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An endogenous rate of time preference, the Penrose effect, and dynamic optimality of environmental quality

(intertemporal preference ordering/endogenous rate of time preference/imputed price/duality principles)

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ABSTRACT **In the present paper, the endogenous theory of time preference is extended to analyze those processes of capital accumulation and changes in environmental quality that are dynamically optimum with respect to the intertemporal preference ordering of the representative individual of the society in question. The analysis is carried out within the conceptual framework of the dynamic analysis of environmental quality, as has been developed by a number of economists for specific cases of the fisheries and forestry commons. The duality principles on intertemporal preference ordering and capital accumulation are extended to the situation where processes of capital accumulation are subject to the Penrose effect, which exhibit the marginal decrease in the effect of investment in private and social overhead capital upon the rate at which capital is accumulated. The dynamically optimum time-path of economic activities is characterized by the proportionality of two systems of imputed, or efficient, prices, one associated with the given intertemporal ordering and another associated with processes of accumulation of private and social overhead capital. It is particularly shown that the dynamically optimality of the processes of capital accumulation involving both private and social overhead capital is characterized by the conditions that are identical with those involving private capital, with the role of social overhead capital only indirectly exhibited.**

1. Introduction

The concept of sustainable development has recently been extensively discussed in relation to the evaluation and design of economic development in many developing nations. It involves a number of theoretical and policy issues, such as the irreversibility of environmental degradation, intergenerational equity, and externalities, both static and dynamic, of environmental impacts, which are not readily handled by the standard method in economic analysis. In the last two decades, however, we have seen an emergence of a significant number of the conceptual frameworks and analytic techniques specifically designed to analyze economic and social implications of environmental degradation and the processes of economic development that are in harmony with the natural environment, as, for example, are succinctly described by Clark (1).

In this paper, I would like to address the analysis of the processes of capital accumulation and changes in environmental quality over time that are dynamically optimum with respect to the intertemporal preference ordering of the representative individual of the society in question. The analysis will be conducted within the conceptual framework of the dynamic analysis of environmental quality, as has been developed for the general situation by Mäler (2) and for specific cases of the fisheries and forestory commons by Gordon (3), Schaefer (4), Clark and Munro (5), Clark (1), and Tahvonen (6), among others. In the present paper, I will be particularly concerned with the implications of the intertemporal preference orderings with endogenous rates of time preference and the irreversibility of processes of capital accumulation as expressed by the Penrose effect (7).

The analysis will be based upon the formula concerning the system of efficient, or imputed, prices associated with a time-path of consumption that is dynamically optimum, as explicitly derived in Epstein (8). It will be shown to be valid for the presence of the Penrose effect expressing the marginal decrease in the effect of investment in private and social overhead capital upon the rate at which capital is accumulated. The technique of the Ramsey–Cass–Koopmans theory of optimum capital accumulation, as developed in Ramsey (9), Cass (10), and Koopmans (11), may be, with slight modification, applied to the case where the rate of time preference is endogenously determined and the process of capital accumulation is subject to the Penrose effect.

The endogenous theory of intertemporal preference ordering was originated by Koopmans (12) for the discrete time case, and was later extended by Uzawa (7, 13) to the continuous time case. It has been recently discussed in terms of abstract theory of functional analysis [e.g., by Epstein (14, 15), Epstein and Haynes (16), Lucas and Stokey (17), and Stokey and Lucas (18)].

The analysis then will be extended to the case where the natural environment plays the role of social overhead capital, as introduced by Uzawa (19), and one concerns the allocation of investment between private capital and social overhead capital that would result in the time-path of consumption that is dynamically optimum with respect to the intertemporal preference ordering with ednogenous rate of time preference.

2. Intertemporal Preference Orderings and Efficient Prices

An intertemporal preference ordering is a binary relation $\gtrsim$ defined over the set of all conceivable time-paths of consumption, to be denoted by Ω. Each time-path of consumption in the set Ω, $x = (x_t)$, is a vector-valued function x_t defined for all t, $0 \leqq t < +\infty$. It is assumed that x_t is positive and piecewise continuously differentiable. The intertemporal preference ordering $\gtrsim$ is assumed to be irreflexible, transitive, monotone, continuous, convex, and time-invariant.

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We consider a special class of intertemporal preference orderings, for each of which the instantaneous utility $u_t = u(x_t)$ at each time t is well defined as a function of the vector of consumption x_t at time t. It is assumed that the utility function $u_t = u(x_t)$ is positive valued, continuously twice-differentiable, and strictly concave.

It is also assumed that the intertemporal preference ordering with which we are concerned may be represented by a certain utility functional, $U = U(u)$, $u = (u_t)$, such that $x \gtrsim x'$, if, and only if, $U(u) \geqq U(u')$, $u = [u(x_t)]$, $u' = [u(x_t')]$.

Let us now consider a time-path of instantaneous utilities $u = (u_t)$, and denote by U_t the utility functional for the truncated time-path at time t, ${}^tu = (u_{t+\tau})$, $u = (u_\tau)$; $U_t = U({}^tu)$, $t \geqq 0$.

The utility functional $U = U(u)$ now is assumed to satisfy the following differential equation:

$$\dot{U}_t = \beta(u_t, U_t) - u_t, \quad [1]$$

where the function $\beta(u, U)$ specifies the way by which future utilities are discounted to the present.

It is assumed that the discounting function $\beta(u, U)$ is defined all $(u, U) > 0$, continuously twice differentiable, and satisfies the following conditions: $\beta = \beta(u, U) > 0$, $\beta(0, U) = \beta(u, 0) = 0$, $\beta_u < 0$, $\beta_U > 0$, and $\beta(u, U)$ is convex and strictly quasi-convex in the sense that $\beta_{uu} > 0$, $\beta_{UU} > 0$, $\beta_{uu}\beta_{UU} - \beta_{uU}^2 \geqq 0$.

