Spontaneous Symmetry Breaking in Interdependent Networked Game

Supplementary Information

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Theoretical Analysis: Strategy-Couple Pair Approximation Method. Pair approximation method has proved its efficiency and importance in traditional spatial game studies to qualitatively predict the trends [1–4]. We believe that the extension of such a approach will also be very significant in our research of interdependent networked game. Since the games are simultaneously implemented on both networks (see Fig. 1), we mainly focus on the evolution of coupled strategies. In this sense, the new name Strategy-Couple Pair Approximation(SCPA) would be more appropriate.

Instead of considering different individuals on the single network [5–8], there exist four types of couples in the system: C-C, C-D, D-C and D-D couples (see Fig. S1). Notice the possible existence of spontaneous symmetry breaking phenomenon, the C-D and D-C couples should denote different meanings. For simplicity, C-D is regarded as the case that the node on Network Up chooses the strategy C, and its corresponding one selects D.

However, if the node on Network Up is a defector (D) and its companion on the other network is a cooperator (C), D-C is the correct notation for this situation. In addition, to keep the expression simple, C-C, C-D, D-C and D-D will be represented by 1, 2, 3 and 4 in the following equations, respectively.

$$\begin{aligned}
p(\begin{bmatrix} I_u & J_u \\ I_d & J_d \end{bmatrix} &\rightarrow \begin{bmatrix} J_u & J_u \\ I_d & J_d \end{bmatrix}) \\
&= \sum_{X,Y,Z} \sum_{U,V,W} f(G_{J_u} - G_{I_u}) \frac{p(X,I)p(Y,I)p(Z,I)p(I,J)p(J,U)p(J,V)p(J,W)}{{p_I}^3 {p_J}^3} \\
&= \sum_{X,Y,Z} \sum_{U,V,W} u(I,J)
\end{aligned}$$

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p(\begin{bmatrix} I_u & J_u \\ I_d & J_d \end{bmatrix} &\rightarrow \begin{bmatrix} I_u & J_u \\ J_d & J_d \end{bmatrix}) \\
&= \sum_{X,Y,Z} \sum_{U,V,W} f(G_{J_d} - G_{I_d}) \frac{p(X,I)p(Y,I)p(Z,I)p(I,J)p(J,U)p(J,V)p(J,W)}{{p_I}^3 {p_J}^3} \\
&= \sum_{X,Y,Z} \sum_{U,V,W} d(I,J)
\end{aligned}$$

where $p(S, S')$ represents the possibility of finding the couples of S and S', and f is the transition probability (see Eq. (2)in the main text). The sum runs over all the possible types of neighbor couples for focal couple, namely, X, Y, Z for I, and U, V , W for J (see Fig. S1). To make notations simple in what follows, the sum will be represented by $u(I, J)$ and $d(I, J)$. (Note that the functions of u and d not only depend on I and J, but also rely on all their neighbor couples, the exact expressions of $u(I, J)$ and $d(I, J)$ should be $u(I, J)|_{X,Y,Z,U,V,W}$ and $d(I, J)|_{X,Y,Z,U,V,W}$). With respect to the individual fitness G, it is not only determined by his own payoff P, but also by the payoff of his companion P' on the other network (see the text for details). While in this analysis method, the fitness of players belonging to couples I and J will be rewritten as follows,

$$\begin{aligned}
G_{I_u} &= (1-\alpha) * P_{I_u} + \alpha * P_{I_d}, && \text{(S.1)} \\
G_{I_d} &= (1-\alpha) * P_{I_d} + \alpha * P_{I_u}, && \text{(S.2)} \\
G_{J_u} &= (1-\alpha) * P_{J_u} + \alpha * P_{J_d}, && \text{(S.3)} \\
G_{J_d} &= (1-\alpha) * P_{J_d} + \alpha * P_{J_u}. && \text{(S.4)}
\end{aligned}$$

