File S1

Proof of Result 1

1. When $y = x$, the mean fitnesses in the two demes E_x and E_y are equal:

$$w_x = 1 + sx = 1 + sy = w_y, \tag{S1.1}$$

and, from (8), the equilibrium equation is, (with $\mu = \mu_B$),

$$(1+sx)x = (1-m)\big[(1-\mu)(1+s)x+\mu(1-x)\big]+m\big[(1-\mu)(1-x)+\mu(1+s)x\big]. \tag{S1.2}$$

Thus

$$(1+sx)x = (1+s)x\big[(1-m)(1-\mu)+m\mu\big]+(1-x)\big[\mu(1-m)+m(1-\mu)\big], \tag{S1.3}$$

or

$$x + sx^2 = (1+s)x\big[1-m-\mu+2m\mu\big]+(1-x)\big[m+\mu-2m\mu\big]. \tag{S1.4}$$

This is equivalent to

$$Q(x) = sx^2 + \big[(s+2)(m+\mu-2m\mu)-s\big]x-(m+\mu-2m\mu) = 0. \tag{S1.5}$$

Now, as $0 < m,\ \mu < 1$, we have

$$(m+\mu-2m\mu) = m(1-\mu)+\mu(1-m) > 0. \tag{S1.6}$$

Therefore

$$Q(0) = -(m+\mu-2m\mu) < 0 \tag{S1.7}$$

and

$$Q(1) = (s+1)(m+\mu-2m\mu) > 0. \tag{S1.8}$$

As $Q(\pm\infty) > 0$, we conclude that the equation (S1.5) has a unique root x^* with $0 < x^* < 1$. Thus there is a unique symmetric polymorphism $(\mathbf{x}^*, \mathbf{y}^*)$, given by (13).

2. Near the equilibrium $(\mathbf{x}^*, \mathbf{y}^*)$, on the boundary where only B is present, $(z - x)$ is small, and from (10), the internal local stability of $(\mathbf{x}^*, \mathbf{y}^*)$ in the boundary is determined by the factor

$$C^* = \frac{(1-2\mu)(1+s)}{(1+sx^*)(1+sz^*)}. \tag{S1.9}$$

As $x^* = z^*$, $C^* < 1$ if $(1+s) < (1+sx^*)^2$, and as $s > 0$ this is true if $s(x^*)^2 + 2x^* > 1$. From the equilibrium equation (S1.5), as $Q(x^*) = 0$ we have

$$\begin{aligned} s(x^*)^2 + 2x^* &= -\big[(s+2)(m+\mu-2m\mu) - s\big]x^* + (m+\mu-2m\mu) + 2x^* \\ &= -(s+2)(m+\mu-2m\mu-1)x^* + (m+\mu-2m\mu). \end{aligned} \tag{S1.10}$$

Thus $s(x^*)^2 + 2x^* > 1$ if and only if

$$(s+2)(1-m-\mu+2m\mu)x^* > (1-m-\mu+2m\mu). \tag{S1.11}$$

But $(1-m-\mu+2m\mu) = (1-m)(1-\mu) + m\mu > 0$ as $0 < m,\ \mu < 1$, and so $C^* < 1$ provided $\mathbf{x}^* > \frac{1}{s+2}$. As $Q(1) > 0$ and $Q(x^*) = 0$, it is sufficient to show that $Q\big(\frac{1}{s+2}\big) < 0$. Indeed

$$\begin{aligned} Q\left(\frac{1}{s+2}\right) &= \frac{s}{(s+2)^2} + \big[(s+2)(m+\mu-2m\mu) - s\big]\frac{1}{s+2} - (m+\mu-2m\mu) \\ &= \frac{s}{(s+2)^2} - \frac{s}{s+2} = -\frac{s(s+1)}{(s+2)^2} < 0. \end{aligned} \tag{S1.12}$$

3. We compute $Q\big(\frac{1}{2}\big)$ using (14),

$$Q\left(\frac{1}{2}\right) = \frac{s}{4} + \frac{1}{2}\big[(s+2)(m+\mu-2m\mu) - s\big] - (m+\mu-2m\mu). \tag{S1.13}$$

In fact,

$$Q\left(\frac{1}{2}\right) = -\frac{s}{4}\big[1 - 2(m+\mu-2m\mu)\big]. \tag{S1.14}$$

But $1 - 2(m+\mu-2m\mu) = (1-2m)(1-2\mu) > 0$ when $0 < m,\ \mu < \frac{1}{2}$, in which case $Q\big(\frac{1}{2}\big) < 0$ and $x^* > \frac{1}{2}$ as $Q(1) > 0$.

File S2

Proof of Result 2

If an asymmetric polymorphism exists, then (11) holds, namely, (with $\mu = \mu_B$),

$$1 + sy = \frac{(1-2\mu)(1+s)}{1+sx}. \tag{S2.1}$$

That is,

$$y = \frac{s(1-x) - 2\mu(1+s)}{s(1+sx)}, \qquad 1 - y = \frac{s(1+s)x + 2\mu(1+s)}{s(1+sx)}. \tag{S2.2}$$

Substituting these relations into the equilibrium equation for x from (8), we find, after some simplification, that

$$x = \frac{1-m}{1+sx}\big[(1-\mu)(1+s)x + \mu(1-x)\big] + \frac{m}{s}(sx + 2\mu + \mu s). \tag{S2.3}$$

Equation (S2.3) is equivalent to the quadratic equation

$$T(x) = (1-m)s^2x^2 - sx\big[s(1-m) - \mu(s+2)(1-2m)\big] - \mu(2m+s) = 0. \tag{S2.4}$$

As μ, m, s are positive and $m < 1$, we have $T(0) < 0$ and $T(\pm\infty) > 0$, implying that $T(x)$ has two real roots, one positive and one negative. Now

$$\begin{aligned} T(1) &= (1-m)s^2 - s\big[s(1-m) - \mu(s+2)(1-2\mu)\big] - \mu(2m+s) \\ &= \mu\big[s(s+2)(1-2m) - (2m+s)\big]. \end{aligned} \tag{S2.5}$$

$T(1;m)$ is a linear function of m and

$$\begin{aligned} &T(1;0) = \mu s(s+1) > 0 \\ &T(1;\tfrac{1}{2}) = -\mu(2m+s) < 0 \\ &T(1;m_0) = 0. \end{aligned} \tag{S2.6}$$

Hence if $0 < m < m_0$, $T(1;m) > 0$ and a unique $0 < \hat{x} < 1$ exists such that $T(\hat{x}) = 0$. In order for $\hat{x}$ to be an equilibrium, its corresponding $\hat{y}$ should satisfy $0 < \hat{y} < 1$, where

$$1 - \hat{y} = \frac{1+s}{1+s\hat{x}}\frac{s\hat{x}+2\mu}{s} \tag{S2.7}$$

and $0 < \hat{y} < 1$ if and only if

$$(1+s)(s\hat{x}+2\mu) < s(1+s\hat{x}) \tag{S2.8}$$

or

$$\hat{x} < \frac{s - 2\mu(1+s)}{s}. \tag{S2.9}$$

So $0 < \hat{x} < 1$ if $0 < \mu < \mu_0 = \frac{1}{2}\frac{s}{s+1}$, and $\big[s - 2\mu(1+s)\big] > 0$. We compute $T\big(\frac{s-2\mu(1+s)}{s}\big)$, which equals

$$(1-m)\big[s-2\mu(1+s)\big]^2 - \big[s-2\mu(1+s)\big]\big[s(1-m)-\mu(s+2)(1-2m)\big] - \mu(2m+s). \tag{S2.10}$$

So

$$T\big(\tfrac{s-2\mu(1+s)}{s}\big) = 2\mu^2(1+s)(s+2m) + s\mu(s+2)(1-2m) - \mu(2m+s). \tag{S2.11}$$

But when $0 < m < m_0$,

$$T(1) = s\mu(s+2)(1-2m) - \mu(2m+s) > 0, \tag{S2.12}$$

therefore $T\big(\frac{s-2\mu(1+s)}{s}\big) > 0$, and (S2.9) holds.

