $\widetilde{K}_2$ and $\gamma$ such that $\mathbb{P}\{|U_i| > u\} \leq \widetilde{K}_1 \exp(-\widetilde{K}_2 u^\gamma)$ for all $u > 0$. We also suppose $\sigma^2 = \mathbb{E}\{(U_i - \mu_0)^2\} \geq M$ for some positive constant $M$, where $\mu_0 = \mathbb{E}(U_i)$. Throughout the following proofs, $C$ denote a generic positive constant which only depending on $\widetilde{K}_1$, $\widetilde{K}_2$ and $\gamma$ that may be different in different uses. Let $V_i = U_i - \mu$ and $S^2 = \frac{1}{n}\sum_{i=1}^n V_i^2$.

Lemma 5. *Suppose that $|\mu - \mu_0| = O(1)$, then*

$$\mathbb{P}\Big\{S^2 \leq \frac{\mathbb{E}\{(U_i - \mu)^2\}}{2}\Big\} \leq \exp(-Cn^{1\wedge\frac{\gamma}{4}}).$$

**Proof**: Note that

$$\begin{aligned} &\mathbb{P}\Big\{S^2 \leq \frac{\mathbb{E}\{(U_i - \mu)^2\}}{2}\Big\} \\ &\leq \mathbb{P}\Big\{\frac{1}{n}\sum_{i=1}^n \Big[(U_i - \mu_0)^2 - \sigma^2\Big] \leq -\frac{\sigma^2}{4}\Big\} + \mathbb{P}\Big\{\frac{2(\mu_0 - \mu)}{n}\sum_{i=1}^n (U_i - \mu_0) \leq -\frac{\sigma^2}{4}\Big\} \\ &=: I_1 + I_2. \end{aligned}$$

By Lemma 1, $I_1 \leq \exp(-Cn^{\frac{1}{1+\tilde{\delta}}})$ where $\tilde{\delta} = \max\{\frac{4}{\gamma} - 1, 0\}$ and $I_2 \leq \exp\{-C(n|\mu - \mu_0|^{-1})^{\frac{1}{1+\delta}}\}$ where $\delta = \max\{\frac{2}{\gamma} - 1, 0\}$. Then, we can obtain the result. □

LEMMA 6. *Let $|\mu-\mu_0| = O(n^{-w})$ for some $w>0$, then*

$$\mathbb{P}\bigg\{|\lambda| > \frac{4|n^{-1}\sum_{i=1}^n V_i|}{3S^2}\bigg\} \leq \begin{cases} \exp(-Cn^{\gamma w}), & \text{if } \gamma\geq 2 \text{ and } w\leq \frac{1}{2+\gamma};\\ \exp(-Cn^{\frac{\gamma}{\gamma+2}}), & \text{if } \gamma\geq 2 \text{ and } w> \frac{1}{2+\gamma};\\ \exp(-Cn^{\gamma w}), & \text{if } \gamma< 2 \text{ and } w\leq \frac{1}{4};\\ \exp(-Cn^{\frac{\gamma}{4}}), & \text{if } \gamma< 2 \text{ and } w> \frac{1}{4}.\end{cases}$$

*where $\lambda$ is defined by $0=\frac{1}{n}\sum_{i=1}^n \frac{V_i}{1+\lambda V_i}$.*

**Proof**: By the same argument in the proof of Lemma 4, we only need to consider $\mathbb{P}\{S^{-2}|\frac{1}{n}\sum_{i=1}^n V_i|\max_l|V_l|\geq\frac{1}{4}\}$. Pick $\epsilon\in(0,w]$, then

$$\begin{aligned}&\mathbb{P}\bigg\{\frac{|n^{-1}\sum_{i=1}^n V_i|}{S^2}\max_l|V_l|\geq\frac{1}{4}\bigg\}\\ &\leq \mathbb{P}\bigg\{\bigg|\frac{1}{n}\sum_{i=1}^n V_i\bigg|\geq \tilde{C}n^{-\epsilon}\bigg\}+\exp(-Cn^{\epsilon\gamma})+\exp(-Cn^{1\wedge\frac{\gamma}{4}}).\end{aligned}$$

Note that $\mathbb{P}\{|\frac{1}{n}\sum_{i=1}^n V_i|\geq\tilde{C}n^{-\epsilon}\}\leq\mathbb{P}\{|\frac{1}{n}\sum_{i=1}^n(U_i-\mu_0)|\geq\frac{\tilde{C}n^{-\epsilon}}{2}\}$ as $n$ is sufficiently large and $\tilde{C}$ is sufficiently large. Then, by Lemma 1,

$$\mathbb{P}\bigg\{\bigg|\frac{1}{n}\sum_{i=1}^n V_i\bigg|\geq\tilde{C}n^{-\epsilon}\bigg\}\leq\begin{cases}\exp(-Cn^{1-2\epsilon}), & \text{if } (1-2\epsilon)(1+2\delta)<1;\\ \exp(-Cn^{\frac{1-\epsilon}{1+\delta}}), & \text{if } (1-2\epsilon)(1+2\delta)\geq 1.\end{cases}$$

