Supplementary Information:

Influence of the number of topologically interacting neighbors on swarm dynamics

Yilun Shang[1] & Roland Bouffanais[1]

[1] *Singapore University of Technology and Design, 20 Dover Drive, Singapore 138682*

Proof for Theorem A. The proof will be divided into two parts. In part 1 we show the almost guaranteed consensus, while in part 2 we address the rate of convergence of consensus.

Part 1. In the framework of our study, the condition $k \geq 1$ is always fulfilled as topologically interacting agents are considered. Since reaching consensus for the dynamical system governed by Eq. (3) is a monotone increasing property with respect to the number of edges of $G(t)$ [1], it suffices to show the case $k = 1$.

Let $\xi_{ij}(t)$ be a random variable representing the connection between agent i and agent j at time t. More specifically, $P(\xi_{ij}(t) = 1) = k/(N-1)$ and $P(\xi_{ij}(t) = 0) = 1 - k/(N-1)$ for all $i, j \in \{1, \cdots, N\}$ $(i \neq j)$ and $t \geq 0$. Let $M \geq 1$ be an integer. By the law of large numbers, we have

$$\begin{aligned} & P\left(\sum_{m=1}^{M} \xi_{ij}(m) \leq \frac{Mk}{2(N-1)}\right) \\ = & P\left(\frac{1}{M}\sum_{m=1}^{M} \xi_{ij}(m) - \frac{k}{N-1} \leq -\frac{k}{2(N-1)}\right) \\ \leq & P\left(\left|\frac{1}{M}\sum_{m=1}^{M} \xi_{ij}(m) - \frac{k}{N-1}\right| \geq \frac{k}{2(N-1)}\right) \\ \leq & \frac{4(N-1-k)}{Mk}. \end{aligned} \tag{1}$$

Hence, we obtain

$$\begin{aligned} & P\left(\bigcap_{i\neq j}\left\{\sum_{m=1}^{M} \xi_{ij}(m) > \frac{Mk}{2(N-1)}\right\}\right) \\ = & 1 - P\left(\bigcup_{i\neq j}\left\{\sum_{m=1}^{M} \xi_{ij}(m) \leq \frac{Mk}{2(N-1)}\right\}\right) \\ \geq & 1 - \frac{4N(N-1)(N-1-k)}{Mk}, \end{aligned} \tag{2}$$

which tends to the unity as $M \to \infty$.

Now define a graph $\tilde{G}$ of order N whose adjacency matrix (a_{ij}) is given by

$$a_{ij} = \begin{cases} 1, & \int_0^\infty \xi_{ij}(t)dt = \infty; \\ 0, & \int_0^\infty \xi_{ij}(t)dt < \infty. \end{cases} \tag{3}$$

Note that $\int_0^M \xi_{ij}(t)dt = \sum_{m=1}^{M} \xi_{ij}(m)$. Thus, Eq. (2) implies that $\tilde{G}$ is a completely connected digraph for almost all sequences $G_1, G_2, \cdots, G_m, \cdots$.

Fix any such sequence $G_1, G_2, \cdots, G_m, \cdots$, we will show that the consensus can be reached for the dynamical system governed by Eq. (3). Let $\Phi(t) = [\phi_1(t), \cdots, \phi_N(t)]^T$ be a rearrangement of the vector $\Theta(t) = [\theta_1(t), \cdots, \theta_N(t)]^T$ such that

$$\phi_1(t) \leq \phi_2(t) \leq \cdots \leq \phi_N(t). \tag{4}$$

Note that this new vector still satisfies the governing equation for the dynamics of the system

$$\dot{\Phi}(t) = \frac{1}{k}(-L(t))\Phi(t) = -L(t)\Phi(t), \tag{5}$$

except that, now, the matrix $L(t)$ is the result of some conjugation transform made by some permutation matrix at time t. To avoid introducing unnecessary new notations, it still appears as $L(t)$ in Eq. (5). It is clear that $\dot{\phi}_1(t) = \sum_j \xi_{1j}(\phi_j(t) - \phi_1(t)) \geq 0$ and $\dot{\phi}_N(t) = \sum_j \xi_{Nj}(\phi_j(t) - \phi_N(t)) \leq 0$. Recall from Eq. (4) that $\phi_1(t) \leq \phi_N(t)$. Therefore, $\phi_1(t)$ and $\phi_N(t)$ are monotonic and bounded functions. We hence obtain the existence of some ϕ_1^* and ϕ_N^* such that

$$\phi_1(t) \to \phi_1^* \quad \text{and} \quad \phi_N(t) \to \phi_N^*, \tag{6}$$

as t tends to infinity.

