Web-based Supplementary Materials for "A note on score tests for homogeneity in two-component parametric models for discrete data" by D. Todem, W-W Hsu and KM Kim

Web Appendix A: Proof of the Proposition

i) Proof of the first statement:

Note that $\mu_i = h^{-1}(x_i'\beta)$ and under the null $\pi_i = \tilde{g}(\mu_i)$, which by substitution gives $\pi_i = \tilde{g}(h^{-1}(x_i'\beta))$ for all i. Thus, a natural function $f(.)$ relating π_i to linear predictor $z_i'\gamma$ is defined such that $f = \tilde{g} \circ h^{-1}$, in which case $\pi_i = \tilde{g} \circ h^{-1}(z_i'\gamma)$, where $\circ$ represents the composite function operator.

ii) Proof of the second statement:

To show that the second statement of the proposition is true, we first consider the situation where $\tilde{g}(.)$ as a function of μ_i is invertible. Therefore, the composition $\tilde{g} \circ h^{-1}$ is invertible and the null hypothesis $\tilde{g} \circ h^{-1}(z_i'\gamma) = \tilde{g} \circ h^{-1}(x_i'\beta), \quad \text{for all } i$, then reduces to the simple form $x_i'\beta = z_i'\gamma \quad \text{for all} \quad i$. This ultimately reduces to a linear contrast of parameters β and γ. In particular, when $x_i = z_i$, for all i, the null hypothesis is given by $\mathrm{H}_0 : \beta = \gamma$. Now we assume that $\tilde{g}(.)$ is not invertible. If $x_i = z_i$, for all i, the identifiability of the non-degenerate distribution G requires that $\beta = \gamma$ under the null. When $x_i \neq z_i$, for some i, the identifiability condition of the non-degenerate distribution G ensures that the null reduces to a linear contrast that involves only the parameters β and γ under the null.

Web Appendix B: Calculation of the information matrix $\mathcal{I}(\theta, \alpha)$ for a mixture of a degenerate distribution at point y^* and a non-degenerate distribution G, assuming $x_i = z_i$

For any parametric non-degenerate distribution, the information matrix is defined as,

$$\mathcal{I}(\theta, \alpha) = \begin{pmatrix} \mathcal{I}_{\theta\theta}(\theta, \alpha) & \mathcal{I}_{\theta\alpha}(\theta, \alpha) \\ \mathcal{I}_{\alpha\theta}(\theta, \alpha) & \mathcal{I}_{\alpha\alpha}(\theta, \alpha) \end{pmatrix}$$

where, $\mathcal{I}_{\theta\theta}(\theta,\alpha) = -E[\partial^2\ell(\theta,\alpha)/\partial\theta\partial\theta']$, $\mathcal{I}_{\theta\alpha}(\theta,\alpha) = -E[\partial^2\ell(\theta,\alpha)/\partial\theta\partial\alpha'] = \mathcal{I}'_{\alpha\theta}(\theta,\alpha)$, $\mathcal{I}_{\alpha\alpha}(\theta,\alpha) = -E[\partial^2\ell(\theta,\alpha)/\partial\alpha\partial\alpha']$. The calculation of these expectations is often tedious for any general parametric non-degenerate distribution. So we restrict our calculations to the observed information matrix. For this, we shall use the following notations, $\dot{\pi}_{i,\alpha} = \partial\pi_i/\partial\alpha, \dot{\pi}_{i,\theta} = \partial\pi_i/\partial\theta, \ddot{\pi}_{i,\alpha\alpha} = \partial^2\pi_i/\partial\alpha\partial\alpha', \ddot{\pi}_{i,\theta\alpha} = \partial^2\pi_i/\partial\theta\partial\alpha', \ddot{\pi}_{i,\theta\theta} = \partial^2\pi_i/\partial\theta\partial\theta', \tilde{b}_{i,\theta} = \frac{g_i(y_i)}{1-g_i(y^*)}, \dot{\tilde{b}}_{i,\theta} = \partial\tilde{b}_{i,\theta}/\partial\theta, \ddot{\tilde{b}}_{i,\theta\theta} = \partial^2\tilde{b}_{i,\theta}/\partial\theta\partial\theta'$. By using these notations, the second-order derivatives of log-likelihood function are then given by,