File S3

Proof of Result 3

The asymmetric equilibrium $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is determined by the solutions of the quadratic equation

$$T(x;\mu_B) = (1-m)s^2x^2 - sx\big[s(1-m) - \mu_B(s+2)(1-2m)\big] - \mu_B(2m+s) = 0. \quad (S3.1)$$

When $\mu_B = 0$, equation (S3.1) reduces to

$$T(x;0) = -(1-m)s^2x(1-x) = 0, \quad (S3.2)$$

giving the two solutions $\hat{x} = 0$ and $\hat{x} = 1$. As $T(0;\mu_B) < 0$, when $\mu_B > 0$ the solution $x = 0$ shifts to a negative solution of (S3.1). Hence, when μ_B is positive and small, the positive root $\hat{x}(\mu_B)$ of $T(x;\mu_B) = 0$ is close to $x = 1$. That is, when μ_B is small the corresponding asymmetric equilibrium is close to the fixation of AB where $\hat{x} = \hat{y} = 1$. Moreover, by continuity, if μ_B is small, their stability is the same. Near fixation of AB, $w = 1-x$ and $z = 1-y$ are small, and up to non-linear terms, when $\mu_B = 0$, we have

$$\begin{aligned} w' &= \frac{1-m}{1+s}w + m(s+1)z \\ z' &= \frac{m}{1+s}w + (1-m)(s+1)z. \end{aligned} \quad (S3.3)$$

The characteristic polynomial $P(\lambda)$ of (S3.3) is

$$P(\lambda) = \lambda^2 - (1-m)\left[(1+s) + \frac{1}{1+s}\right]\lambda + (1-2m) \quad (S3.4)$$

and

$$P(1) = 1 - (1-m)\frac{(1+s)^2+1}{1+s} + 1 - 2m. \quad (S3.5)$$

In fact, it can be easily seen that

$$(1+s)P(1) = -s^2(1-m). \quad (S3.6)$$

As $P(+\infty) > 0$ and $P(1) < 0$, since $s > 0$, $0 < m < 1$, $P(\lambda)$ has a root larger than 1. Thus, when μ_B is small, fixation in AB is internally locally unstable and so is the asymmetric equilibrium when μ_B is small.

File S4

Proof of Result 4

A straightforward computation shows that the 4×4 matrix $\mathbf{L}_{\text{ex}}$ can be written as

$$\mathbf{L}_{\text{ex}} = \begin{pmatrix} (1-m)A & (1-m)B & mC & mD \\ (1-m)D & (1-m)C & mB & mA \\ mA & mB & (1-m)C & (1-m)D \\ mD & mC & (1-m)B & (1-m)A \end{pmatrix}, \qquad (S4.1)$$

where

$$\begin{aligned} (1+sx^*)A &= (1+s)(1-\mu_b) + r(1-x^*)\big[(s+2)\mu_b - (s+1)\big] \\ (1+sx^*)B &= (1+s)rx^* + \mu_b\big[1-(s+2)rx^*\big] \\ (1+sx^*)C &= (1-\mu_b) + rx^*\big[(s+2)\mu_b - 1\big] \\ (1+sx^*)D &= (1+s)\mu_b + r(1-x^*)\big[1-(s+2)\mu_b\big]. \end{aligned} \qquad (S4.2)$$

Observe that "formally" A, B, C, D are linear in μ_b. Let A_0 be the value of A when $\mu_b = 0$ and A_1 be its value when $\mu_b = 1$. Similarly we have $B_0, B_1, C_0, C_1, D_0, D_1$. In fact,

$$\begin{aligned} (1+sx^*)A_0 &= (1+s)\big[1-r(1-x^*)\big] \\ (1+sx^*)A_1 &= r(1-x^*) \\ (1+sx^*)B_0 &= (1+s)rx^* \\ (1+sx^*)B_1 &= 1-rx^* \\ (1+sx^*)C_0 &= 1-rx^* \\ (1+sx^*)C_1 &= (1+s)rx^* \\ (1+sx^*)D_0 &= r(1-x^*) \\ (1+sx^*)D_1 &= (1+s)\big[1-r(1-x^*)\big]. \end{aligned} \qquad (S4.3)$$

As $0 < r < 1$, $0 < x^* < 1$ we have A_i, B_i, C_i, D_i positive for $i = 0, 1$. Hence, as A, B, C, D are linear in μ_b, A, B, C, D are all positive for $0 < \mu_b < 1$. Moreover we have

$$C_0 = B_1, \qquad C_1 = B_0, \qquad D_0 = A_1, \qquad D_1 = A_0. \qquad (S4.4)$$

Let $S(\lambda) = \det(\mathbf{L}_{\text{ex}} - \lambda\mathbf{I})$ be the characteristic polynomial of $\mathbf{L}_{\text{ex}}$. The structure of $\mathbf{L}_{\text{ex}}$ given in (S4.1) entails that $S(\lambda)$ factors into the product of two quadratic polynomials $S_1(\lambda)$ and $S_2(\lambda)$:

$$S(\lambda) = S_1(\lambda)S_2(\lambda), \qquad (S4.5)$$

where

$$S_1(\lambda) = \lambda^2 - \lambda\big[(1-m)(A+C) + m(B+D)\big] + (1-2m)(AC-BD)$$
$$S_2(\lambda) = \lambda^2 - \lambda\big[(1-m)(A+C) - m(B+D)\big] + (1-2m)(AC-BD). \tag{S4.6}$$

See Balkau and Feldman (1973) for analogous calculations with migration modification. Consider first the roots of $S_1(\lambda) = 0$. These are real since the discriminant of $S_1(\lambda) = 0$ is

$$\big[(1-m)(A+C) + m(B+D)\big]^2 - 4(1-2m)(AC-BD) =$$
$$= \big[(1-m)(A-C) + m(B-D)\big]^2 + 4m(1-m)(AD+BC) + 4(1-m)^2BD + 4m^2AC, \tag{S4.7}$$

which is positive since A, B, C, D are positive and $0 < m < 1$.

In addition,

$$AC - BD = \big[(1-\mu_b)A_0 + \mu_b A_1\big]\big[(1-\mu_b)(C_0 + \mu_b C_1\big]$$
$$- \big[(1-\mu_b)B_0 + \mu_b B_1\big]\big[(1-\mu_b)D_0 + \mu_b D_1\big]. \tag{S4.8}$$

Since $C_0 = B_1$, $C_1 = B_0$, $D_0 = A_1$, $D_1 = A_0$, (S4.8) reduces to

$$AC - BD = (1-2\mu_b)(A_0B_1 - A_1B_0). \tag{S4.9}$$

Substituting A_0, A_1, B_0, B_1 from (S4.3) we have

$$(1+sx^*)^2(AC-BD) = (1-2\mu_b)(1+s)(1-r). \tag{S4.10}$$

Since we assume $0 < m$, $\mu_b < \frac{1}{2}$, the two roots of $S_1(\lambda) = 0$ are positive. Both of these roots are less than 1 if and only if $S_1(1) > 0$ and $S_1'(1) > 0$.

$$S_1'(1) = 2 - \big[(1-m)(A+C) + m(B+D)\big]. \tag{S4.11}$$

As $C_0 = B_1, C_1 = B_0, D_0 = A_1, D_1 = A_0$, we have

$$(A+C) = (1-\mu_b)(A_0 + C_0) + \mu_b(A_1 + C_1)$$
$$= (1-\mu_b)(A_0 + C_0) + \mu_b(B_0 + D_0), \tag{S4.12}$$

$$(B+D) = (1-\mu_b)(B_0 + D_0) + \mu_b(B_1 + D_1)$$
$$= (1-\mu_b)(B_0 + D_0) + \mu_b(A_0 + C_0). \tag{S4.13}$$