For any given time-path of utilities, ${}^0u = (u_\tau)$ the value of the utility functional $V = U({}^0u)$ may be obtained by the following procedure: $V = U({}^0u)$ if, and only if, the solution path (U_t) to the basic differential Eq. **1** with initial condition $U_0 = V$ satisfies the following condition:

$$\int_0^\infty u_t e^{-\hat{\Delta}_t} dt = V, \ \hat{\Delta}_t$$

$$= \int_0^t \delta(u_\tau, U_\tau) d\tau, \ \delta(u, U) = \beta(u, U)/U \quad [2]$$

In Uzawa (19), it has been proved that, for any time-path of utilities $u = (u_t)$, the value of utility functional $V = U(u)$ thus defined is uniquely determined. Indeed, $V = U(u)$ is equal to the minimum value of initial condition $U_0 = V$ for which the solution (U_t) to differential Eq. **1** exists and is positive for all t, $0 \leqq t < +\infty$.

It can also be shown that Eq. **2** is satisfied if, and only if, $\lim_{t\to\infty} U_t e^{-\hat{\Delta}_t} = 0$, which in turn implies that

$$\lim_{t\to\infty} U_t e^{-\Delta_t} = 0, \ \Delta_t = \int_0^t \beta_U(u_\tau, U_\tau) d\tau.$$

It may be noted that, because of the convexity of $\beta(u, U)$, $0 < \delta(u, U) < \beta_U(u, U)$.

It is now possible to obtain an explicit formula for the system of efficient prices for a given time-path of utilities, $u = (u_t)$, to be denoted by $p(u) = [p_t(u)]$: $p_t(u) = [1 - \beta_U(u_t, U_t)]e^{-\Delta_t}$, where (U_t) is the solution path to differential Eq. **1** associated with the time-path $u = (u_t)$ and Δ_t is the accumulated marginal rate of discount.

To illustrate the concept of efficient prices, suppose there exist two time-paths of utilities, $u^0 = (u_t^0)$ and $u^1 = (u_t^1)$, which are indifferent in terms of the given intertemporal preference ordering: $u^0 \sim u^1$.

Let $u(\theta) = [u_t(\theta)]$ be any smooth curve connecting u^0 and u^1 along the indifferent surface; namely, $u(\theta)$ is defined for all θ, $0 \leqq \theta \leqq 1$, positive valued, continuously, piecewise continuously twice differentiable with respect to θ, and $u(\theta) \sim u^0 \sim u^1$, for all $0 \leqq \theta \leqq 1$, where $u(0) = u^0$, $u(1) = u^1$.

Then, the system of efficient prices $p(u) = [p_t(u)]$ satisfies the following conditions:

$$\int_0^\infty u_t'(\theta) p_t[u(\theta)] dt = 0, \int_0^\infty u_t''(\theta) p_t[u(\theta)] dt \geqq 0, \ (0 \leqq \theta \leqq 1).$$

Analogous to the static situation, the system of efficient prices $p(u) = [p_t(u)]$ plays the role of a separating hyperplane to the indifferent surface. Namely, if a given time-path of utilities $u^0 = (u_t^0)$ minimizes the expenditures evaluated at a certain given price system $p = (p_t)$, $p_t > 0$: $p \cdot u = \int_0^\infty p_t u_t dt$ among all time-paths of utilities which are indifferent with or preferred to the given time-path $u^0 = (u_t^0)$, then $p_t = \ell p_t(u^0)$, for all $t \geqq 0$, with a positive constant ℓ.

On the other hand, if the proportionality condition is satisfied, together with the transversality condition: $\lim_{t\to+\infty} p_t u_t^0 = 0$, then u^0 minimizes the expenditures among all the time-paths of utilities which are indifferent with or preferred to u^0.

The relationships of our approach to the classic Euler–Lagrange method are transparent. Let us denote $\xi_t = e^{-\Delta_t}/p_t$, and differentiate it logarithmically with respect to t, to obtain

$$\dot{\xi}/\xi = \beta_U - \dot{p}/p, \ 1 - \beta_u = \xi.$$

3. Capital Accumulation and Efficient Prices

We denote by k_t the stock of capital at each time t, and assume that the stock of capital is the only factor of production relevant in the process of production. The production function is denoted by $f(k)$ and it is assumed that $f(k)$ is defined for all non-negative k, non-negative-valued, continuously twice differentiable, and satisfies the following conditions: $f(k) > 0$, $f'(k) > 0$, $f''(k) < 0$, for all $k > 0$; and $f(0) = 0$, $f(+\infty) = +\infty$: $f'(0) = \infty$, $f'(+\infty) = 0$.

The dynamic process of capital accumulation is described by the following differential equation:

$$\dot{k}_t = f(k_t) - x_t, \ k_0 = K, \quad [3]$$

where x_t is the level of consumption at time t and K is the initial stock of capital.

In the basic differential Eq. **3**, it is required that k_t remains non-negative. Hence, for a given positive time-path of consumption, $x = (x_t)$, there exists the minimum value of the initial stock of capital, K^0, for which the positive solution k_t to Eq. **3** exists for all time t. Such an K^0 may be denoted by $K^0(x)$ to emphasize the dependency upon the given time-path of consumption $x = (x_t)$.

A time-path of consumption $x = (x_t)$ is termed feasible with respect to the stock of capital K, if the solution (k_t) to the basic differential Eq. **3** with initial condition $k_0 = K$, exists for all time t. Thus, $K^0 = K^0(x)$ defined above gives us the minimum stock of capital with respect to which the given time-path consumption $x = (x_t)$ is feasible.

PROPOSITION 1. *Let $x = (x_t)$ be a given time-path of consumption, where $x_t > 0$, for all t. A stock of capital K is the associated stock of capital with respect to x—i.e., $K = K^0(x)$, if, and only if, $\lim_{t\to+\infty} k_t e^{-\nabla_t} = 0$, where ∇_t is the accumulated marginal rate of discount: $\nabla_t = \int_0^t f'(k_\tau) d\tau$, and (k_t) is the solution to Eq. 3 with initial condition K.*

Proof. If the above condition is not satisfied, then there always exists a positive number ε such that $k_t e^{-\nabla_t} > \varepsilon > 0$, for all t.