$$\frac{\partial^2\ell(\theta,\alpha)}{\partial\theta\partial\theta'} = \sum_{i=1}^{n}\Bigg\{\delta(y_i)\frac{\pi_i\ddot{\pi}_{i,\theta\theta} - \dot{\pi}_{i,\theta}\dot{\pi}'_{i,\theta}}{\pi_i^2} + (1-\delta(y_i))\frac{-\ddot{\pi}_{i,\theta\theta}(1-\pi_i) - \dot{\pi}_{i,\theta}\dot{\pi}'_{i,\theta}}{(1-\pi_i)^2} + (1-\delta(y_i))\frac{\tilde{b}_{i,\theta}\ddot{\tilde{b}}_{i,\theta} - \dot{\tilde{b}}_{i,\theta}\dot{\tilde{b}}'_{i,\theta}}{\tilde{b}^2_{i,\theta}}\Bigg\},$$

$$\frac{\partial^2\ell(\theta,\alpha)}{\partial\theta\partial\alpha'} = \sum_{i=1}^{n}\Bigg\{\delta(y_i)\frac{\pi_i\ddot{\pi}_{i,\theta\alpha} - \dot{\pi}_{i,\theta}\dot{\pi}'_{i,\alpha}}{\pi_i^2} + (1-\delta(y_i))\frac{-\ddot{\pi}_{i,\theta\alpha}(1-\pi_i) - \dot{\pi}_{i,\theta}\dot{\pi}'_{i,\alpha}}{(1-\pi_i)^2}\Bigg\},$$

$$\frac{\partial^2\ell(\theta,\alpha)}{\partial\alpha\partial\alpha'} = \sum_{i=1}^{n}\Bigg\{\delta(y_i)\frac{\pi_i\ddot{\pi}_{i,\alpha\alpha} - \dot{\pi}_{i,\alpha}\dot{\pi}'_{i,\alpha}}{\pi_i^2} + (1-\delta(y_i))\frac{-\ddot{\pi}_{i,\alpha\alpha}(1-\pi_i) - \dot{\pi}_{i,\alpha}\dot{\pi}'_{i,\alpha}}{(1-\pi_i)^2}\Bigg\}.$$

Web Appendix C: Calculation of $\mathcal{I}(\hat{\theta},0)$ when G is a Poisson model and $y^*=0$

If the non-degenerate distribution is a Poisson process, we have $g_i(0) = \exp\{-\mu_i\}$ where $\mu_i = \exp\{x_i'\beta\} = \exp\{x_i'\theta\}$, with $\theta = \beta$. A natural parameterization for π_i is given by $\pi_i = \exp\{-\exp\{z_i'\gamma\}\}$. Assuming $x_i = z_i$, estimates of entries of the information matrix are given by,

$$\mathcal{I}_{\theta\theta}(\hat{\theta},\mathbf{0}) = \boldsymbol{x}'\text{diag}\Big\{\hat{\mu}_i(2\exp\{-\hat{\mu}_i\}-1)\Big\}_{i=1}^{n}\boldsymbol{x},$$
$$\mathcal{I}_{\theta\alpha}(\hat{\theta},\mathbf{0}) = \boldsymbol{x}'\text{diag}\Big\{-\frac{\hat{\mu}_i^2\exp\{-\hat{\mu}_i\}}{1-\exp\{-\hat{\mu}_i\}}\Big\}_{i=1}^{n}\boldsymbol{x},$$
$$\mathcal{I}_{\alpha\alpha}(\hat{\theta},\mathbf{0}) = \boldsymbol{x}'\text{diag}\Big\{\frac{\hat{\mu}_i^2\exp\{-\hat{\mu}_i\}}{1-\exp\{-\hat{\mu}_i\}}\Big\}_{i=1}^{n}\boldsymbol{x},$$