Hence,

$$(1-m)(A+C)+m(B+D)=(1-m_b)(A_0+C_0)+m_b(B_0+D_0), \tag{S4.14}$$

where

$$m_b = m+\mu_b-2m\mu_b. \tag{S4.15}$$

Substituting for A_0, B_0, C_0, D_0, gives

$$S_1'(1)=(1+sx^*)^{-1}\big[r+rsx^*+s(1-r)(2x^*-1)+m_b(1-r)(s+2)\big]. \tag{S4.16}$$

Now $s>0$, $0<r<1$, $m_b=m(1-\mu_b)+\mu_b(1-m)>0$, and $x^*>\frac{1}{2}$ if $0<m$, $\mu_B<\frac{1}{2}$. Therefore $S'(1)>0$ provided $0<m$, $\mu_B<\frac{1}{2}$. Using (S4.11) and (S4.16) it is easily seen that $S(1)>0$ if

$$(1+sx^*)^{-2}(1-r)\left\{(x^*)^2s^2+sx^*\big[-s+m_b(s+2)\big]-sm_b\right\}>0. \tag{S4.17}$$

Using the equation $Q(x^*)=0$ from (14), we have

$$s(x^*)^2+\big[(s+2)m_B-s\big]x^*-m_B=0, \tag{S4.18}$$

where

$$m_B=m+\mu_B-2m\mu_B. \tag{S4.19}$$

Therefore (S4.17) is satisfied if and only if

$$(m_b-m_B)(1+sx^*)^{-2}(1-r)s\big[x^*(2+s)-1\big]>0. \tag{S4.20}$$

As $x^*>\frac{1}{2}$, by Result 1, and $0<m<\frac{1}{2}$, (S4.20) holds if and only if $m_b>m_B$, which is true if and only if $\mu_b>\mu_B$.

It is not obvious that the roots of $S_2(\lambda)=0$ are real. However, as the matrix $\mathbf{L}^*_{\text{ex}}$ is positive, the Perron-Frobenius theory ensures that its largest eigenvalue in magnitude is positive. Therefore we just have to ensure that when both eigenvalues are real and positive they are less than 1; when they are real, both are positive or both are negative since $(1-2m)(1-2\mu_b)(AC-BD)$ is positive for $0 < m,\ \mu_b < \frac{1}{2}$. The conditions for this are that both $S_2(1)$ and $S_2'(1)$ are positive. But

$$\begin{aligned} S_2(1) &= 1 - \big[(1-m)(A+C) - m(B+D)\big] + (1-2m)(AC-BD) \\ &> 1 - \big[(1-m)(A+C) + m(B+D)\big] + (1-2m)(AC-BD) = S_1(1), \end{aligned} \tag{S4.21}$$

and $S_1(1) > 0$ when $0 < m,\ \mu_b < \frac{1}{2}$ and $\mu_b > \mu_B$, so also $S_2(1) > 0$. Similarly

$$\begin{aligned} S_2'(1) &= 2 - \big[(1-m)(A+C) - m(B+D)\big] \\ &> 2 - \big[(1-m)(A+C) + m(B+D)\big] = S_1'(1). \end{aligned} \tag{S4.22}$$

Thus, when $S_1'(1) > 0$ also $S_2'(1) > 0$.

File S5

Proof of Result 5

At a symmetric equilibrium $y = x$, and also, by (32), $\tilde{y} = \tilde{x}$. Thus (30) and (31) imply that

$$\tilde{x} = \frac{\big[(s+1)(1-m_B) - m_B\big]x + m_B}{sx+1} \qquad (S5.1)$$

and

$$x = \frac{\big[(1-m_B) - m_B(s+1)\big]\tilde{x} + m_B(1+s)}{(1+s) - s\tilde{x}}. \qquad (S5.2)$$

Substituting (S5.1) into (S5.2) gives the quadratic equation

$$(s+2)m_B\left\{sx^2 + \big[2 - m_B(s+2)\big]x - (1-m_B)\right\} = 0. \qquad (S5.3)$$

As $0 < m, \mu_B < 1$, $s > 0$ and $m_B = m(1-\mu_B) + \mu_B(1-m) > 0$, x satisfies the equation $R(x) = 0$ with $R(x)$ given in (36). As $0 < m_B < 1$ we have $R(0) < 0$, and as $R(\pm\infty) > 0$, $R(x) = 0$ has two real roots, one positive and one negative. Observe that

$$R(1) = s + \big[2 - m_B(s+2)\big] - (1-m_B) = (1-m_B)(s+1) > 0 \qquad (S5.4)$$

and

$$R\left(\frac{1}{2}\right) = \frac{s}{4} + \frac{1}{2}\cdot\big[2 - m_B(s+2)\big] - (1-m_B) = \frac{s}{4}(1-2m)(1-2\mu_B) \qquad (S5.5)$$

as $1 - 2m_B = 1 - 2m - 2\mu_B + 4m\mu_B = (1-2m)(1-2\mu_B)$. Therefore when $0 < m, \mu_B < \frac{1}{2}$ we have $R(\frac{1}{2}) > 0$ and $0 < \bar{x} < \frac{1}{2}$.

File S6

Proof of Result 6

In view of (33), the symmetric equilibrium $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is internally stable if

$$\frac{(1-2\mu)^2(1+s)^2}{(1+s\bar{x})^2\big[1+s(1-\tilde{x})\big]^2} < 1, \tag{S6.1}$$

as $\bar{x} = \bar{y}$ and $\tilde{x} = \tilde{y}$, where, by (S5.1)

$$\tilde{x} = \frac{\big[(s+1)(1-m_B) - m_B\big]\bar{x} + m_B}{s\bar{x}+1}. \tag{S6.2}$$

Thus

$$1 + s(1-\tilde{x}) = (1+s) - s \cdot \frac{\big[(1+s) - m_B(2+s)\big]\bar{x} + m_B}{s\bar{x}+1}. \tag{S6.3}$$

Hence

$$(1+s\bar{x})\big[1+s(1-\tilde{x})\big] = (1+s)(1+s\bar{x}) - s\big[(1+s) - m_B(s+2)\big]\bar{x} - sm_B, \tag{S6.4}$$

or

$$(1+s\bar{x})\big[1+s(1-\tilde{x})\big] = (1+s) + sm_B\big[(s+2)\bar{x} - 1\big]. \tag{S6.5}$$

For condition (S6.1) to be satisfied, since $(1+s\bar{x})\big[1+s(1-\tilde{x})\big] > 0$, it is sufficient that

$$(1+s) + sm_B\big[(s+2)\bar{x} - 1\big] > (1+s), \tag{S6.6}$$

or that $\bar{x} > \frac{1}{s+2}$. But

$$R\left(\frac{1}{s+2}\right) = \frac{s}{(s+2)^2} + \big[2 - m_B(s+2)\big]\frac{1}{s+2} - (1-m_B), \tag{S6.7}$$

or

$$R\left(\frac{1}{s+2}\right) = \frac{s}{(s+2)^2} + \frac{2}{s+2} - 1 = -\frac{s(1+s)}{(s+2)^2} < 0. \tag{S6.8}$$

Thus $R(\frac{1}{s+2}) < 0$ and $R(1) > 0$, and so $\bar{x} > \frac{1}{s+2}$ as desired.