If couple J succeeds in impacting couple I, then the probabilities of various couple pairs will also change, which can result in 16 ordinary differential equations. Consider the symmetry of the equations and sum of the probabilities equalling to 1 (two elementary conditions for pair approximation method [1–4]), the equations can be reduced as a set of 9 equations:

$$\dot{p}(1,1) = \sum_{X,Y,Z}\sum_{U,V,W} 2[(e_1+1)p(1,3)u(3,1) - e_1p(1,3)u(1,3) - e_1p(1,4)u(1,4) + e_1p(2,3)u(3,2)] + \sum_{X,Y,Z}\sum_{U,V,W} 2[(e_1+1)p(1,2)d(2,1) - e_1p(1,2)d(1,2) - e_1p(1,4)d(1,4) + e_1p(2,3)d(2,3)] \quad (S.5)$$

$$\begin{aligned}\dot{p}(1,2) = \sum_{X,Y,Z}\sum_{U,V,W} &[(e_1+1)p(1,4)u(4,1) + (e_2+1)p(2,3)u(3,2) + e_2p(1,3)u(3,1) + e_1p(2,4)u(4,2) \\ &- e_2p(1,3)u(1,3) - e_2p(1,4)u(1,4) - e_1p(2,3)u(2,3) - e_1p(2,4)u(2,4)] + \\ \sum_{X,Y,Z}\sum_{U,V,W} &[-(e_1+1)p(1,2)d(2,1) + e_2p(1,2)d(2,1) + e_1p(1,2)d(1,2) - (e_2+1)p(1,2)d(1,2) \\ &+ e_2p(2,3)d(2,3) + e_1p(1,4)d(1,4) - e_2p(1,4)d(1,4) - e_1p(2,3)d(2,3)] \end{aligned} \quad (S.6)$$

$$\begin{aligned}\dot{p}(1,3) = \sum_{X,Y,Z}\sum_{U,V,W} &[-(e_3+1)p(1,3)u(1,3) - (e_1+1)p(1,3)u(3,1) + e_3p(1,3)u(3,1) + e_3p(2,3)u(3,2) \\ &+ e_1p(1,3)u(1,3) + e_1p(1,4)u(1,4) - e_3p(1,4)u(1,4) - e_1p(2,3)u(3,2)] + \\ \sum_{X,Y,Z}\sum_{U,V,W} &[(e_1+1)p(1,4)d(4,1) + (e_3+1)p(2,3)d(2,3) + e_3p(1,2)d(2,1) + e_1p(3,4)d(4,3) \\ &- e_3p(1,2)d(1,2) - e_3p(1,4)d(1,4) - e_1p(2,3)d(3,2) - e_1p(3,4)d(3,4)] \end{aligned} \quad (S.7)$$

$$\begin{aligned}\dot{p}(1,4) = \sum_{X,Y,Z}\sum_{U,V,W} &[-(e_4+1)p(1,4)u(1,4) - (e_1+1)p(1,4)u(4,1) + e_4p(1,3)u(3,1) + e_4p(2,3)u(3,2) \\ &+ e_1p(2,3)u(2,3) + e_1p(2,4)u(2,4) - e_4p(1,3)u(1,3) - e_1p(2,4)u(4,2)] + \\ \sum_{X,Y,Z}\sum_{U,V,W} &[-(e_4+1)p(1,4)d(1,4) - (e_1+1)p(1,4)d(4,1) + e_4p(2,3)d(2,3) + e_4p(1,2)d(2,1) \\ &+ e_1p(2,3)d(3,2) + e_1p(3,4)d(3,4) - e_4p(1,2)d(1,2) - e_1p(3,4)d(4,3)] \end{aligned} \quad (S.8)$$

$$\dot{p}(2,2) = \sum_{X,Y,Z}\sum_{U,V,W} 2[(e_2+1)p(2,4)u(4,2) - e_2p(2,3)u(2,3) - e_2p(2,4)u(2,4) + e_2p(1,4)u(4,1)] + \sum_{X,Y,Z}\sum_{U,V,W} 2[(e_2+1)p(1,2)d(1,2) - e_2p(1,2)d(2,1) - e_2p(2,3)d(2,3) + e_2p(1,4)d(1,4)], \quad (S.9)$$

$$\dot{p}(2,3) = \sum_{X,Y,Z}\sum_{U,V,W}[-(e_3+1)p(2,3)u(2,3) - (e_2+1)p(2,3)u(3,2) + e_3p(2,4)u(4,2) + e_3p(1,4)u(4,1) + e_2p(1,3)u(1,3) + e_2p(1,4)u(1,4) - e_3p(2,4)u(2,4) - e_2p(1,3)u(3,1)] + \sum_{X,Y,Z}\sum_{U,V,W}[-(e_3+1)p(2,3)d(2,3) - (e_2+1)p(2,3)d(3,2) + e_3p(1,2)d(1,2) + e_3p(1,4)d(1,4) + e_2p(1,4)d(4,1) + e_2p(3,4)d(4,3) - e_3p(1,2)d(2,1) - e_2p(3,4)d(3,4)] \quad (S.10)$$

