Web-Based Supplementary Materials for "Inverse Probability of Censoring Weighted Estimates of Kendall's τ for Gap Time Analyses" by Lakhal-Chaieb, M.L., Cook, R.J. & Lin, X.

Web Appendix A: Asymptotic expression for $\sqrt{n}(\hat{\gamma}-\gamma)$

One has:

$$\begin{aligned}
\sqrt{n}\left\{\hat{\gamma}-\gamma\right\} &= \sqrt{n}\left[\binom{n}{2}^{-1}\sum_{i<j}\frac{L_{ij}\xi_{ij}}{\hat{p}_{ij}}-\gamma\right] \\
&= \sqrt{n}\binom{n}{2}^{-1}\sum_{i<j}\left[\frac{L_{ij}\xi_{ij}}{p_{ij}}-\gamma\right]+\sqrt{n}\binom{n}{2}^{-1}\sum_{i<j}L_{ij}\xi_{ij}\left[\frac{1}{\hat{p}_{ij}}-\frac{1}{p_{ij}}\right] \\
&= A+B
\end{aligned}$$

Calculations, similar to those presented in Appendix B of Lakhal-Chaieb et al. (2009), show that B is asymptotically equivalent to

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\int_0^\infty\frac{q(u)}{\Pr(\tilde{X}+\tilde{Y}>u)}dM_k^c(u),$$

where

$$q(u)=E\left[\frac{L_{12}\xi_{12}}{p_{12}}\{1(\tilde{X}_1+\tilde{Y}_{12}>u)+1(\tilde{X}_2+\tilde{Y}_{12}>u)\}\right].$$

Standard theory of U-statistics shows that the distribution of $\sqrt{n}\{\hat{\gamma}-\gamma\}$ is asymptotically normal with mean zero and variance equal to $Var(A)+Var(B)+2Cov(A,B)$.

The first term A is a U-statistics of order 2. Thus, its asymptotic variance is equal to

$$lim_{n\sim\infty}\frac{4}{n^3}\sum_{i<j<k}\left[\frac{L_{ij}\xi_{ij}}{p_{ij}}-\gamma\right]\left[\frac{L_{ik}\xi_{ik}}{p_{ik}}-\gamma\right]+\left[\frac{L_{ij}\xi_{ij}}{p_{ij}}-\gamma\right]\left[\frac{L_{jk}\xi_{jk}}{p_{jk}}-\gamma\right]+\left[\frac{L_{ik}\xi_{ik}}{p_{ik}}-\gamma\right]\left[\frac{L_{jk}\xi_{jk}}{p_{jk}}-\gamma\right].$$

On the other hand, standard martingale variance calculation yields

$$Var(B)=\int_0^\infty\frac{q^2(u)}{\Pr(\tilde{X}+\tilde{Y}>u)}d\Lambda^c(u).$$

To compute $Cov(A,B)$, we follow the lines of Cheng et al. (1995) and write

$$L_{ij} = \delta_{Y_j} I(\tilde{Y}_i > \tilde{Y}_j) + \delta_{Y_i} I(\tilde{Y}_i < \tilde{Y}_j).$$

Hence, one has:

$$\begin{aligned}
q(u) &= 2E\left[\frac{\delta_{Y_1} I(\tilde{Y}_1 < \tilde{Y}_2)\xi_{12}}{G(\tilde{X}_1+\tilde{Y}_1)G(\tilde{X}_2+\tilde{Y}_1)}\{I(\tilde{X}_1+\tilde{Y}_1>u)+I(\tilde{X}_2+\tilde{Y}_1>u)\}\right],\\
A &= \frac{2}{n^{3/2}}\sum_{i\neq j}\left[\frac{\delta_{Y_j} I(\tilde{Y}_i > \tilde{Y}_j)\xi_{ij}}{G(\tilde{X}_i+\tilde{Y}_j)G(\tilde{X}_j+\tilde{Y}_j)} - \gamma/2\right]\\
Cov(A,B) &= Cov\left[\frac{2}{n^{3/2}}\sum_{i\neq j}\left\{\frac{\delta_{Y_j} I(\tilde{Y}_i > \tilde{Y}_j)\xi_{ij}}{G(\tilde{X}_i+\tilde{Y}_j)G(\tilde{X}_j+\tilde{Y}_j)} - \gamma/2\right\};\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\int_0^\infty \frac{q(u)}{\Pr(\tilde{X}+\tilde{Y}>u)}dM_k^c(u)\right]\\
&= \frac{2}{n^2}\sum_{i\neq j}E\left\{\frac{\delta_{Y_j} I(\tilde{Y}_i > \tilde{Y}_j)\xi_{ij}}{G(\tilde{X}_i+\tilde{Y}_j)G(\tilde{X}_j+\tilde{Y}_j)} \times \int_0^\infty \frac{q(u)}{\Pr(\tilde{X}+\tilde{Y}>u)}\left[dM_i^c(u)+dM_j^c(u)\right]\right\}
\end{aligned}$$