File S7
Proof of Result 7

As the transformation T of the population state is $T = T_2 \circ T_1$, where T_i corresponds to phase i, with selection of type i, for $i = 1, 2$, and as $\tilde{\mathbf{x}} = T_1\bar{\mathbf{x}}$, $\bar{\mathbf{x}} = T_2\tilde{\mathbf{x}}$, following the analysis for the case without cycles, the linear approximation matrix $\mathbf{L}_{\text{ex}}$ becomes

$$\mathbf{L}_{\text{ex}} = \mathbf{L}_{\text{ex}}^2 \cdot \mathbf{L}_{\text{ex}}^1, \tag{S7.1}$$

where, as in (S4.1) and (S4.2), we have

$$\mathbf{L}_{\text{ex}}^1 = \begin{pmatrix} (1-m)\bar{A} & (1-m)\bar{B} & m\bar{C} & m\bar{D} \\ (1-m)\bar{D} & (1-m)\bar{C} & m\bar{B} & m\bar{A} \\ m\bar{A} & m\bar{B} & (1-m)\bar{C} & (1-m)\bar{D} \\ m\bar{D} & m\bar{C} & (1-m)\bar{B} & (1-m)\bar{A} \end{pmatrix}, \tag{S7.2}$$

$$\mathbf{L}_{\text{ex}}^2 = \begin{pmatrix} (1-m)\widetilde{A} & (1-m)\widetilde{B} & m\widetilde{C} & m\widetilde{D} \\ (1-m)\widetilde{D} & (1-m)\widetilde{C} & m\widetilde{B} & m\widetilde{A} \\ m\widetilde{A} & m\widetilde{B} & (1-m)\widetilde{C} & (1-m)\widetilde{D} \\ m\widetilde{D} & m\widetilde{C} & (1-m)\widetilde{B} & (1-m)\widetilde{A} \end{pmatrix}, \tag{S7.3}$$

and

$$\begin{aligned} (1+s\bar{x})\overline{A} &= (1+s)(1-\mu_b) + r(1-\bar{x})\big[(s+2)\mu_b - (s+1)\big] \\ (1+s\bar{x})\overline{B} &= (1+s)r\bar{x} + \mu_b\big[1-(s+2)r\bar{x}\big] \\ (1+s\bar{x})\overline{C} &= (1-\mu_b) + r\bar{x}\big[(s+2)\mu_b - 1\big] \\ (1+s\bar{x})\overline{D} &= (1+s)\mu_b + r(1-\bar{x})\big[1-(s+2)\mu_b\big], \end{aligned} \tag{S7.4}$$

$$\begin{aligned} \big[1+s(1-\tilde{x})\big]\widetilde{A} &= (1-\mu_b) + r(1-\tilde{x})\big[(2+s)\mu_b - 1\big] \\ \big[1+s(1-\tilde{x})\big]\widetilde{B} &= (1+s)\mu_b + r\tilde{x}\big[1-(s+2)\mu_b\big] \\ \big[1+s(1-\tilde{x})\big]\widetilde{C} &= (1+s)(1-\mu_b) + r\tilde{x}\big[(s+2)\mu_b - (s+1)\big] \\ \big[1+s(1-\tilde{x})\big]\widetilde{D} &= \mu_b + r(1-\tilde{x})\big[(s+1) - (s+2)\mu_b\big]. \end{aligned} \tag{S7.5}$$

When we multiply $\mathbf{L}_{\text{ex}}^2$ by $\mathbf{L}_{\text{ex}}^1$ we find that the product $\mathbf{L}_{\text{ex}}$ has the following structure:

$$\mathbf{L}_{\text{ex}} = \begin{pmatrix} a & e & h & d \\ b & f & g & c \\ c & g & f & b \\ d & h & e & a \end{pmatrix}, \tag{S7.6}$$

where

$$\begin{aligned}
a &= (1-m)^2 \widetilde{A}\overline{A} + (1-m)^2 \widetilde{B}\overline{D} + m^2 \widetilde{C}\overline{A} + m^2 \widetilde{D}\overline{D} \\
b &= (1-m)^2 \widetilde{D}\overline{A} + (1-m)^2 \widetilde{C}\overline{D} + m^2 \widetilde{B}\overline{A} + m^2 \widetilde{A}\overline{D} \\
c &= m(1-m)\left[\widetilde{A}\overline{A} + \widetilde{B}\overline{D} + \widetilde{C}\overline{A} + \widetilde{D}\overline{D}\right] \\
d &= m(1-m)\left[\widetilde{D}\overline{A} + \widetilde{C}\overline{D} + \widetilde{B}\overline{A} + \widetilde{A}\overline{D}\right] \\
e &= (1-m)^2 \widetilde{A}\overline{B} + (1-m)^2 \widetilde{B}\overline{C} + m^2 \widetilde{C}\overline{B} + m^2 \widetilde{D}\overline{C} \\
f &= (1-m)^2 \widetilde{D}\overline{B} + (1-m)^2 \widetilde{C}\overline{C} + m^2 \widetilde{B}\overline{B} + m^2 \widetilde{A}\overline{C} \\
g &= m(1-m)\left[\widetilde{A}\overline{B} + \widetilde{B}\overline{C} + \widetilde{C}\overline{B} + \widetilde{D}\overline{C}\right] \\
h &= m(1-m)\left[\widetilde{D}\overline{B} + \widetilde{C}\overline{C} + \widetilde{B}\overline{B} + \widetilde{A}\overline{C}\right].
\end{aligned} \tag{S7.7}$$

Let $D(\lambda) = \det(\mathbf{L}_{\text{ex}} - \lambda\mathbf{I})$ be the characteristic polynomial of $\mathbf{L}_{\text{ex}}$. From (S7.6), $D(\lambda)$ factors into 2×2 determinants:

$$D(\lambda) = \begin{vmatrix} a+d-\lambda & e+h & 0 & 0 \\ b+c & f+g-\lambda & 0 & 0 \\ c & g & f-g-\lambda & b-c \\ d & h & e-h & a-d-\lambda \end{vmatrix}. \tag{S7.8}$$

Therefore $D(\lambda)$ can be written

$$D(\lambda) = D_1(\lambda)D_2(\lambda), \tag{S7.9}$$

where

$$\begin{aligned}
D_1(\lambda) &= \lambda^2 - (a+d+f+g)\lambda + (a+d)(f+g) - (b+c)(e+h) \\
D_2(\lambda) &= \lambda^2 - (a-d+f-g)\lambda + (a-d)(f-g) - (b-c)(e-h).
\end{aligned} \tag{S7.10}$$

As $0 < m < 1$ and $\overline{A}, \overline{B}, \overline{C}, \overline{D}$ and $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ are all positive, the matrix $\mathbf{L}_{\text{ex}}$ is a positive matrix and its largest eigenvalue in magnitude is positive. Observe that the discriminant of $D_1(\lambda)$ is

$$(a+d+f+g)^2 - 4\big[(a+d)(f+g) - (b+c)(e+h)\big], \tag{S7.11}$$

which is positive and equal to

$$[(a+d) - (f+g)]^2 + 4(b+c)(e+h). \tag{S7.12}$$

In addition, $(a+d+f+g)$ is positive. Therefore $D_1(\lambda)$ has real roots, and its largest root in magnitude is positive. Thus this positive root is less than 1 if $D_1(1) > 0$ and $D_1'(1) > 0$.