Let us denote by $k_t(\theta)$ the solution to Eq. **3** with initial condition $K(\theta) = K - \theta\varepsilon$, and $0 \leqq \theta \leqq 1$. Then we have $\dot{k}_t(\theta) = f[k_t(\theta)] - x_t$, $k_0 = K(\theta) = K - \theta\varepsilon$, multiplying by $e^{-\nabla_t(\theta)}$, and integrating from 0 to t, we obtain

$$k_t'(\theta) e^{-\nabla_t(\theta)} = -\varepsilon, \text{ for all } t. \quad [4]$$

By applying the mean-value theorem, there exists θ_t such that $k_t(1) - k_t(0) = k_t'(\theta_t)$, $0 \leqq \theta_t \leqq 1$, for all t, which implies that $k_t(1) > 0$, for all t. Namely, the given time-path of consumption x is feasible with respect to initial condition $K(1) = K - \varepsilon < K$.

Since the solution k_t to Eq. **3** is increased when initial condition K is increased, Eq. **4** implies that $0 < k_t(1) < k_t(0) = k_t$, $\nabla_t(1) > \nabla_t(0)$, $0 \leqq \lim_{t\to+\infty} k_t(1)e^{-\nabla_t(1)} < \lim_{t\to+\infty} k_t(0)e^{-\nabla_t(0)}$. Q.E.D.

PROPOSITION 2. For a given time-path of consumption $x = (x_t)$, ($x_t > 0$, for all t), if stock of capital K is the associated stock of capital with respect to x, i.e., $K = \hat{K}^0(x)$, the following equality holds: $\int_0^\infty x_t e^{-\hat{\nabla}_t}\, dt = K$, where $\hat{\nabla}_t$ is the accumulated average rate of discount defined.

Proof. We first multiply both sides of Eq. **1** by $e^{-\hat{\nabla}_t}$, to obtain $[\dot{k}_t - f(k_t)]e^{-\hat{\nabla}_t} = -x_t e^{-\hat{\nabla}_t}$, which by integrating from 0 to t and noting the initial condition $k_0 = K$, we obtain $k_t e^{-\hat{\nabla}_t} = K - \int_0^t x_\tau e^{-\hat{\nabla}_t}\, d\tau$, for all t.

Suppose K is the associated stock of capital for the given time-path of consumption x, then, by Proposition 1, we have $\lim_{t\to+\infty} k_t e^{-\nabla_t} = 0$ and $\lim_{t\to+\infty} k_t e^{-\hat{\nabla}_t} = 0$, which imply the required equality. Q.E.D.

We should now like to see how the system of efficient prices may be obtained for processes of intertemporal production involving capital accumulation. Let K be the given stock of capital ($K > 0$), and let us denote by $X(K)$ the set of all time-paths of consumption $x = (x_t)$ which are feasible with respect to initial condition K. We may formally write $X(K) = \{x = (x_t): x_t > 0\ (t \geqq 0),\ K^0(x) \leqq K\}$. $X(K)$ is the set of all $x = (x_t)$ such that the solution (k_t) to the basic differential Eq. **3** with initial condition K exists at all time t.

The set $X(K)$ is non-empty; the time-path $x_t \equiv 0$ is always in $X(K)$. The concavity assumption on $f(k)$ implies that $X(K)$ is a convex set.

A time-path of consumption $x = (x_t)$ is termed efficient with respect to initial stock of capital K, if $x \in X(K)$ and there exists no $x' = (x_t')$ such that $x' \in X(K)$ and $x' > x$.

It is evident that, if x is efficient with respect to K, then $K = K^0(x)$—i.e., K is the minimum value of the initial condition for which a non-negative solution to **3** exists for all t.

Suppose now there exist two time-paths of consumption, $x^0 = (x_t^0)$ and $x^1 = (x_t^1)$, both of which are efficient in $X(K)$, where $K > 0$ is a given stock of capital initially endowed. Let us denote by $x(\theta) = [x_t(\theta)]$, $0 \leqq \theta \leqq 1$, a smooth curve connecting x^0 and x^1 on the efficiency frontier; namely, $x(\theta)$, as a function of θ defined on [0,1], is continuously twice differentiable, $x(\theta)$ is efficient in $X(K)$ ($0 \leqq \theta \leqq 1$), $x(0) = x^0$, $x(1) = x^1$.

We denote by $k(\theta) = [k_t(\theta)]$ the solution to Eq. **3** with initial condition K and time-path of consumption $x(\theta)$—i.e., $\dot{k}_t(\theta) = f[k_t(\theta)] - x_t(\theta)$, $k_0(\theta) = K$. The efficiency assumption on $x(\theta)$ implies that $\lim_{t\to\infty} k_t(\theta)e^{-\nabla_t(\theta)} = 0$ ($0 \leqq \theta \leqq 1$), where $\nabla_t(\theta)$ is the accumulated marginal rate of discount at $k(\theta)$.

We obtain

$$\dot{k}_t'(\theta) - f'[k_t(\theta)]k_t'(\theta) = -x_t'(\theta), \quad k_0'(\theta) = 0. \qquad [5]$$

By multiplying both sides of Eq. **5** by $e^{-\nabla_t(\theta)}$ and integrating from 0 to t, we get

$$k_t'(\theta)e^{-\nabla_t(\theta)} = -\int_0^t x_t'(\theta)e^{-\nabla_\tau(\theta)}d\tau. \qquad [6]$$

We next show that, for $0 < \theta < 1$,

$$\lim_{t\to\infty} k_t'(\theta)e^{-\nabla_t(\theta)} = 0. \qquad [7]$$

Suppose, to the contrary, there exist a θ_0 for which the equality **7** is not satisfied. For the sake of simplicity, we substitute θ for $\theta - \theta_0$ or $\theta_0 - \theta$, so that, for some positive number ε, $\lim_{t\to\infty} k_t'(0)e^{-\nabla_t(0)} > \varepsilon > 0$.