Define $\Psi(t) = [\psi_1(t), \cdots, \psi_N(t)]^T$ and recall that $k = 1$. Then, the k-nearest neighbor model entails that the outdegree of any vertex in $G(t)$ is equal to 1. Hence, it is easy to see that there exists a diagonal matrix B with diagonal elements equal to 1 or -1 such that

$$\Psi(t) = B\Phi(t), \tag{7}$$

and $\dot{\psi}_i(t) \geq 0$ for all i. Since $\psi_i(t) \leq |\phi_N(0)|$ (i.e., bounded), $\psi_i(t)$ converges for all i. It follows from (7) that $\phi_i(t)$ also converges. We write

$$\Phi(t) \to \Phi^* = [\phi_1^*, \cdots, \phi_N^*]^T. \tag{8}$$

Next, we claim that $\theta_i(t)$ converges for $i = 1, \cdots, N$. This can be shown as follows. Note that there exists an $\varepsilon_0 > 0$, such that for any $\varepsilon < \varepsilon_0$, any pair of intervals in the family $\{(\phi_i^* - \varepsilon, \phi_i^* + \varepsilon)\}_{i=1}^N$ is either coincident or disjoint. For such ε, there exists $T > 0$ such that for $t > T$,

$$\{\theta_i(t)\}_{i=1}^N = \{\phi_i(t)\}_{i=1}^N \in \cup_{i=1}^N (\phi_i^* - \varepsilon, \phi_i^* + \varepsilon), \tag{9}$$

by invoking (8). Since $\theta_i(t)$ is continuous, for any $t_1, t_2 > T$ we obtain $|\theta_i(t_1) - \theta_i(t_2)| < 2\varepsilon$. Therefore, by the Cauchy convergence criterion we have

$$\Theta(t) \to \Theta^* = [\theta_1^*, \cdots, \theta_N^*]^T, \tag{10}$$

for some θ_i^* $(i = 1, \cdots, N)$.

Finally, we need to show that all the above θ_i^* are equal. From Eq. (3) we have

$$\dot{\theta}_i = \sum_{j=1}^N \xi_{ij}(\theta_j - \theta_i). \tag{11}$$

In the following, we will use the method of proof by contradiction. Without loss of generality, we assume that

$$\theta_{j_0}^* > \theta_{i_0}^*. \tag{12}$$

Then there exists some $T > 0$ such that

$$\theta_{j_0}(t) - \theta_{i_0}(t) \geq \frac{\theta_{j_0}^* - \theta_{i_0}^*}{2} := \delta > 0, \tag{13}$$

holds for any $t > T$. Using Eqs. (11) and (13) we obtain

$$\begin{aligned}\int_T^\infty \xi_{i_0 j_0} dt &\leq \frac{1}{\delta}\int_T^\infty \xi_{i_0 j_0}(\theta_{j_0} - \theta_{i_0}) dt \\ &= \frac{1}{\delta}\int_T^\infty \dot{\theta}_{i_0} dt \\ &= \frac{\theta_{i_0}^* - \theta_{i_0}(T)}{\delta}.\end{aligned} \tag{14}$$

It then follows from the definition (3) that $a_{i_0 j_0} = 0$, and hence $\{i_0, j_0\}$ is not an edge in $\tilde{G}$. This yields a contradiction since we know that $\tilde{G}$ is a complete digraph. Therefore, we have $\theta_1^* = \cdots = \theta_N^*$, which is the final consensus value for all the agents.

Part 2. To determine the rate of convergence, we will look at the system via a Lyapunov function's point of view [2, 3]. Define the agreement subspace as

$$\text{span}\{\mathbf{1}\} := \{x = (x_1, \cdots, x_N) \in \mathbb{R}^N | \\ x_i = x_j \text{ for all } i, j\}, \tag{15}$$

where $\mathbf{1} = (1, \cdots, 1)^T \in \mathbb{R}^N$.