As the largest eigenvalue of $\mathbf{L}_{\text{ex}}$ is positive, for stability of $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ we require that if the eigenvalues associated with $D_2(\lambda)$ are real and at least one is positive, they are both less than 1. Again the conditions for this are $D_2(\lambda) > 0$ and $D_2'(1) > 0$. Observe that

$$\begin{aligned} D_1'(1) &= 2 - (a + d + f + g) \\ D_2'(1) &= 2 - (a - d + f - g) = D_1'(1) + 2(d + g) > D_1'(1). \end{aligned} \tag{S7.13}$$

In view of (S7.13), for the largest eigenvalue of $\mathbf{L}_{\text{ex}}$ to be less than one, we require

$$D_1(1) > 0, \qquad D_1'(1) > 0, \qquad D_2(1) > 0. \tag{S7.14}$$

We now compute the constant terms of $D_1(\lambda)$ and $D_2(\lambda)$. We already know, based on the properties of the matrices $\mathbf{L}_{\text{ex}}^1$ and $\mathbf{L}_{\text{ex}}^2$ that the constant terms of both $D_1(\lambda)$ and $D_2(\lambda)$ are the same and are equal to

$$(1 - 2m)^2 \left(\overline{A}\,\overline{C} - \overline{B}\,\overline{D}\right) \left(\widetilde{A}\widetilde{C} - \widetilde{B}\widetilde{D}\right). \tag{S7.15}$$

With the same technique used to compute (S4.10), we deduce that

$$(1 + s\overline{x})^2 \left(\overline{A}\,\overline{C} - \overline{B}\,\overline{D}\right) = (1 - 2\mu_b)\,(1 + s)(1 - r), \tag{S7.16}$$

and similarly

$$[1 + s\,(1 - \tilde{x})]^2 \left(\widetilde{A}\widetilde{C} - \widetilde{B}\widetilde{D}\right) = (1 - 2\mu_b)(1 + s)(1 - r). \tag{S7.17}$$

Therefore the constant terms of both $D_1(\lambda)$ and $D_2(\lambda)$ are the same and are equal to

$$(1 - 2m)^2\,(1 - 2\mu_b)^2\,(1 + s\overline{x})^{-2}\,[1 + s\,(1 - \tilde{x})]^{-2}\,(1 + s)^2\,(1 - r)^2\,, \tag{S7.18}$$

which is positive, and so $D_1(\lambda)$ has two positive roots. Also, as a, b, c, d are all positive,

$$\begin{aligned} D_2(1) &= 1 - (a - d + f - g) + (a - d)(f - g) - (b - c)(e - h) \\ &> 1 - (a + d + f + g) + (a + d)(f + g) - (b + c)(e + h) = D_1(1). \end{aligned} \tag{S7.19}$$

Hence for the symmetric equilibrium to be externally stable, we require that $D_1(1)$ and $D_1'(1)$ are both positive.

Now from (S6.5) we know that

$$(1+s\bar{x})\big[1+s(1-\tilde{x})\big] = (1+s) + sm_B\big[(s+2)\bar{x}-1\big]. \tag{S7.20}$$

As $\bar{x} > \frac{1}{s+2}$ we have

$$(1+s\bar{x})\left[1+s\left(1-\tilde{x}\right)\right] > (1+s). \tag{S7.21}$$

Thus the equal constant terms of $D_1(\lambda)$ and $D_2(\lambda)$ given in (S7.18) are positive and less than 1. As a result it is impossible for the two positive roots of $D_1(\lambda)$ to both be larger than 1, and they are both less than 1 provided $D_1(1) > 0$. Hence the external stability of the symmetric equilibrium requires that

$$D_1(1) = 1 - (a+d+f+g) + (a+d)(f+g) - (b+c)(e+h) \tag{S7.22}$$

is positive (the last summand in (S7.22) is given in (S7.18)). We now compute $a+d+f+g$.

Computation of $(a+d+f+g)$

We have

$$\begin{aligned}(a+d+f+g) = &\left[(1-m)^2\widetilde{A} + m^2\widetilde{C} + m(1-m)\left(\widetilde{B}+\widetilde{D}\right)\right]\overline{A}\\ &+\left[(1-m)^2\widetilde{C} + m^2\widetilde{A} + m(1-m)\left(\widetilde{B}+\widetilde{D}\right)\right]\overline{C}\\ &+\left[(1-m)^2\widetilde{B} + m^2\widetilde{D} + m(1-m)\left(\widetilde{A}+\widetilde{C}\right)\right]\overline{D}\\ &+\left[(1-m)^2\widetilde{D} + m^2\widetilde{B} + m(1-m)\left(\widetilde{A}+\widetilde{C}\right)\right]\overline{B}.\end{aligned} \tag{S7.23}$$

As $\overline{A}, \overline{B}, \overline{C}, \overline{D}$ and also $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ given in (S7.4) and (S7.5), respectively, are all linear functions of μ_b, where $0 \le \mu_b \le 1$, we can represent them as $\overline{A} = (1-\mu_b)\overline{A}_0 + \mu_b\overline{A}_1$, etc. Hence

$$\begin{aligned}(1+s\bar{x})\overline{A}_0 &= (1+s)\left[1-r\left(1-\bar{x}\right)\right] = (1+s\bar{x})\overline{D}_1\\ (1+s\bar{x})\overline{B}_0 &= (1+s)r\bar{x} = (1+s\bar{x})\overline{C}_1\\ (1+s\bar{x})\overline{C}_0 &= 1-r\bar{x} = (1+s\bar{x})\overline{B}_1\\ (1+s\bar{x})\overline{D}_0 &= r(1-\bar{x}) = (1+s\bar{x})\overline{A}_1,\end{aligned} \tag{S7.24}$$

$$\begin{aligned}\left[1+s\left(1-\tilde{x}\right)\right]\widetilde{A}_o &= 1-r\left(1-\tilde{x}\right) = \left[1+s\left(1-\tilde{x}\right)\right]\widetilde{D}_1\\ \left[1+s\left(1-\tilde{x}\right)\right]\widetilde{B}_o &= r\bar{x} = \left[1+s\left(1-\tilde{x}\right)\right]\widetilde{C}_1\\ \left[1+s\left(1-\tilde{x}\right)\right]\widetilde{C}_o &= (1+s)\left[1-r\tilde{x}\right] = \left[1+s\left(1-\tilde{x}\right)\right]\widetilde{B}_1\\ \left[1+s\left(1-\tilde{x}\right)\right]\widetilde{D}_o &= (1+s)r\left(1-\tilde{x}\right) = \left[1+s\left(1-\tilde{x}\right)\right]\widetilde{A}_1.\end{aligned} \tag{S7.25}$$

Since

$$
\begin{aligned}
m_b &= m + \mu_b - 2m\mu_b = m(1-\mu_b) + \mu_b(1-m) \\
1 - m_b &= 1 - m - \mu_b + 2m\mu_b = (1-m)(1-\mu_b),
\end{aligned} \tag{S7.26}
$$

we can write

$$
\begin{aligned}
(a+d+f+g) = &\left[(1-m)(1-m_b)\widetilde{A}_0 + (1-m)m_b\widetilde{D}_0 + m(1-m_b)\widetilde{B}_0 + m\cdot m_b\widetilde{C}_0\right]\overline{A} \\
&+\left[(1-m)(1-m_b)\widetilde{C}_0 + (1-m)m_b\widetilde{B}_0 + m(1-m_b)\widetilde{D}_0 + m\cdot m_b\widetilde{A}_0\right]\overline{C} \\
&+\left[(1-m)(1-m_b)\widetilde{B}_0 + (1-m)m_b\widetilde{C}_0 + m(1-m_b)\widetilde{A}_0 + m\cdot m_b\widetilde{D}_0\right]\overline{D} \\
&+\left[(1-m)(1-m_b)\widetilde{D}_0 + (1-m)m_b\widetilde{A}_0 + m(1-m_b)\widetilde{C}_0 + m\cdot m_b\widetilde{B}_0\right]\overline{B}.
\end{aligned} \tag{S7.27}
$$

Substitute into (S7.27)

$$
\begin{aligned}
\overline{A} &= (1-\mu_b)\overline{A}_0 + \mu_b\overline{D}_0, \qquad \overline{B} = (1-\mu_b)\overline{B}_0 + \mu_b\overline{C}_0, \\
\overline{C} &= (1-\mu_b)\overline{C}_0 + \mu_b\overline{B}_0, \qquad \overline{D} = (1-\mu_b)\overline{D}_0 + \mu_b\overline{A}_0,
\end{aligned} \tag{S7.38}
$$

to obtain

$$
\begin{aligned}
(a+d+f+g) = &(1-m_b)^2\left[\widetilde{A}_0\overline{A}_0 + \widetilde{B}_0\overline{D}_0 + \widetilde{C}_0\overline{C}_0 + \widetilde{D}_0\overline{B}_0\right] \\
&+ m_b(1-m_b)\left[\left(\widetilde{B}_0 + \widetilde{D}_0\right)\left(\overline{A}_0 + \overline{C}_0\right) + \left(\widetilde{A}_0 + \widetilde{C}_0\right)\left(\overline{B}_0 + \overline{D}_0\right)\right] \\
&+ m_b^2\left[\widetilde{A}_0\overline{C}_0 + \widetilde{B}_0\overline{B}_0 + \widetilde{C}_0\overline{A}_0 + \widetilde{D}_0\overline{D}_0\right].
\end{aligned} \tag{S7.29}
$$