where $\boldsymbol{x} = (x_1', x_2', \cdots, x_n')'$ and $\hat{\mu}_i = \exp\{x_i'\hat{\theta}\}$ with $\hat{\theta}$ being the maximum likelihood estimator of θ^* under H_0. In these expressions, $\text{diag}\{a_i\}_{i=1}^n$ represents a diagonal matrix of order n with $a_i, i = 1, \cdots, n$ being the entries on the diagonal.

Web Appendix D: Calculation of $\mathcal{I}(\hat{\theta}, 0)$ when G is a binomial model and $y^* = 0$

If the non-degenerate distribution is a Binomial distribution with number of trials m_i and success probability μ_i, then we have $\mu_i = 1/(1 + \exp\{-x_i'\beta\}) = 1/(1 + \exp\{-x_i'\theta\})$, with $\theta = \beta$. A natural parameterization for π_i is given by $\pi_i = \{1 + \exp\{z_i'\gamma\}\}^{-m_i}$. Assuming $x_i = z_i$, estimates of entries of the information matrix are given by,

$$\mathcal{I}_{\theta\theta}(\hat{\theta}, \mathbf{0}) = \boldsymbol{x}'\text{diag}\left\{\frac{m_i(1-\hat{\mu}_i)^{m_i+2}(\hat{\mu}_i^2(1-\hat{\mu}_i)^{-2} - 1)}{1-(1-\hat{\mu}_i)^{m_i}} + m_i\hat{\mu}_i(1-\hat{\mu}_i)\right\}_{i=1}^n \boldsymbol{x}$$

$$\mathcal{I}_{\theta\alpha}(\hat{\theta}, \mathbf{0}) = \boldsymbol{x}'\text{diag}\left\{-\frac{m_i^2\hat{\mu}_i^2(1-\hat{\mu}_i)^{m_i}}{1-(1-\hat{\mu}_i)^{m_i}}\right\}_{i=1}^n \boldsymbol{x},$$

$$\mathcal{I}_{\alpha\alpha}(\hat{\theta}, \mathbf{0}) = \boldsymbol{x}'\text{diag}\left\{\frac{m_i^2\hat{\mu}_i^2(1-\hat{\mu}_i)^{m_i}}{1-(1-\hat{\mu}_i)^{m_i}}\right\}_{i=1}^n \boldsymbol{x},$$

where $\boldsymbol{x} = (x_1', x_2', \cdots, x_n')'$ and $\hat{\mu}_i = 1/(1 + \exp\{-x_i'\hat{\theta}\})$, with $\hat{\theta}$ being the maximum likelihood estimator for θ^* under H_0.

Web Appendix E: Components $\hat{u}_\alpha$ of the score test statistic for commonly used G distributions

For Poisson and binomial non degenerate distributions with $\theta = \beta$, the components $\hat{u}_\alpha$ of the score test statistic when $y^* = 0$ are respectively given by, $\hat{u}_\alpha = \sum_{i=1}^n \left\{ x_i\hat{\mu}_i \left(\frac{\delta(y_i) - \exp\{-\hat{\mu}_i\}}{1-\exp\{-\hat{\mu}_i\}} \right) \right\}$ and $\hat{u}_\alpha = \sum_{i=1}^n \left\{ x_i m_i\hat{\mu}_i \left(\frac{\delta(y_i)-(1-\hat{\mu}_i)^{m_i}}{1-(1-\hat{\mu}_i)^{m_i}} \right) \right\}$. In these expressions, $\hat{\mu}_i$ takes values $\exp\{x_i'\hat{\theta}\}$ and $\{1 + \exp\{-x_i'\hat{\theta}\}\}^{-1}$ for the Poisson and binomial non degenerate distributions, respectively.