Let us denote by $k_t^0(\theta)$ the solution to Eq. **3** with initial condition $K(\theta) = K - (\theta/2)\varepsilon$. Namely,

$$\dot{k}_t^0(\theta) = f[k_t^0(\theta)] - x_t(\theta), \quad k_0^0(\theta) = K - (\theta/2)\varepsilon. \qquad [8]$$

Differentiating both sides of Eq. **8** with respect to θ, multiplying by $e^{-\nabla_t^0(\theta)}$, and integrating from 0 to ∞ yield $\lim_{t\to\infty} k_t^0(\theta)e^{-\nabla_t^0(\theta)} = -\varepsilon/2 - \int_0^\infty x_t^{0\prime}(\theta)e^{-\nabla_t^0(\theta)}\, dt$.

It may be noted that $k_t^0(0) = K_t(0)$, $\nabla_t^0(0) = \nabla_t(0)$, and $\lim_{t\to\infty} k_t^{0\prime}(\theta)e^{-\nabla_t^0(\theta)} = -\varepsilon/2 - \int_0^\infty x_t(\theta)e^{-\nabla_t^0(\theta)}\, dt > \varepsilon/2 > 0$, ($0 \leq \theta \leq \bar{\theta}$), $\bar{\theta}$ for sufficiently small, but positive.

By applying the mean-value theorem, there exists θ_t^* such that $k_t^0(\theta) - K_t^0(0) = \theta K_t^{0\prime}(\theta_t^*)$, $0 < \theta_t^* < \theta$, which implies that $\lim_{t\to\infty} k_t^0(\theta)e^{-\nabla_t^0(\theta)} > 0$, for sufficiently small $\theta > 0$, meaning that $x(\theta)$ is feasible with respect to $K(\theta)$; a contradiction.

By taking the limit, as $t \to \infty$ of both sides of Eq. **6** and by noting relation **7**, we obtain the following basic equality: $\int_0^\infty x_t'(\theta)e^{-\nabla_t(\theta)}\, dt = 0$.

We next differentiate Eq. **5** with respect to θ and rearrange to get

$$\dot{k}_t''(\theta) - f'[k_t(\theta)]k_t''(\theta) = f''[k_t(\theta)]k_t'(\theta)^2 - x_t''(\theta),\ k_t''(\theta) = 0,$$

$$k_t''(\theta)e^{-\nabla_t(\theta)} = \int_0^t f''[k_\tau(\theta)]k_\tau'(\theta)^2 e^{-\nabla_\tau(\theta)}\, d\tau - \int_0^t x_\tau''(\theta)e^{-\nabla_\tau(\theta)}\, d\tau.$$

An argument similar to the one by which we have obtained Eq. **7** may be applied to show that $\lim_{t\to\infty} k''(\theta)e^{-\nabla_t(\theta)} \geqq 0$, $\int_0^\infty x_t''(\theta)e^{-\nabla_t(\theta)}\, dt \leqq 0$ with strict inequality when $x^0 \neq x^1$. We have now established the following.

PROPOSITION 3. Let $x = (x_t)$, $x_t > 0$ ($t \geqq 0$), be a given time-path of consumption and $k = (k_t)$ be the solution to Eq. **3** with the initial condition $K = K^0(x)$. Then the time-path, $p(x) = [p_t(x)]$, defined by $p_t(x) = e^{-\nabla_t(k)}$, represents the system of efficient prices at $x = (x_t)$—i.e., $p(x)$ satisfies the following conditions:

$$\int_0^\infty x_t'(\theta)p_t[x(\theta)]dt = 0, \int_0^\infty x_t''(\theta)p_t[x(\theta)]dt \leqq 0,$$

where $x(\theta) = [x_t(\theta)]$, $0 \leqq \theta \leqq 1$, is any smooth curve connecting any given pair of time-paths of consumption, where X^0 and X^1, on the efficiency frontier corresponding to the given initial stock of capital K.

4. Dynamic Duality Principle: A Simple Model of Capital Accumulation

Let K be the stock of capital endowed in the economy at time 0, and let the production function $f(k)$ satisfy the neo-classical assumptions as specified in Section 3. A time-path of consumption, $x = (x_t)$, ($x_t > 0$), is feasible with respect to initial stock of capital K if there exists a solution to differential Eq. **3**.

The time-path of utilities, $u = (u_t)$, associated with time-path of consumption, $x = (x_t)$, is simply given by $u_t = u(x_t)$, where the utility function $u(x)$ satisfies the assumptions described in Section 2.

The time-path of utility functionals, (U_t), where U_t represents the value of the utility functional of the time-path of utilities truncated at t, is specified by differential Eq. **1**.

The level of utility functional $U_0 = U(u)$ for the time-path of utilities $u = (u_t)$ itself is to be determined so as to satisfy the transversality condition.

A time-path of capital accumulation, (k_t), or of consumption, (x_t), is defined dynamically optimum when the value of utility functional $U(u)$, $u = [u(x_t)]$, is maximized among all feasible time-paths of consumption with the given initial stock of capital K.

The dynamic duality principle implies that a feasible time-path of consumption $x = (x_t)$ is dynamically optimum if, and only if, the two systems of imputed prices, $[p_t(u)]$ and $[p_t(x)]$ are proportional—i.e.,

$$u'(x_t)(1 - \beta[u_t, U_t]e^{-\Delta_t} = \ell f'(k_t)e^{-\nabla_t}, \qquad [9]$$

where $u_t = u(x_t)$ and ℓ is a positive constant.

The structure of dynamically optimum time-paths of capital accumulation then is analyzed by logarithmically differentiating both sides of Eq. **9** with respect to time t, as in detail discussed in Uzawa (19).