$$\dot{p}(2,4) = \sum_{X,Y,Z}\sum_{U,V,W}[-(e_4+1)p(2,4)u(2,4) - (e_2+1)p(2,4)u(4,2) + e_4p(2,4)u(4,2) + e_4p(1,4)u(4,1) + e_2p(2,3)u(2,3) + e_2p(2,4)u(2,4) - e_4p(2,3)u(2,3) - e_2p(1,4)u(4,1)] + \sum_{X,Y,Z}\sum_{U,V,W}[(e_4+1)p(1,4)d(1,4) + (e_2+1)p(2,3)d(3,2) + e_4p(1,2)d(1,2) + e_2p(3,4)d(3,4) - e_4p(1,2)d(2,1) - e_4p(2,3)d(2,3) - e_2p(1,4)d(4,1) - e_2p(3,4)d(4,3)] \quad (S.11)$$

$$\dot{p}(3,3) = \sum_{X,Y,Z}\sum_{U,V,W}2[(e_3+1)p(1,3)u(1,3) - e_3p(1,3)u(3,1) - e_3p(2,3)u(3,2) + e_3p(1,4)u(1,4)] + \sum_{X,Y,Z}\sum_{U,V,W}2[(e_3+1)p(3,4)d(4,3) - e_3p(2,3)d(3,2) - e_3p(3,4)d(3,4) + e_3p(1,4)d(4,4)] \quad (S.12)$$

$$\dot{p}(3,4) = \sum_{X,Y,Z}\sum_{U,V,W}[(e_4+1)p(1,4)u(1,4) + (e_3+1)p(2,3)u(2,3) + e_4p(1,3)u(1,3) + e_3p(2,4)u(2,4) - e_4p(1,3)u(3,1) - e_4p(2,3)u(3,2) - e_3p(1,4)u(4,1) - e_3p(2,4)u(4,2)] + \sum_{X,Y,Z}\sum_{U,V,W}[-(e_4+1)p(3,4)d(3,4) - (e_3+1)p(3,4)d(4,3) + e_4p(1,4)d(4,1) + e_4p(3,4)d(4,3) + e_3p(2,3)d(3,2) + e_3p(3,4)d(3,4) - e_4p(2,3)d(3,2) - e_3p(1,4)d(4,1)] \quad (S.13)$$

where the variable $e_l (l = 1, 2, 3, 4)$ represents the number of l couples among X, Y, Z (a more accurate demonstration should be $e_l(X,Y,Z)$).

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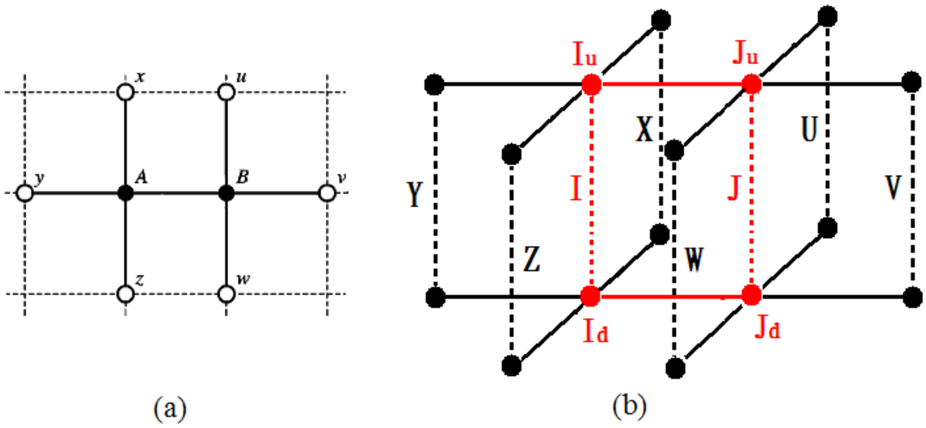

Figure S1. Demonstration of the relevant configurations for traditional pair approximation (a) and strategy-couple pair approximation (b). In (a), the configuration of nodes will determine the prediction results, since the probability of changing strategy fully depend on the situation of neighbor nodes on the single network [1]. However, for the interdependent networks (b), the probability of changing strategy for focal couples (marked by red dashed lines) rely on their neighbor couples (denoted by black dashed lines). That is, X, Y, Z are neighbor couples for couple I, and U, V, W are neighbor couples for couple J.