Equation (S7.29) can also be written as

$$
\begin{aligned}
(a+d+f+g) = &(1-2m_b)\left[\widetilde{A}_0\overline{A}_0 + \widetilde{B}_0\overline{D}_0 + \widetilde{C}_0\overline{C}_0 + \widetilde{D}_0\overline{B}_0\right] \\
&+ m_b\left[\left(\widetilde{B}_0 + \widetilde{D}_0\right)\left(\overline{A}_0 + \overline{C}_0\right) + \left(\widetilde{A}_0 + \widetilde{C}_0\right)\left(\overline{B}_0 + \overline{D}_0\right)\right] \\
&+ m_b^2\left[\left(\widetilde{A}_0 + \widetilde{C}_0\right)\left(\overline{A}_0 + \overline{C}_0\right) + \left(\widetilde{B}_0 + \widetilde{D}_0\right)\left(\overline{B}_0 + \overline{D}_0\right)\right. \\
&\qquad \left. - \left(\widetilde{A}_0 + \widetilde{C}_0\right)\left(\overline{B}_0 + \overline{D}_0\right) + \left(\widetilde{B}_0 + \widetilde{D}_0\right)\left(\overline{A}_0 + \overline{C}_0\right)\right],
\end{aligned} \tag{S7.30}
$$

or as

$$
\begin{aligned}
(a+d+f+g) = &(1-2m_b)\left[\widetilde{A}_0\overline{A}_0 + \widetilde{B}_0\overline{D}_0 + \widetilde{C}_0\overline{C}_0 + \widetilde{D}_0\overline{B}_0\right] \\
&+ m_b\left[\left(\widetilde{B}_0 + \widetilde{D}_0\right)\left(\overline{A}_0 + \overline{C}_0\right) + \left(\widetilde{A}_0 + \widetilde{C}_0\right)\left(\overline{B}_0 + \overline{D}_0\right)\right] \\
&+ m_b^2\left(\widetilde{A}_0 + \widetilde{C}_0 - \widetilde{B}_0 - \widetilde{D}_0\right)\left(\overline{A}_0 + \overline{C}_0 - \overline{B}_0 - \overline{D}_0\right).
\end{aligned} \tag{S7.31}
$$

From (S7.24) and (S7.25),

$$(1+s\bar{x})[1+s(1-\tilde{x})]\left[\widetilde{A}_0\overline{A}_0+\widetilde{B}_0\overline{D}_0+\widetilde{C}_0\overline{C}_0+\widetilde{D}_0\overline{D}_0\right]= \\ =(1+s)\left[2(1-r)+r^2\right]+r^2s(s+1)\bar{x}-r^2s\tilde{x}-r^2s^2\bar{x}\tilde{x}, \tag{S7.32}$$

$$(1+s\bar{x})[1+s(1-\tilde{x})]\left[\left(\widetilde{B}_0+\widetilde{D}_0\right)\left(\overline{A}_0+\overline{C}_0\right)+\left(\widetilde{A}_0+\widetilde{C}_0\right)\left(\overline{B}_0+\overline{D}_0\right)\right]= \\ =2r^2(1+s\bar{x})[1+s(1-\bar{x})]+r(1-r)(s+2)\left[(s+2)+s\left(\bar{x}-\tilde{x}\right)\right], \tag{S7.33}$$

$$(1+s\bar{x})[1+s(1-\tilde{x})]\left(\widetilde{A}_0+\widetilde{C}_0-\widetilde{B}_0-\widetilde{D}_0\right)\left(\overline{A}_0+\overline{C}_0-\overline{B}_0-\overline{D}_0\right)= \\ =(s+2)^2(1-r)^2. \tag{S7.34}$$

Remember that by (S7.18)

$$(1+s\bar{x})^2\left[1+(1-\tilde{x})\right]^2\left[(a+d)(f+g)-(b+c)(e+h)\right]= \\ =(1-2m)^2(1-2\mu_b)^2(s+1)^2(1-r)^2. \tag{S7.35}$$

But

$$(1-2m)(1-2\mu_b)=1-2(m+\mu_b-2m\mu_b)=1-2m_b. \tag{S7.36}$$

Therefore

$$(1+s\bar{x})^2\left[1+(1-\tilde{x})\right]^2\left[(a+d)(f+g)-(b+c)(e+h)\right]= \\ =(1-2m_b)^2(s+1)^2(1-r)^2. \tag{S7.37}$$

Combining all of this, we get that $D_1(1)=1-(a+d+f+g)+(a+d)(f+g)-(b+c)(e+h)$, which we compute as

$$\begin{aligned}
&1-(1-2m_b)\frac{(1+s)\left[2(1-r)+r^2\right]+r^2s(s+1)\bar{x}-r^2s\tilde{x}-r^2s^2\bar{x}\tilde{x}}{(1+s\bar{x})[1+s(1-\tilde{x})]} \\
&\quad-m_b\frac{2r^2(1+s\bar{x})[1+s(1-\tilde{x})]+r(1-r)(s+2)[(s+2)+s(\bar{x}-\tilde{x})]}{(1+s\bar{x})[1+s(1-\tilde{x})]} \\
&\quad-m_b^2\frac{(s+2)^2(1-r)^2}{(1+s\bar{x})[1+s(1-\tilde{x})]} \\
&\quad+\frac{(1-2m_b)^2(s+1)^2(1-r)^2}{(1+s\bar{x})^2\left[1+s\left(1-\tilde{x}\right)\right]^2}.
\end{aligned} \tag{S7.38}$$

Observe that

$$r^2[(1+s)+s(s+1)\bar{x}-s\tilde{x}-s^2\bar{x}\tilde{x}]=r^2(1+s\bar{x})[1+s(1-\tilde{x})], \tag{S7.39}$$

so (S7.38) simplifies to

$$1 - r^2 - (1 - 2m_b)\frac{2(1+s)(1-r)}{(1+s\bar{x})[1+s(1-\tilde{x})]} - m_b\frac{r(1-r)(s+2)[(s+2)+s(\bar{x}-\tilde{x})]}{(1+s\bar{x})[1+s(1-\tilde{x})]}$$
$$- m_b^2\frac{(s+2)^2(1-r)^2}{(1+s\bar{x})[1+s(1-\tilde{x})]} + \frac{(1-2m_b)^2(s+1)^2(1-r)^2}{(1+s\bar{x})^2[1+s(1-\tilde{x})]^2}. \qquad (S7.40)$$

Clearly $D_1(1)$ of (S7.40) has a factor of $(1-r)$, and in fact

$$D_1(1) = (1-r)f(r), \qquad (S7.41)$$

where $f(r)$ is a linear function of r, for $0 \le r \le 1$. Now

$$f(1) = 2 - (1-2m_b)\frac{2(1+s)}{(1+s\bar{x})[1+s(1-x)]} - m_b\frac{(s+2)[(s+2)+s(\bar{x}-\tilde{x})]}{(1+s\bar{x})[1+s(1-\tilde{x})]}. \qquad (S7.42)$$

Following (S6.7) we have

$$(1+s\bar{x})[1+s(1-\tilde{x})] = (1+s) + sm_B[(s+2)\bar{x}-1]. \qquad (S7.43)$$

We also have an equivalent expression for (S7.43) in terms of $\tilde{x}$, namely

$$(1+s\bar{x})[1+s(1-\tilde{x})] = (1+s) + sm_B[(s+1)-(s+2)\tilde{x}]. \qquad (S7.44)$$