To incorporate the Penrose effect in our model, let us introduce the real investment A as an explicit variable in our model. The output $f(k_t)$ at each time t then is divided between consumption x_t and investment A_t:

$$f(k_t) = x_t + A_t.$$

Then the accumulation of capital is described by

$$\dot{k}_t = \phi(A_t, k_t), \qquad [10]$$

where $\phi(A, k)$ is the Penrose function satisfying the following conditions, as introduced in Uzawa (19):

The function $\phi(A, k)$ is defined for all $A \geqq 0$ and $k > 0$, continuously twice differentiable, concave with respect to (A, k), and satisfies $\phi_A(A, k) > 0$, $\phi_k(A, k) > 0$, $\phi_{AA} < 0$, $\phi_{kk} < 0$, $\phi_{AA}\phi_{kk} - \phi_{Ak}^2 \geqq 0$.

For any given efficient time-path of consumption $x = (x_t)$, the system of imputed prices is given by

$$p_t(x) = \phi_A(A_t, k_t)f'(k_t)e^{-\nabla_t}, \nabla_t = \int_0^t [\phi_A(A_\tau, k_\tau)r(k_\tau) + \phi_k(A_\tau, k_\tau)]d\tau.$$

The duality principle **8** is now modified as

$$u'(x_t)(1 - \beta_u[u_t, U_t)]e^{-\Delta_t} = \ell\phi_A(A_t, k_t)f'(k_t)e^{-\nabla_t}. \qquad [11]$$

To examine the time-path of capital accumulation k_t and the truncated time-path of utilities U_t for which the duality conditions are satisfied, let us introduce the new time-path of imputed prices $\xi = (\xi_t)$: $\xi_t = \ell e^{\Delta_t - \nabla_t}$ $(t \geqq 0)$. Then relation **11** may be written

$$(1 - \beta_u)/\phi_A = \xi, \qquad [12]$$

where time suffix t is omitted.

Differentiating Eq. **12** logarithmically with respect to t, we obtain

$$\dot{\xi}/\xi = \beta_U - (r\phi_A + \phi_k), \qquad [13]$$

while k_t and U_t have to satisfy the following dynamic equations

$$\dot{k} = \phi(A, k), \qquad [14]$$

$$\dot{U} = \beta(u, U) - u \quad [u_t = u(x_t)], \qquad [15]$$

with initial conditions $k^0 = K(x)$, $U^0 = U(u)$.

Let us first consider the simple case where both $f(k)$ and $\beta(u, U)$ are homogenous—i.e., $f'(k) = r$: constant, $r > 0$, $\beta(u, U) = \delta(z)U$, $z = u/U$, where the average rate of discount function $\delta(z)$, $z = u/U$, satisfies the following conditions: $\delta(z) > 0$, $\delta'(z) < 0$, $\delta''(z) > 0$ $(z > 0)$.

It will be also assumed that the Penrose function $\phi(A, k)$ is homogeneous of order one so that $\phi(A, k) = \phi(\alpha)k$, $\alpha = A/k$, where $\phi(\alpha)$ satisfies the following conditions: $\phi'(\alpha) > 0$, $\phi''(\alpha) < 0$ $(\alpha \geqq 0)$.

Furthermore, we consider the case where $u(x) = x$ $(x \geqq 0)$. Then, the marginality condition **12** is simply written as

$$\{1 - \delta'(z)\}/\phi'(\alpha) = \xi, \qquad [16]$$

and the feasibility condition may be reduced to

$$r = zw + \alpha, w = U/k, z = x/U. \qquad [17]$$

The system of differential Eqs. **13–15** may be reduced to the following

$$\dot{\xi}/\xi = \hat{\delta}(z) - [\phi(\alpha) + (r - \alpha)\phi'(\alpha)], \hat{\delta}(z) = \delta(z) - \delta'(z)z, \qquad [18]$$

$$\dot{k}/k = \phi(\alpha), \qquad [19]$$

$$\dot{U}/U = \delta(z) - z. \qquad [20]$$

At each time t, the optimum values of z and α are determined by the pair of Eqs. **16** and **17**, where the values of $w = U/k$ and ξ are given. By taking differentials of both sides of Eqs. **16** and **17** and solving with respect to dz and $d\alpha$, we obtain

$$\begin{pmatrix} d & z \\ d & \alpha \end{pmatrix} = \frac{1}{\delta'' - \xi w \phi''}\begin{pmatrix} z\phi'' & \xi - \phi'' \\ -z\delta'' & w\phi'' \end{pmatrix}\begin{pmatrix} d & w \\ d & \xi \end{pmatrix}, \begin{pmatrix} dz \\ d\alpha \end{pmatrix} = \begin{pmatrix} - & - \\ - & + \end{pmatrix}\begin{pmatrix} dw \\ d\xi \end{pmatrix}.$$

The determination of (z, α) is illustrated in Fig. 1, where the first quadrant describes the relationships between the investment/capital ratio, $\alpha = A/k$, and the marginal increase in the rate of capital accumulation, $\phi'(\alpha)$, as depicted by the downward sloping curve, AA. The third quadrant in Fig. 1 describes the relationships between the consumption/utility functional ratio, $z = x/U$, and $1 - \delta'(z)$, the latter being measured along the abscissa in the negative direction, as depicted by the

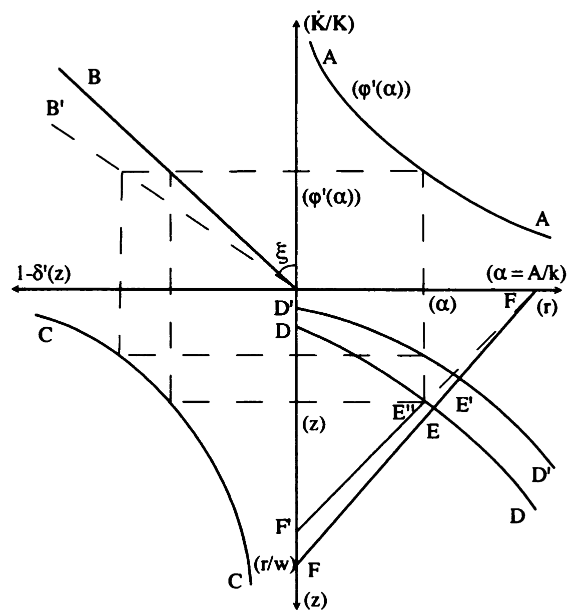

FIG. 1. Determination of (α, z).