Calculations similar to those presented in Appendices 1 & 2 of Cheng et al. (1995) yield the following expression for $Cov(A,B)$:

$$-\frac{2}{n^2}\sum_{i\neq j}E\left\{\int_0^\infty \frac{\delta_{Y_j} I(\tilde{Y}_j < \tilde{Y}_i)\xi_{ij}}{G(\tilde{X}_i+\tilde{Y}_j)G(\tilde{X}_j+\tilde{Y}_j)}\{I(\tilde{X}_j+\tilde{Y}_j>u)+I(\tilde{X}_i+\tilde{Y}_j>u)\}\frac{q(u)}{\Pr(\tilde{X}+\tilde{Y}>u)}d\Lambda^c(u)\right\}$$

which converges to

$$-\int_0^\infty \frac{q^2(u)}{\Pr(\tilde{X}+\tilde{Y}>u)}d\Lambda^c(u)$$

Putting all these pieces together leads to the asymptotic variance of $\sqrt{n}(\hat{\gamma}-\gamma)$.

Web Appendix B: Asymptotic presentation of $\sqrt{n}\{\hat{\tau}_2-\tau\}$

One has:

$$\sqrt{n}\{\hat{\tau}_2-\tau\} = \sqrt{n}\left\{\frac{\binom{n}{2}^{-1}\sum_{i<j}\frac{L_{ij}\psi_{ij}}{\hat{p}_{ij}}}{\binom{n}{2}^{-1}\sum_{i<j}\frac{L_{ij}}{\hat{p}_{ij}}} - \tau\right\}$$

$$= \sqrt{n}\left\{\left[\binom{n}{2}^{-1}\sum_{i<j}\frac{L_{ij}\psi_{ij}}{\hat{p}_{ij}} - \tau\right] - \tau\left[\binom{n}{2}^{-1}\sum_{i<j}\frac{L_{ij}}{\hat{p}_{ij}} - 1\right]\right\} + o_p(1)$$

$$= \sqrt{n}\binom{n}{2}^{-1}\sum_{i<j}\frac{L_{ij}(\psi_{ij}-\tau)}{\hat{p}_{ij}} + o_p(1)$$

Web Appendix C: Expression of $\tau_{\mathcal{A}}$

Let (U, V) be a pair of random variables following the Clayton copula with uniform marginals and a parameter $\theta > 1$. For $(u, v) \in [0, 1]^2$, one has

$$\Pr(U \leq u; V \leq v) = C_\theta(u, v) = \left[u^{-(\theta-1)} + v^{-(\theta-1)} - 1\right]^{-1/(\theta-1)}.$$

Kendall's tau restricted to $\mathcal{A} = \{a \leq U \leq b\}$ is equal to $\tau_{\mathcal{A}} = 4\int_0^1\int_a^b H_{\mathcal{A}}(u, v)h_{\mathcal{A}}(u, v)dudv - 1$, where $H_{\mathcal{A}}$ and $h_{\mathcal{A}}$ are the cumulative distribution and the density functions of $(U, V|a \leq U \leq b)$, respectively. On the other side, for $(u, v) \in [a, b] \times [0, 1]$, one has

$$H_{\mathcal{A}}(u, v) = P(U \leq u; V \leq v|a \leq U \leq b) = \frac{P(a \leq U \leq u; V \leq v)}{P(a \leq U \leq b)} = \frac{C_\theta(u, v) - C_\theta(a, v)}{b - a}$$

and hence $h_{\mathcal{A}}(u, v) = C_\theta^{11}(u, v)/(b - a)$ where $C_\theta^{ij}(u, v) = \partial^{i+j}C_\theta(u, v)/\partial^i u\partial^j v$. So

$$\tau_{\mathcal{A}} = \frac{4}{(b-a)^2}\int_0^1\int_a^b C_\theta^{11}(u, v)\left[C_\theta(u, v) - C_\theta(a, v)\right]dudv - 1 = \frac{4}{(b-a)^2}(\mathcal{I}_1 - \mathcal{I}_2) - 1.$$

Tedious but straightforward computations show that $\mathcal{I}_1 = \theta(b^2 - a^2)/\{2(\theta+1)\}$ and

$$\mathcal{I}_2 = \frac{1}{\theta - 1}\int_0^\infty (a^{-(\theta-1)} + t)^{-1/(\theta-1)}(b^{-(\theta-1)} + t)^{-\theta/(\theta-1)}dt - \frac{a^2}{2}.$$

The expression of $\tau_{\mathcal{A}}$ when $\theta > 1$ is obtained by putting together $\mathcal{I}_1$ and $\mathcal{I}_2$. Expression for $\tau_{\mathcal{A}}$ when $\theta < 1$ is obtained by similar computations.