Also, whereas $\bar{x} > \frac{1}{s+2}$, we have $\tilde{x} < \frac{s+1}{s+2}$. Applying all of this to (S7.42) and using the fact that

$$(1+s\bar{x})[1+s(1-\tilde{x})] = (1+s) + \frac{1}{2}sm_B[s+(s+2)(\bar{x}-\tilde{x})], \qquad (S7.45)$$

we get that

$$\begin{aligned}(1+s\bar{x})[1+s(1-\tilde{x})]f(1) &= 2(s+1) + sm_B[s+(s+2)(\bar{x}-\tilde{x})] \\ &\quad - 2(1-2m_b)(s+1) - m_b(s+2)[(s+2)+s(\bar{x}-\tilde{x})] \\ &= s^2m_B - m_b\left[(s+2)^2 - 4(s+1)\right] + s(s+2)(m_B-m_b)(\bar{x}-\tilde{x}) \\ &= s^2(m_B-m_b) + s(s+2)(m_B-m_b)(\bar{x}-\tilde{x}).\end{aligned} \qquad (S7.46)$$

Thus

$$(1+s\bar{x})[1+s(1-\tilde{x})]f(1) = s(m_B-m_b)[s+(s+2)(\bar{x}-\tilde{x})]. \qquad (S7.47)$$

But as $(s+2)\bar{x} > 1$, $(s+2)\tilde{x} < (s+1)$,

$$s + (s+2)(\bar{x} - \tilde{x}) = [(s+2)\bar{x} - 1] + [(s+1) - (s+2)\tilde{x}] > 0. \tag{S7.48}$$

It follows that the sign of $f(1)$ is the same as the sign of $(m_B - m_b)$.

We now compute $f(0)$:

$$\begin{aligned} f(0) = 1 - (1-2m_b)\frac{2(1+s)}{(1+s\bar{x})[1+s(1-\tilde{x})]} - m_b^2\frac{(s+2)^2}{(1+s\bar{x})[1+s(1-\tilde{x})]} \\ + \frac{(1-2m_b)^2(s+1)^2}{(1+s\bar{x})^2[1+s(1-\tilde{x})]^2}. \end{aligned} \tag{S7.49}$$

Using the expression (S7.43) for the product of the two mean fitnesses, we get

$$\begin{aligned} (1+s\bar{x})^2[1+s(1-\tilde{x})]^2 f(0) = &\{(1+s) + sm_B[(s+2)\bar{x}-1]\}^2 \\ &- 2(1-2m_b)(s+1)\{(1+s) + sm_B[(s+2)\bar{x}-1]\} \\ &- m_b^2(s+2)\{(1+s) + sm_B[(s+2)\bar{x}-1]\} \\ &+ (1-2m_b)^2(s+1)^2. \end{aligned} \tag{S7.50}$$

In (S7.50) we replace the $\bar{x}^2$ term using the equilibrium equation (36) to give

$$\begin{aligned} (1+s\bar{x})^2[1+s(1-\tilde{x})]^2 f(0) = (m_B - m_b)s\Big\{&m_b s(s+1) \\ &- m_B^2(s+2)^2 + m_B\Big[(s+1)(s+4) - m_b(s+2)^2\Big] \\ &+ m_B(s+2)\bar{x}\Big[m_B(s+2)^2 + m_b(s+2)^2 - 4(s+1)\Big]\Big\}. \end{aligned} \tag{S7.51}$$

The right-hand side of (S7.51) is $(m_B - m_b)s$ multiplied by

$$m_b s(s+1) + m_B(s+2)^2(m_B+m_b)[\bar{x}(s+2)-1] + m_B(s+1)[(s+4) - 4\bar{x}(s+2)]. \tag{S7.52}$$

We will show that (S7.52) is always positive. In fact, (S7.52) is equal to

$$\begin{aligned} m_b s(s+1) + m_B \cdot m_b(s+2)^2\big[\bar{x}(s+2)-1\big] + m_B^2(s+2)^2\big[\bar{x}(s+2)-1\big] \\ + m_B(s+1)\big[(s+4) - 4\bar{x}(s+2)\big]. \end{aligned} \tag{S7.53}$$

From the equilibrium equation (36) we get that

$$m_B\big[(s+2)\bar{x} - 1\big] = s\bar{x}^2 + 2\bar{x} - 1. \tag{S7.54}$$

Hence (S7.53) is equal to

$$m_b s(s+1) + m_B \cdot m_b\,(s+2)^2\left[\bar{x}(s+2)-1\right]+ \\ + m_B\,(s+2)^2\left[s\bar{x}^2+2\bar{x}-1\right] + m_B(s+1)\big[(s+4)-4\bar{x}(s+2)\big]. \tag{S7.55}$$

The last two terms have a factor m_B that multiplies

$$\begin{aligned}(s+1)(s+4)-(s+2)^2+(s+2)^2\,\bar{x}(2+s\bar{x})-4\bar{x}(s+1)(s+2) &= \\ = s+(s+2)\bar{x}\big[(s+2)(2+s\bar{x})-4(s+1)\big] & \\ = s+(s+2)\bar{x}\big[(s+2)s\bar{x}-2s\big] & \\ = s\left[(s+2)^2\,\bar{x}^2-2(s+2)\bar{x}+1\right] = s\big[(s+2)\bar{x}-1\big]^2, &\end{aligned} \tag{S7.56}$$

which is positive. To sum up, $f(0)$ also has the same sign of $(m_B - m_b)$, and so

$$D_1(1) = (1-r)s(m_B - m_b)\Delta(r), \tag{S7.57}$$

where $\Delta(r)$ is a linear function of r that is positive for all $0 \le r \le 1$. As $(m_B - m_b) = (1-2m)(\mu_B - \mu_b)$, this proves the following result.

File S8

Proof of Result 8

i. In a constant environment, the mean fitness w^* at the symmetric equilibrium $(\bar{\mathbf{x}}^*, \bar{\mathbf{y}}^*)$ is $w^* = 1+sx^*$, and it is a decreasing function of μ_B if $\frac{\partial x^*}{\partial \mu_B}$ is negative, or equivalently if $\frac{\partial x^*}{\partial m_B}$ is negative (since $m_B = m+\mu_B(1-2m)$ and $0 \leq m < \frac{1}{2}$). Using the equilibrium equation (14),

$$\frac{\partial x^*}{\partial m_B} = \frac{1-(s+2)x^*}{2sx^* + \big[(s+2)m_B - s\big]}. \tag{S8.1}$$

As $x^* > \frac{1}{s+2}$, in order for $\frac{\partial x^*}{\partial m_B}$ to be negative, it is sufficient that

$$x^* > \frac{s-m_B(s+2)}{2s}. \tag{S8.2}$$

This follows easily from the fact that $Q(x)$ of (14) satisfies $Q(0) < 0$, $Q(x^*) = 0$, and $Q\left(\frac{s-m_B(s+2)}{2s}\right) < 0$.

ii. With a fitness cycle of period 2, the mean fitness $\bar{w}$ at the symmetric equilibrium $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is

$$\bar{w} = (1+s) + sm_B\big[(s+2)\bar{x} - 1\big]. \tag{S8.3}$$

$\bar{w}$ is an increasing function of μ_B if $\frac{\partial \bar{w}}{\partial \mu_B} > 0$ or equivalently if $\frac{\partial \bar{w}}{\partial m_B} > 0$. Now

$$\frac{\partial \bar{w}}{\partial m_B} = s\big[(s+2)\bar{x} - 1\big] + s(s+2)m_B\frac{\partial \bar{x}}{\partial m_B}. \tag{S8.4}$$

Thus $\frac{\partial \bar{w}}{\partial m_B} > 0$ provided $\frac{\partial \bar{x}}{\partial m_B} > 0$. Using the equilibrium equation $R(x) = 0$ for $\bar{x}$, we have

$$\frac{\partial \bar{x}}{\partial m_B} = \frac{\bar{x}(s+1)-1}{2s\bar{x} + \big[2 - m_B(s+2)\big]}. \tag{S8.5}$$

Since $\big[\bar{x}(s+1)-1\big] > 0$, we conclude that $\frac{\partial \bar{x}}{\partial m_B} > 0$ if

$$\bar{x} > \frac{m_B(s+2)-2}{2s}, \tag{S8.6}$$

which follows from $R(0) < 0$, $R(\bar{x}) = 0$, and $R\left(\frac{m_B(s+2)-2}{2s}\right) < 0$.