downward sloping curve, CC. The CC curve is concave toward the origin. The OB line in the second quadrant has a slope of ξ with the ordinate. Thus, the combinations of α and z for which the optimum condition **16** is satisfied may be depicted by the DD curve in the fourth quadrant, which is obtained by the transformation through three curves, AA, BB, and CC. On the other hand, condition **17** is represented by the straight line FF in the fourth quadrant, with the intercepts r and r/w. The intersection E of two curves, DD and FF, gives the combination of α and z, for which conditions **16** and **17** are satisfied. As is easily seen from Fig. 1, an increase in ξ results in a shift from DD to D′D′, implying an increase in α and a decrease in z. On the other hand, an increase in w shifts FF line to F′F′ line, resulting a decrease in both α and z.

Let us now show that the system of differential Eqs. **18–20** has a unique steady state (ξ^0, k^0, U^0). To see this, the system of differential Eqs. **18–20** may be reduced to the one involving (w, ξ); namely, Eq. **18** and the following:

$$\dot{w}/w = \delta(z) - z - \phi(\alpha). \qquad \textbf{[21]}$$

By taking the differentials of right-hand sides of both Eqs. **18** and **21**, we obtain

$$(dz/d\alpha)_{\dot{w}=0} = -\phi'(\alpha)/[1 - \delta'(z)] < 0,$$

$$(dz/d\alpha)_{\dot{\alpha}=0} = -\{(r-\alpha)\phi''(\alpha)\}/[z\delta''(z)] > 0, \quad \text{for } \alpha < r.$$

Hence, the values of z^0 and α corresponding to the stationary state (w^0, ξ^0) for the system of differential Eqs. **18** and **21** are uniquely determined as the solution to the following pair of equations:

$$\delta(z^0) - z^0 - \phi(\alpha) = 0,$$

$$\hat{\delta}(z^0) = \phi(\alpha) + (r - \alpha)\phi'(\alpha).$$

The stationary state (w^0, ξ^0) is then given by $\xi^0 = [1 - \delta'(z^0)]/\phi'(\alpha)$, $w^0 = (r - \alpha)/z^0$.

The phase diagrams for the system of differential Eqs. **18** and **21** are easily analyzed. We have the following relations: $(d\xi/dw)_{\dot{w}=0} = -z(\delta'' - \phi''\xi^2)/\{\phi'(\xi - w)\} \gtrless 0,\ \xi \gtrless w$; $(d\xi/dw)_{\dot{\xi}=0} = -(\delta'' - \phi''\xi^2)\phi'/\{(\xi - w)z\delta''\phi''w\} \lesseqgtr 0,\ \xi \lesseqgtr w$.

Hence, the structure of the solution paths to the system of differential Eqs. **18** and **21** may be illustrated in terms of Fig. 2, where the first quadrant describes the relationships between the investment/capital ratio, $\alpha = A/k$, and the rate of capital accumulation $\phi(\alpha)$, as depicted by the curve OA. The second quadrant describes the relationships between the consumption/utility ratio, $z = x/U$, and the rate of utility discount $\delta(z)$, as depicted by the BB curve. The CC curve in the fourth quadrant depicts the combinations of (α, z) for which $\dot{\xi} = 0$, while the DD curve depicts those for which $\dot{w} = 0$. The intersection E of the CC and DD curves gives us the uniquely determined stable solution to the system of differential Eqs. **18** and **21**. The corresponding path of consumption will give us the dynamically optimum solution. This will be seen as follows.

The stationarity conditions **20** and **21** may be rearranged

$$U_0/k_0 = w = (r - \alpha)/[\delta(\alpha) - \phi(\alpha)]. \qquad \textbf{[22]}$$

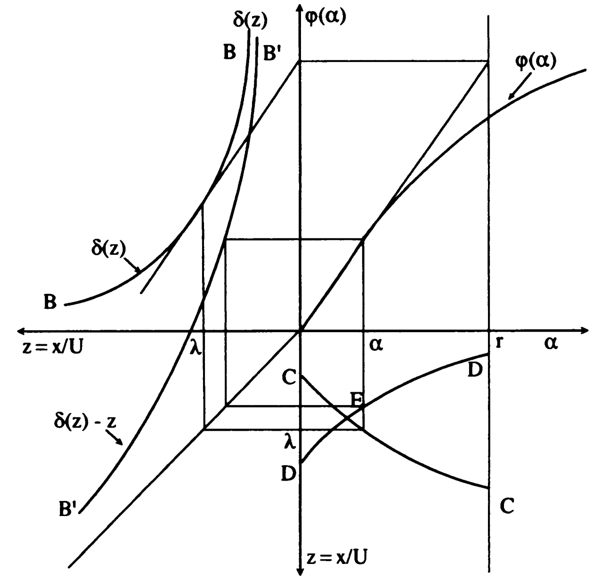

FIG. 2. Determination of stationary state.

By differentiating Eq. **17** with respect to α, we obtain $(dz/d\alpha) = -z/(r - \alpha)$. Hence, by differentiating Eq. **22** with respect to α, we obtain $dw/d\alpha = \{-\delta + \phi + z\delta' + (r - \alpha)\phi'\}/(r - \alpha)^2$. Therefore,

$$dw/d\alpha = 0 \text{ if, and only if, } \delta - z\delta' = \phi + (r - \alpha)\phi'. \qquad \textbf{[23]}$$

On the other hand, a simple calculation shows that $d^2w/d\alpha^2 \sim -z\delta''/w + zw\delta'' < 0$, where symbol $\sim$ indicates both sides have the same sign, implying that Eq. **23** gives us the maximum value of w subject to Eq. **17**.