Web Appendix F: Additional simulation results when covariates x_i and z_i are not equal

Further simulation studies were conducted to compare the three tests when covariates x_i and z_i are not equal. We specifically considered the situation where x_i is a subset of z_i. Specifically, we take $x_i = (1, x_{1i})'$ and $z_i = (1, x_{1i}, x_{2i})'$, where $x_{1i} \sim \text{Uniform}(0, 1)$ and $x_{2i} \sim \text{Uniform}(-2, 2)$.

We then performed the proposed covariate-adjusted score test assuming the working mixing weight model $\omega_i = \{\pi_i - \exp\{-\mu_i\}\}\{1 - \exp\{-\mu_i\}\}^{-1}$ under the alternative, where $\pi_i = \exp\{-\exp\{\gamma_0 + \gamma_1 x_{1i} + \gamma_2 x_{2i}\}\}$ and $\mu_i = \exp\{\beta_0 + \beta_1 x_{1i}\}$. The test of van den Broek (1995) was performed assuming $\omega_i = \gamma_0$, and that of Jansakul and Hinde (2009), assuming $\omega_i = \gamma_0 + \gamma_1 x_{1i} + \gamma_2 x_{2i}$. With these parameterizations, the null hypotheses to be evaluated were then given by: $\text{H}_0 : \gamma_0 = \beta_0, \gamma_1 = \beta_1, \gamma_2 = 0$, for our formulation; $\text{H}_0 : \gamma_0 = 0$, for van den Broek's test; and $\text{H}_0 : \gamma_0 = 0, \gamma_1 = 0, \gamma_2 = 0$, for Jansakul and Hinde's test. The maximum likelihood estimate $\hat{\beta}$ of the true value of $\beta = (\beta_0, \beta_1)'$ under the null was obtained from a homogeneous Poisson model with mean $\mu_i = \exp\{\beta_0 + \beta_1 x_{1i}\}$. Results of these studies are given in Table 5. Findings from these studies are similar to those that assume $x_i = z_i$.

Table 5

Empirical power of score test statistics to detect various forms ω_i^* of heterogeneity coupled with a non-degenerate Poisson model with mean $\mu_i^* = \exp\{\beta_0^* - 1.45x_{1i}\}$, $x_{1i} \sim$ Uniform$(0,1)$, at 5% significance level

	n=50			n=100			n=200		
β_0^*	-0.75	0	0.75	-0.75	0	0.75	-0.75	0	0.75
$\omega_i^* = 0.35 + 0.1x_{1i} - 0.15x_{2i}$									
vdB test	0.071	0.145	0.520	0.122	0.266	0.845	0.184	0.534	0.992
J&H test	0.105	0.202	0.594	0.193	0.413	0.925	0.333	0.755	0.998
Prop. c-a test	0.103	0.204	0.605	0.199	0.426	0.918	0.333	0.761	0.998
$\omega_i^* = \frac{\exp\{-15+30x_{1i}-5x_{2i}\}}{1+\exp\{-15+30x_{1i}-5x_{2i}\}}$									
vdB test	0.043	0.064	0.109	0.044	0.082	0.153	0.058	0.105	0.216
J&H test	0.122	0.177	0.377	0.179	0.331	0.628	0.275	0.542	0.864
Prop. c-a test	0.134	0.237	0.643	0.210	0.455	0.911	0.338	0.769	0.996
$\omega_i^* = \Phi(-15 + 30x_{1i} + 5x_{2i})$									
vdB test	0.055	0.064	0.100	0.040	0.062	0.126	0.058	0.118	0.237
J&H test	0.105	0.194	0.364	0.165	0.316	0.587	0.295	0.557	0.878
Prop. c-a test	0.115	0.257	0.636	0.189	0.444	0.913	0.354	0.769	0.998

Note: $x_{2i} \sim$ Uniform$(-2,2)$.
vdB: van den Broek, test with df=1. J&H: Jansakul & Hinde, test with df=3.
Prop. c-a: Proposed covariate-adjusted, test with df=3.