For $t \in [m-1, m)$, rewrite $L_m := L(t)$, which is the Laplacian of the digraph G_m. The trajectory of (3) can be viewed as [4, 5]

$$\Theta(m+1) = \mathrm{e}^{-\frac{1}{k}L_m}\Theta(m), \quad m = 0, 1, 2, \cdots \tag{16}$$

Let $\{\Theta^\perp(m)\}_{m\ge1}$ be the projection of $\{\Theta(m)\}_{m\ge1}$ on the subspace $(\text{span}\{\mathbf{1}\})^\perp$ orthogonal to the agreement subspace span$\{\mathbf{1}\}$. Hence, $\Theta^\perp(m)^T\mathbf{1} = 0$. We aim to estimate the convergence rate of $\Theta^\perp(m)^T\Theta^\perp(m) \to 0$, as m tends to infinity.

Let us define the Lyapunov function as

$$V(\Theta^\perp(m)) = \frac{1}{N}\Theta^\perp(m)^T\hat{L}\Theta^\perp(m), \tag{17}$$

where $\hat{L} = NI_N - \mathbf{1}\mathbf{1}^T$, I_N being the N-dimensional identity matrix. Using Eqs. (16) and (17), we obtain

$$\begin{aligned}
&E\left[V(\Theta^\perp(m+1)) - V(\Theta^\perp(m))|\Theta^\perp(m)\right] \\
&= \frac{1}{N}E[\Theta^\perp(m)^T\mathrm{e}^{-\frac{1}{k}L_m^T}\hat{L}\mathrm{e}^{-\frac{1}{k}L_m}\Theta^\perp(m) \\
&\quad -\Theta^\perp(m)^T\hat{L}\Theta^\perp(m)|\Theta^\perp(m)] \\
&= E\Big[\Theta^\perp(m)^T\mathrm{e}^{-\frac{1}{k}(L_m^T+L_m)}\Theta^\perp(m) \\
&\quad -\frac{1}{N}\Theta^\perp(m)^T(\mathrm{e}^{-\frac{1}{k}L_m^T}\mathbf{1})(\mathrm{e}^{-\frac{1}{k}L_m^T}\mathbf{1})^T\Theta^\perp(m) \\
&\quad -\Theta^\perp(m)^T\Theta^\perp(m) \\
&\quad +\frac{1}{N}\Theta^\perp(m)^T\mathbf{1}\mathbf{1}^T\Theta^\perp(m)\,\Big|\,\Theta^\perp(m)\Big] \\
&= \Theta^\perp(m)^T E\left[\mathrm{e}^{-\frac{1}{k}(L_m^T+L_m)} - \frac{1}{N}\left\|\mathrm{e}^{-\frac{1}{k}L_m^T}\mathbf{1}\right\|^2 I_N - I_N\right] \\
&\quad \cdot\Theta^\perp(m),
\end{aligned} \tag{18}$$

where in the last equation above we used the property $\Theta^\perp(m)^T\mathbf{1} = 0$.

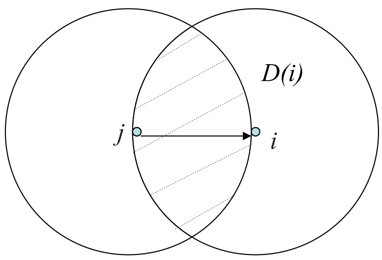

FIG. 1. An illustration that node i belongs to the set of k-nearest neighbors of node j. $D(i)$ is the disk of radius ij center at i.

We claim that almost all graphs in $G(N, k)$ have balanced induced subgraphs [2] when $k \ge 3$ and N is large. Indeed, we note that the probability that $G \in G(N, k)$ contains a balanced induced subgraph is lower bounded by the probability that G contains a bidirectional edge. The latter one can be estimated as follows. For any node $j \in G$, we can pick a node i such that $\{j, i\}$ is an edge in G starting from j ending at i. In other words, i belongs to the set of k-nearest neighbors of j. Since the area of shaded region constitutes more than one-third of the area of the disk $D(i)$ (see Fig. 1), we conclude that, on average, there will be at least 3 nodes within the disk $D(i)$ when N is large.

Therefore, $\{i,j\}$ must be an edge when $k \geq 3$ and thus $\{i,j\}$ and $\{j,i\}$ form a bidirectional edge in G. We have $P(G$ contains a balanced induced subgraph$) \geq P(G$ contains a bidirectional edge$) = 1$.