The analysis developed above may be extended to the case where the marginal product of capital is variable. Namely, we consider the case where $f'(k) > 0, f''(k) < 0\ (k \geqq 0)$.

The basic system of differential equations now may be summarized as:

$$\dot{U}/U = \delta(z) - z,$$

$$\dot{k}/k = \phi(\alpha)$$

$$\dot{\xi}/\xi = [\delta(z) - z\delta'(z)] - [\phi(\alpha) + (r - \alpha)\phi'(\alpha)].$$

where $f(k) = x + \alpha k$, $z = x/U$, $[1 - \delta'(z)] = \xi\phi'(\alpha)$.

We obtain

$$\begin{pmatrix} \delta''/U & \varphi''\xi \\ 1 & k \end{pmatrix}\begin{pmatrix} dx \\ d\alpha \end{pmatrix} = \begin{pmatrix} \delta''z/U & 0 & -\varphi' \\ 0 & r-\alpha & 0 \end{pmatrix}\begin{pmatrix} dU \\ dk \\ d\xi \end{pmatrix},$$

$$\begin{pmatrix} dx \\ d\alpha \end{pmatrix} = 1/\Delta_1\begin{pmatrix} \delta''zk/U & -(r-\alpha)\varphi''\xi & -k\varphi' \\ -\delta''z/U & (-\alpha)\delta''/U & \varphi' \end{pmatrix}\begin{pmatrix} dU \\ dk \\ d\xi \end{pmatrix},$$

where $\Delta_1 = \delta''k/U - \varphi''\xi > 0$.

On the other hand,

$$\begin{pmatrix} d\dot{U} \\ d\dot{k} \\ d\dot{\xi} \end{pmatrix} = \frac{1}{\Delta_1}\begin{pmatrix} -\frac{(1-\delta')\delta''zk}{U^2} + \Delta_1\frac{(1-\delta')z}{U} & \frac{(1-\delta')\varphi''\xi(r-\alpha)}{U} & \frac{(1-\delta')k\varphi'}{U} \\ -\frac{\delta''z\varphi'}{U} & \frac{\delta''\varphi'(r-\alpha)}{U} & \varphi'^2 \\ -\frac{\delta''z}{U}\left(\frac{\delta''zk}{U} - (r-\alpha)\varphi''\right) + \Delta_1\frac{\delta''z^2}{U} & \frac{\delta''\varphi''}{U}((r-\alpha)(z\xi - (r-\alpha)) - \Delta_1 r'\varphi' & \varphi'\left(\frac{\delta''zk}{U} - (r-\alpha)\varphi''\right) \end{pmatrix}\begin{pmatrix} dU \\ dk \\ d\xi \end{pmatrix}$$

whose determinant D may be calculated to show that $D \sim r'\varphi'\Delta_1^2 < 0$. On the other hand, the trace of the above matrix is easily shown to be positive.

Hence, the system of differential Eqs. **18–20** is catenary. The stationary state is uniquely determined, and exactly one characteristic root is real and negative.

For any given pair of (U, k), there uniquely exists a value of imputed price ξ such that the solution path to the system **18–20** converges to the stationary state, corresponding to the dynamically optimum path.

5. Dynamic Optimality of Investment in Social Overhead Capital

The dynamic duality principles discussed in the previous sections may be easily extended to the case where the intertemporal preference ordering is affected by the natural environment, which is conceptualized as the stock of social overhead capital, as introduced in Uzawa (19).

We denote by V_t the stock of social overhead capital existing in the society at time t, while the stock of private capital is denoted by k_t, as in Section 4. The output $f(k_t)$ at each time t is now divided between consumption x_t, investment in private capital A_t, and investment in social overhead capital B_t: $f(k_t) = x_t + A_t + B_t$, where the production function $f(k)$ is assumed to satisfy all the neoclassical conditions postulated in Section 3.

The rate of increase in the stock of private capital, $\dot{k}_t$, is determined in terms of the Penrose function $\phi(A_t, k_t)$: $\dot{k}_t = \phi(A_t, k_t)$, where the Penrose function $\phi(A, k)$ is assumed to satisfy the conditions specified in Section 4.

We also assume that the effect of investment in social overhead capital is subject to the Penrose effect, so that the rate of increase in the stock of social overhead capital, $\dot{V}_t$, is determined in terms of the Penrose function $\Psi(B, V)$: $\dot{V}_t = \Psi(B_t, V_t)$. The Penrose function $\Psi(B, V)$ concerning the accumulation of social overhead capital is assumed to satisfy the concavity conditions.

The intertemporal preference ordering is assumed to possess the structure specified in Section 2. It will be represented by the utility functional U_t associated with the truncated time-path of utilities ${}^t u = (u_{t+\tau})$, which satisfies the basic differential Eq. **10** in Section 4. The instantaneous level of utility u_t is assumed to depend upon the stock of social overhead capital V_t: $u_t = u(x_t, V_t)$, where the following concavity conditions are satisfied: $u_x > 0, u_V > 0, u_{xx} < 0, u_{VV} < 0, u_{xx}u_{VV} - u_{xV}^2 \geqq 0$.

Let us denote the imputed prices of private capital and social overhead capital at each time t by ξ_t and η_t, respectively. Then, it will be easily shown that the imputed prices are subject to the following dynamic equations:

$$\dot{\xi}/\xi = -\phi_k - r\phi_A, \quad r = f_k, \tag{24}$$

$$\dot{\eta}/\eta = -\Psi_V - s\Psi_B, \quad s = u_V/u_x. \tag{25}$$

To derive formulas for the systems of efficient prices, let us consider two time-paths of consumption and the stocks of private and social overhead capital, (x_t^0, k_t^0, V_t^0) and (x_t^1, k_t^1, V_t^1), such that $(k_0^0, V_0^0) = (k_0^1, V_0^1) = k^0, V^0)$, and both (x_t^0) and (x_t^1) are efficient.