Hence,

$$\mathbb{P}\bigg\{\frac{|n^{-1}\sum_{i=1}^n V_i|}{S^2}\max_l|V_l|\geq\frac{1}{4}\bigg\}\leq\begin{cases}\exp(-Cn^{1-2\epsilon}), & \text{if } \gamma\geq 2 \text{ and } \epsilon\geq\frac{1}{2+\gamma};\\ \exp(-Cn^{\epsilon\gamma}), & \text{if } \gamma\geq 2 \text{ and } \epsilon<\frac{1}{2+\gamma};\\ \exp(-Cn^{\epsilon\gamma}), & \text{if } \gamma<2 \text{ and } \epsilon\leq\frac{1}{4};\\ \exp(-Cn^{1-2\epsilon}), & \text{if } \gamma<2 \text{ and } \epsilon>\frac{4-\gamma}{8};\\ \exp(-Cn^{\frac{\gamma}{4}}), & \text{if } \gamma<2 \text{ and } \epsilon\in(\frac{1}{4},\frac{4-\gamma}{8}].\end{cases}$$

Note that $\epsilon\in(0,w]$, we can obtain the result. □

PROOF OF PROPOSITION 4: The main idea is similar to the proof of

Proposition 3. By Taylor expansion, we have

$$\begin{aligned}\ell_j(0) &= n\Big(\frac{1}{n}\sum_{i=1}^n Z_{ij}^2\Big)^{-1}\Big(\frac{1}{n}\sum_{i=1}^n Z_{ij}\Big)^2 - n\Big(\frac{1}{n}\sum_{i=1}^n Z_{ij}^2\Big)^{-1}\Big\{\frac{1}{n}\sum_{i=1}^n \frac{\lambda^2 Z_{ij}^3}{(1+c_{i2}\lambda Z_{ij})^3}\Big\}^2\\ &\quad + \frac{2}{3}\sum_{i=1}^n \frac{\lambda^3 Z_{ij}^3}{(1+c_{i1}\lambda Z_{ij})^3}\\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Define

$$\mathcal{A} = \Big\{|\lambda| < \frac{4|n^{-1}\sum_{i=1}^n Z_{ij}|}{3n^{-1}\sum_{i=1}^n Z_{ij}^2} \text{ and } \Big|\frac{1}{n}\sum_{i=1}^n Z_{ij}\Big|\max_l |Z_{lj}| < \frac{1}{4n}\sum_{i=1}^n Z_{ij}^2\Big\},$$

then by Lemma 6,

$$\mathbb{P}(\mathcal{A}^c) \le \begin{cases} \exp(-Cn^{\gamma\eta}), & \text{if } \gamma \ge 2 \text{ and } \eta \le \frac{1}{2+\gamma};\\ \exp(-Cn^{\frac{\gamma}{\gamma+2}}), & \text{if } \gamma \ge 2 \text{ and } \eta > \frac{1}{2+\gamma};\\ \exp(-Cn^{\gamma\eta}), & \text{if } \gamma < 2 \text{ and } \eta \le \frac{1}{4};\\ \exp(-Cn^{\frac{\gamma}{4}}), & \text{if } \gamma < 2 \text{ and } \eta > \frac{1}{4}.\end{cases}$$

The left part is similar to the proof of Proposition 3. □

Proof of Theorem 5: Similar to the proof of Theorem 1, we can obtain

$$\begin{aligned}&\mathbb{P}\Big\{\ell_j(0) < \frac{n\|\mathbf{g}^{(j)}(\mathbf{Z};0)\|_\infty^2}{L^2}\Big\}\\ &\le \begin{cases}\exp\Big\{-\dfrac{n\|\mathbf{g}^{(j)}(\mathbf{Z};0)\|_\infty^2}{4H\sigma^2}\Big\} + \exp(-CL^{\gamma_3}),\\ \qquad \text{if } n^{\frac{1}{2}}\|\mathbf{g}^{(j)}(\mathbf{Z};0)\|_\infty \le \sigma(H^{1+\delta}\bar{\Delta})^{\frac{1}{1+2\delta}};\\ \exp\Big\{-\dfrac{1}{4}\Big(\dfrac{n\|\mathbf{g}^{(j)}(\mathbf{Z};0)\|_\infty}{2K}\Big)^{\frac{1}{1+\delta}}\Big\} + \exp(-CL^{\gamma_3}),\\ \qquad \text{if } n^{\frac{1}{2}}\|\mathbf{g}^{(j)}(\mathbf{Z};0)\|_\infty > \sigma(H^{1+\delta}\bar{\Delta})^{\frac{1}{1+2\delta}};\end{cases}\end{aligned}$$

for any $L \to \infty$, where $\delta$, $H$, $\bar{\Delta}$, $\sigma$ and $K$ are defined the same as those appeared in Theorem 1, and $C$ is a positive constant. Then, following the arguments of the proofs of Proposition 2 and Theorem 2, we complete the proof. □