Note that $L_m^T + L_m$ is symmetric and hence has real eigenvalues. First, we assume that G_m itself is balanced. We have $L_m^T \mathbf{1} = 0$ and $L_m^T + L_m$ can be viewed as the Laplacian for the disoriented version of the digraph G_m [1, 6]. We know that the spectrum of $\mathrm{e}^{-\frac{1}{k}(L_m^T+L_m)}$ can be ordered as

$$\begin{aligned} \mathrm{e}^{-\frac{1}{k}\lambda_N(L_m^T+L_m)} &\leq \mathrm{e}^{-\frac{1}{k}\lambda_{N-1}(L_m^T+L_m)} \leq \cdots \\ &\leq \mathrm{e}^{-\frac{1}{k}\lambda_2(L_m^T+L_m)} \leq 1. \end{aligned} \tag{19}$$

The vector $\mathbf{1}$ is the eigenvector corresponding to the largest eigenvalue 1 of the matrix exponential $\mathrm{e}^{-\frac{1}{k}(L_m^T+L_m)}$ (This can easily be seen by expanding the matrix exponential [7]). Noting that $\mathrm{e}^{-\frac{1}{k}L_m^T}\mathbf{1} = 0$ and applying the Courant–Fischer theorem to (18), we obtain

$$\begin{aligned} & E\left[V(\Theta^{\perp}(m+1)) - V(\Theta^{\perp}(m)) | \Theta^{\perp}(m)\right] \\ \leq & \left(\lambda_{N-1}\left(E[\mathrm{e}^{-\frac{1}{k}(L_m^T+L_m)}]\right) - 1\right) \\ & \cdot\Theta^{\perp}(m)^T\Theta^{\perp}(m). \end{aligned} \tag{20}$$

Let $R(N,k)$ be the cardinality of the set of k-nearest neighbor digraphs on N vertices, $L^{(i)}$ be the Laplacian matrix associated with the i-th graph in this set, and $p^{(i)}$ be the probability that the i-th graph appears in the $G(N,k)$ model. Using the Courant–Fischer theorem again gives

$$\begin{aligned} & \lambda_{N-1}\left(E[\mathrm{e}^{-\frac{1}{k}(L_m^T+L_m)}]\right) \\ = & \max_{\substack{\|\Theta\|=1 \\ \Theta\perp\mathbf{1}}} \sum_{i=1}^{R(N,k)} p^{(i)}\Theta^T\mathrm{e}^{-\frac{1}{k}(L^{(i)T}+L^{(i)})}\Theta \\ \leq & E\left[\mathrm{e}^{-\frac{1}{k}\lambda_2(L_m^T+L_m)}\right]. \end{aligned} \tag{21}$$

If the disoriented version of G_m is connected, we have $0 < \mathrm{e}^{-\frac{1}{k}\lambda_2(L_m^T+L_m)} < 1$; otherwise, we have $\mathrm{e}^{-\frac{1}{k}\lambda_2(L_m^T+L_m)} = 1$. Hence

$$0 < E\left[\mathrm{e}^{-\frac{1}{k}\lambda_2(L_m^T+L_m)}\right] < 1. \tag{22}$$

Combining (20) and (21), we get

$$\begin{aligned} & E\left[V(\Theta^{\perp}(m+1)) - V(\Theta^{\perp}(m)) | \Theta^{\perp}(m)\right] \\ \leq & \left(E\left[\mathrm{e}^{-\frac{1}{k}\lambda_2(L_m^T+L_m)}\right] - 1\right)\|\Theta^{\perp}(m)\|^2. \end{aligned} \tag{23}$$

By using the Markov inequality and (23), we obtain for any $\varepsilon > 0$,

$$\begin{aligned} & P\left(\sup_{m\geq M} \Theta^{\perp}(m)^T\Theta^{\perp}(m) \geq \varepsilon\right) \\ \leq & \frac{\left(E\left[\mathrm{e}^{-\frac{1}{k}\lambda_2(L_m^T+L_m)}\right]\right)^M}{\varepsilon}\Theta^{\perp}(0)^T\Theta^{\perp}(0). \end{aligned} \tag{24}$$

In view of (22) and (24), we obtain that the rate of convergence is dictated by the quantity $E\left[\mathrm{e}^{-\frac{1}{k}\lambda_2(L^T+L)}\right]$.