We denote by $[x_t(\theta), k_t(\theta), V_t(\theta)]$ the time-path connecting (x_t^0, k_t^0, V_t^0) and (x_t^1, k_t^1, V_t^1) which is also efficient:

$$\dot{k}_t(\theta) = \phi[A_t(\theta), K_t(\theta)], \quad k_0(\theta) = k^0,$$

$$\dot{V}_t(\theta) = \Psi[B_t(\theta), \quad V_t(\theta)], V_0(\theta) = V^0,$$

where $A_t(\theta)$ and $B_t(\theta)$ are, respectively, investments in private capital and social overhead capital.

We have

$$\dot{k}_t' = \phi_A A_t' + \phi_k K_t', \quad k_0' = 0, \tag{26}$$

$$\dot{V}_t' = \Psi_B B_t' + \Psi_V V_t', \quad V_0' = 0, \tag{27}$$

where symbol ′ indicates the derivative with respect to θ, while parameter θ is omitted, and $\phi_A, \phi_k, \Psi_B, \Psi_V$ are evaluated at time t.

Feasibility conditions may be written $f[k_t(\theta)] = x_t(\theta) + A_t(\theta) + B_t(\theta)$ that, by differentiating with respect to θ, yields $rk_t' = x_t' + A_t' + B_t'$, $r = f'(k_t)$. Similarly, $u_t' = u_x(x_t' + sV_t')$, $s = u_V/u_x$.

Now let us denote by π_t the imputed price of the output at time t. Then, we have $\xi_t\phi_A = \eta_t\Psi_B = \pi_t$.

By multiplying Eqs. **26** and **27**, respectively, by ξ_t and η_t, we obtain $\xi_t\dot{k}_t' + \eta_t\dot{V}_t' = -\pi_t x_t' + \pi_t' r k_t' + \xi_t\phi_k K_t' + \eta_t\Psi_V V_t'$.

We obtain $\dot{\xi}_t k_t' + \dot{\eta}_t V_t' = -(\xi_t\phi_k k_t' + \eta_t\Psi\Psi_V V_t') - \pi_t(rk_t' + sV_t')$, $d(\xi_t k_t + \eta_t V_t')/dt = -\pi_t(x_t' + sV_t' = -\xi_t u_t'(\phi_A/u_x)$.

The transversality conditions imply that $\lim_{t\to\infty}(\xi_t k_t' + \eta_t V_t') = 0$, while $k_0' = V_0' = 0$. Hence,

$$\int_0^\infty u_t' p_t dt = 0, \quad p_t = (\phi_A/u_x)\xi_t. \tag{28}$$

It will also be shown that

$$\int_0^\infty u_t'' p_t dt \leqq 0, \tag{29}$$

To prove Eq. **29**, let us note that $\dot{k}_t'' = S_k + \phi_A A_t'' + \phi_k k_t''$, $\dot{V}_t'' = S_V + \Psi_B B_t'' + \Psi_V V_t''$, $S_k = \phi_{AA}A_t'^2 + 2\phi_{AK}A_t' k_t + \phi_{kk}k_t'^2$, $S_V = \Psi_{BB}B_t'^2 + 2\Psi_{BV}B_t'V_t' + \Psi_{VV}V_t'^2$.

The concavity conditions for the Penrose functions $\phi(A, k)$ and $\Psi(B, V)$ imply that

$$S_k \leqq 0, S_v \leqq 0. \tag{30}$$

On the other hand, $u_t'' = S_u + u_x(x_t'' + sV_t'')$, $S_u = u_{xx}x_t'^2 + 2u_{xv}x_t'V_t' + u_{VV}V_t'^2$. Again the concavity of $u(x, V)$ implies that

$$S_u \leqq 0. \tag{31}$$

We also have $\xi_t\dot{k}_t'' + \eta_t\dot{V}_t'' = \xi_t S_k + \eta_t S_V + \pi_t(A_t'' + B_t'') + \xi_t\phi_k k_t'' + \eta_t\Psi_V V_t''$.

By multiplying both sides of Eqs. **24** and **25**, respectively, by $\xi_t k_t''$ and $\eta_t V_t''$, and adding them, we obtain $\dot{\xi}_t k_t'' + \dot{\eta}_t V_t'' = -\pi_t(rk_t'' + sV_t'') - (\xi_t\phi_k k_t'' + \eta_t\Psi_V V_t'')$,

$$d(\xi_t k_t'' + \eta_t V_t'')/dt = \xi_t S_k + \eta_t S_V + S_u/u_x + f_{kk}k_t'^2 - \xi_t\phi_A u_t''/u_x. \tag{32}$$

The transversality conditions imply that $\lim_{t\to\infty}(\xi_t k_t'' + \eta_t V_t'') = 0$, while $k_0'' = V_0'' = 0$. The relation **32**, together with Eqs. **30** and **31**, implies the required inequality **29**. Q.E.D.

We have now proved that the system of prices (p_t) given by Eq. **28** is the system of efficient, or imputed, prices for the efficient time-path of utilities (u_t). In view of the differential Eq. **22**, the definition of the system of imputed prices (p_t), as given by Eq. **28**, may be explicitly written as follows:

$$p_t = (\phi_A/u_x)e^{-\nabla_t}, \tag{33}$$

where $\nabla_t = \int_0^t [\phi_k(A_\tau, k_\tau) + r(k_\tau)\phi_A(A_\tau, k_\tau)]\, d\tau$.

It may be noted the system of efficient prices, as expressed by Eq. **33**, is determined independently of the time-path of the stock of social overhead capital. The effect of the changing pattern of the stock of social overhead capital is felt only through the changes in the levels of instantaneous utility $u_t = u(x_t, V_t)$.

Valuable comments and technical advise by Morio Kuninori (The Japan Development Bank) are highly appreciated.

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