Proof of Theorem 6: By Taylor expansion, we have

$$\begin{aligned}
&\ell_j(0)\\
&= n\bigg\{\frac{1}{n}\sum_{i=1}^n \mathbf{g}^{(j)}(\mathbf{Z}_i;0)\bigg\}^{\mathrm{T}}\bigg\{\frac{1}{n}\sum_{i=1}^n \mathbf{g}^{(j)}(\mathbf{Z}_i;0)\mathbf{g}^{(j)}(\mathbf{Z}_i;0)^{\mathrm{T}}\bigg\}^{-1}\bigg\{\frac{1}{n}\sum_{i=1}^n \mathbf{g}^{(j)}(\mathbf{Z}_i;0)\bigg\}\\
&\quad - n\bigg[\frac{1}{n}\sum_{i=1}^n \frac{\{\lambda^{\mathrm{T}}\mathbf{g}^{(j)}(\mathbf{Z}_i;0)\}^2\mathbf{g}^{(j)}(\mathbf{Z}_i;0)}{\{1+c_{i2}\lambda^{\mathrm{T}}\mathbf{g}^{(j)}(\mathbf{Z}_i;0)\}^3}\bigg]^{\mathrm{T}}\bigg\{\frac{1}{n}\sum_{i=1}^n \mathbf{g}^{(j)}(\mathbf{Z}_i;0)\mathbf{g}^{(j)}(\mathbf{Z}_i;0)^{\mathrm{T}}\bigg\}^{-1}\\
&\qquad\qquad \times\bigg[\frac{1}{n}\sum_{i=1}^n \frac{\{\lambda^{\mathrm{T}}\mathbf{g}^{(j)}(\mathbf{Z}_i;0)\}^2\mathbf{g}^{(j)}(\mathbf{Z}_i;0)}{\{1+c_{i2}\lambda^{\mathrm{T}}\mathbf{g}^{(j)}(\mathbf{Z}_i;0)\}^3}\bigg]\\
&\quad + \frac{2}{3}\sum_{i=1}^n \frac{\{\lambda^{\mathrm{T}}\mathbf{g}^{(j)}(\mathbf{Z}_i;0)\}^3}{\{1+c_{i1}\lambda^{\mathrm{T}}\mathbf{g}^{(j)}(\mathbf{Z}_i;0)\}^3}.
\end{aligned}$$

Define

$$\begin{aligned}
\mathcal{A} = \bigg\{&\|\lambda\|_2 < \frac{4}{3}\cdot\bigg\|\frac{1}{n}\sum_{i=1}^n \mathbf{g}^{(j)}(\mathbf{Z}_i;0)\bigg\|_2\cdot\lambda_{\min}^{-1}\bigg[\frac{1}{n}\sum_{i=1}^n \mathbf{g}^{(j)}(\mathbf{Z}_i;0)\mathbf{g}^{(j)}(\mathbf{Z}_i;0)^{\mathrm{T}}\bigg]\\
&\text{and } \bigg\|\frac{1}{n}\sum_{i=1}^n \mathbf{g}^{(j)}(\mathbf{Z}_i;0)\bigg\|_2 \max_l \|\mathbf{g}^{(j)}(\mathbf{Z}_l;0)\|_2\\
&\qquad < \frac{1}{4}\cdot\lambda_{\min}\bigg[\frac{1}{n}\sum_{i=1}^n \mathbf{g}^{(j)}(\mathbf{Z}_i;0)\mathbf{g}^{(j)}(\mathbf{Z}_i;0)^{\mathrm{T}}\bigg]\bigg\}.
\end{aligned}$$

Similar to Lemma 6, we have

$$\mathbb{P}(\mathcal{A}^c) \le \begin{cases} \exp(-Cn^{\gamma_3\eta}), & \text{if } \gamma_3 \ge 2 \text{ and } \eta \le \frac{1}{2+\gamma_3};\\ \exp(-Cn^{\frac{\gamma_3}{\gamma_3+2}}), & \text{if } \gamma_3 \ge 2 \text{ and } \eta > \frac{1}{2+\gamma_3};\\ \exp(-Cn^{\gamma_3\eta}), & \text{if } \gamma_3 < 2 \text{ and } \eta \le \frac{1}{4};\\ \exp(-Cn^{\frac{\gamma_3}{4}}), & \text{if } \gamma_3 < 2 \text{ and } \eta > \frac{1}{4}. \end{cases}$$

Then, following the proof of Proposition 3, we can complete the proof. □

## REFERENCES

[1] Owen, A. B. (1988), "Empirical likelihood ratio confidence intervals for a single function," *Biometrika*, 75, 237–249.

[2] — (2001), *Empirical Likelihood*, New York: Chapman and Hall/CRC.

[3] Saulis, L. and Statulevičius, V. A. (1991), *Limit Theorems for Large Deviations*, Dordrecht: Kluwer Academic Publishers.

Guanghua School of Management
Peking University
Beijing, 100871
China
E-mail: changjinyuan1986@pku.edu.cn

Business School
University of Colorado Denver
Campus Box 165
PO Box 173364
Denver, CO 80217-3364
E-mail: CHENGYONG.TANG@ucdenver.edu

Department of Statistics
North Carolina State University
2311 Stinson Drive
Raleigh, NC 27695-8203
USA
E-mail: wu@stat.ncsu.edu