When G_m is not balanced, $\mathbf{1}$ is not the eigenvector corresponding to the largest eigenvalue 1 of the matrix exponential $\mathrm{e}^{-\frac{1}{k}(L_m^T+L_m)}$. Similarly as in (20), by applying the Courant–Fischer theorem to (18) we now get

$$\begin{aligned} & E\left[V(\Theta^{\perp}(m+1)) - V(\Theta^{\perp}(m)) | \Theta^{\perp}(m)\right] \\ \leq & \left(\lambda_N\left(E[\mathrm{e}^{-\frac{1}{k}(L_m^T+L_m)}]\right) - E\left\|\mathrm{e}^{-\frac{1}{k}L_m^T}\mathbf{1}\right\|^2 - 1\right) \\ & \cdot\Theta^{\perp}(m)^T\Theta^{\perp}(m). \end{aligned} \tag{25}$$

Arguing as in (21) and using the interlacing theorem [8], we have

$$\begin{aligned} 0 &< \lambda_N \left(E[\mathrm{e}^{-\frac{1}{k}(L_m^T + L_m)}] \right) \\ &= \max_{\|\Theta\|=1} \sum_{i=1}^{R(N,k)} p^{(i)} \Theta^T \mathrm{e}^{-\frac{1}{k}(L^{(i)T} + L^{(i)})} \Theta \\ &\le E\left[\mathrm{e}^{-\frac{1}{k}\lambda_1(L_m^T + L_m)}\right] \\ &\le E\left[\mathrm{e}^{-\frac{1}{k}\mu_1(L_m^T + L_m)}\right] = 1, \end{aligned} \tag{26}$$

where $\mu_1(L_m^T + L_m)$ represents the smallest eigenvalue of the Laplacian matrix for the disoriented version of $\hat{G}_m$, and $\hat{G}_m$ is a balanced induced subgraph of G_m.

On the other hand, by directly expanding the matrix exponential and noting that $N > k$, we have $0 < \left\| \mathrm{e}^{-\frac{1}{k}L_m^T}\mathbf{1} \right\|^2 < 1$. Therefore, arguing as in (24) we obtain for any $\varepsilon > 0$,

$$\begin{aligned} &P\left(\sup_{m \ge M} \Theta^\perp(m)^T \Theta^\perp(m) \ge \varepsilon \right) \\ &\le \frac{\left(1 - E\left\| \mathrm{e}^{-\frac{1}{k}L_m^T}\mathbf{1} \right\|^2\right)^M}{\varepsilon} \Theta^\perp(0)^T \Theta^\perp(0). \end{aligned} \tag{27}$$

Combining (24) and (27), we conclude that the rate of convergence to consensus is dictated by the quantity $\nu = \max\left\{ E\left[\mathrm{e}^{-\frac{1}{k}\lambda_2(L^T + L)}\right], 1 - E\left\| \mathrm{e}^{-\frac{1}{k}L^T}\mathbf{1} \right\|^2 \right\}$. □

Proof for Corollary B. It suffices to show that $\nu(k) > \nu(k+1)$ for all k. For $i \ge 0$ and $k < N$, a straightforward calculation shows that

$$\left(-\frac{1}{k}L^T\right)^i \mathbf{1} \prec \left(-\frac{1}{k+1}L^T\right)^i \mathbf{1}, \tag{28}$$

where $\prec$ means "entry-wise less than". Hence, we obtain

$$1 - E\left\| \mathrm{e}^{-\frac{1}{k}L^T}\mathbf{1} \right\|^2 > 1 - E\left\| \mathrm{e}^{-\frac{1}{k+1}L^T}\mathbf{1} \right\|^2. \tag{29}$$

We rewrite the graph Laplacian L as $L^{k,N}$ to stress the dependence on both k and N. We claim that for all $N \ge 3$, the following holds:

$$\frac{k+1}{k} < \frac{\lambda_2\left((L^{k+1,N})^T + L^{k+1,N}\right)}{\lambda_2\left((L^{k,N})^T + L^{k,N}\right)}. \tag{30}$$

From Eq. (30), we know that

$$\begin{aligned} &E\left[\mathrm{e}^{-\frac{1}{k}\lambda_2\left((L^{k,N})^T + L^{k,N}\right)}\right] \\ &\qquad > E\left[\mathrm{e}^{-\frac{1}{k+1}\lambda_2\left((L^{k+1,N})^T + L^{k+1,N}\right)}\right], \end{aligned} \tag{31}$$

which together with Eq. (29) yields the desired result, namely $\nu(k) > \nu(k+1)$.

It remains to show the claim (30). This can be shown by induction on N. Suppose Eq. (30) holds for N, and we will show it holds for $N+1$ (the fact that it holds for any finite N can be very easily be tested numerically as is shown below). Since the second smallest Laplacian eigenvalue of a star graph S is zero, we obtain

$$\lambda_2\left((L^{k+1,N})^T + L^{k+1,N}\right) \le \lambda_2\left((L^{k+1,N+1})^T + L^{k+1,N}\right),$$

by working analogously to Property 3.1 in [9]. Similarly, we obtain

$$\lambda_2\left((L^{k,N})^T + L^{k,N}\right) \ge \lambda_2\left((L^{k,N+1})^T + L^{k,N}\right) - 1/n,$$

by using an argument similar to the one used in Property 3.3 [9]. Therefore, by using Eq. (30) we have

$$\frac{k+1}{k} < \frac{\lambda_2\left((L^{k+1,N})^T + L^{k+1,N}\right)}{\lambda_2\left((L^{k,N})^T + L^{k,N}\right)} \leq \frac{\lambda_2\left((L^{k+1,N+1})^T + L^{k+1,N+1}\right)}{\lambda_2\left((L^{k,N+1})^T + L^{k,N+1}\right) - \frac{1}{N}}. \tag{32}$$

When N is large enough, we get from Eq. (32)

$$\frac{k+1}{k} < \frac{\lambda_2\left((L^{k+1,N+1})^T + L^{k+1,N+1}\right)}{\lambda_2\left((L^{k,N+1})^T + L^{k,N+1}\right)}, \tag{33}$$

which concludes the induction step. □

Continuous-time Markovian Process.

Formally, the random network $G(t)$ in (1) switches among m topologies $G_1, \cdots, G_m$ in $G(N,k)$, and $G(t) = G_i$ if and only if the switching signal $s(t) = i \in \mathcal{M} := \{1, \cdots, m\}$. The random process $\{s(t), t \geq 0\}$ is ruled by a Markov process with state space $\mathcal{M}$ and infinitesimal generator $\Gamma = (\gamma_{ij})$ given by

$$\mathbb{P}(s(t+h) = j | s(t) = i) = \begin{cases} \gamma_{ij}h + o(h), & \text{when } s(t) \text{ jumps from } i \text{ to } j, \\ 1 + \gamma_{ii}h + o(h), & \text{otherwise.} \end{cases}$$

Here, $\mathbb{P}$ is the probability measure of interest, γ_{ij} is the transition rate from state i to state j with $\gamma_{ij} \geq 0$ if $i \neq j$, $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$, and $o(h)$ represents an infinitesimal of higher order than h. For practical implementation, we may set γ_{ij} large (thus more likely) if G_i and G_j differ only locally, while set γ_{ij} small (thus less likely) if G_i and G_j widely differ.

[1] Ren, W. & Beard, R. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Trans. Autom. Control* **50**, 655–661 (2005).

[2] Olfati-Saber, R. & Murray, R. M. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Autom. Control* **49**, 1520–1533 (2004).

[3] Bertsekas, D. & Tsitsiklis, J. *Parallel and Distributed Computation* (Prentice-Hall, 1989).

[4] Jadbabaie, A., Lin, J. & Morse, A. S. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. Autom. Contr.* **48**, 988–1001 (2003).

[5] Seneta, E. *Non-negative Matrices and Markov Chains* (Springer, 2006).

[6] Agaev, R. & Chebotarev, P. On the spectra of nonsymmetric Laplacian matrices. *Linear Algebra Appl.* **399**, 157–168 (2005).

[7] Meyer, C. *Matrix Analysis and Applied Linear Algebra* (SIAM, 2001).

[8] Wu, B., Shao, J. & Yuan, X. Deleting vertices and interlacing Laplacian eigenvalues. *Chinese Ann. Math. B* **31**, 231–239 (2010).

[9] Fiedler, M. Algebraic connectivity of graphs. *Czech. Math. J.* **23**, 298–305 (1973).