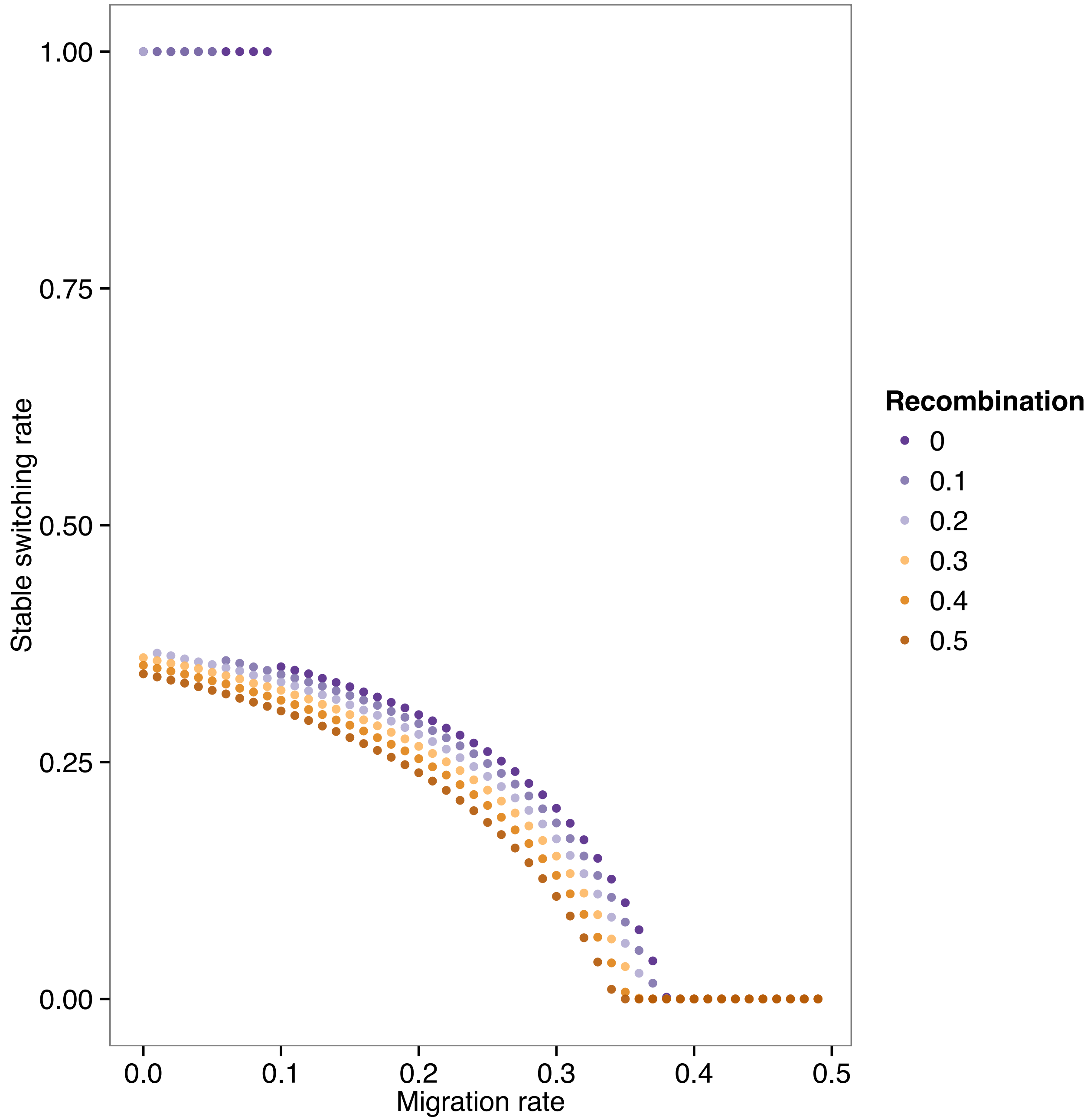
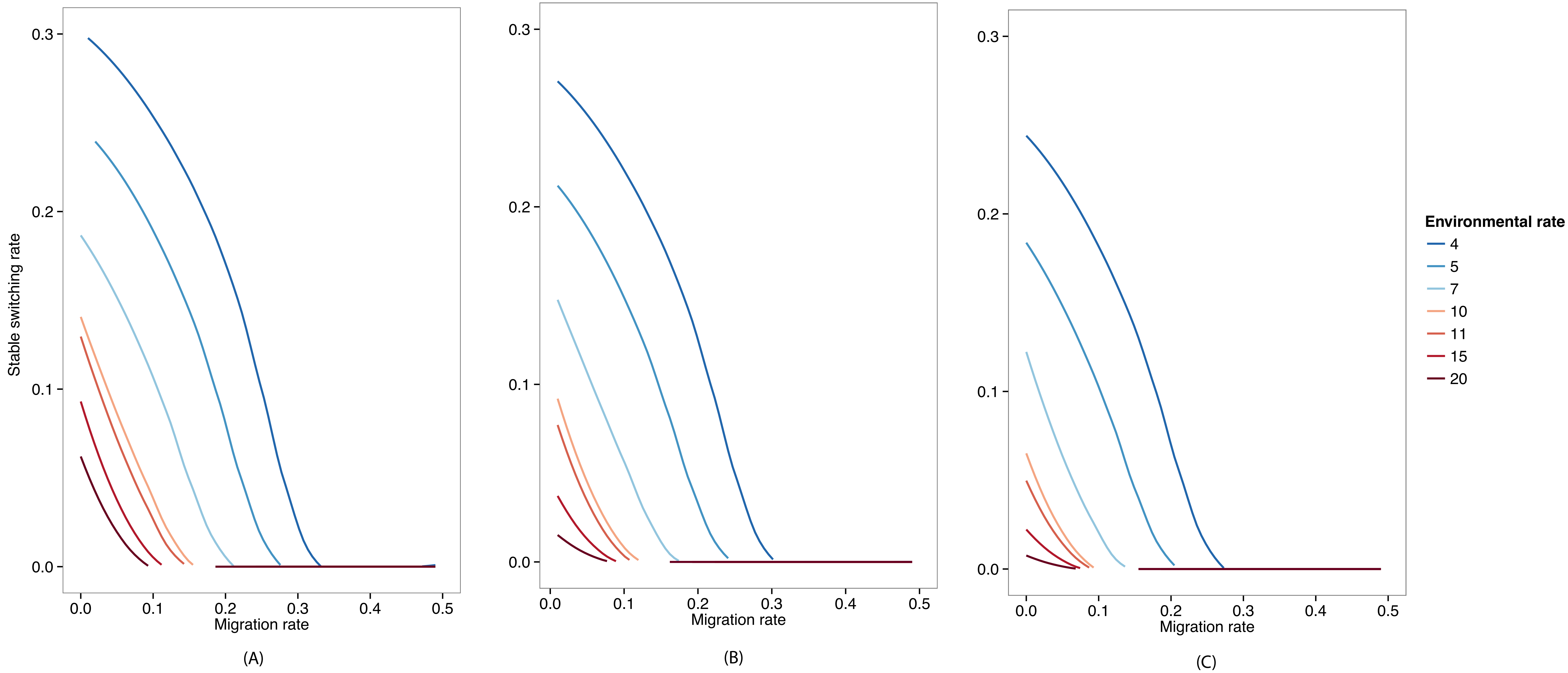
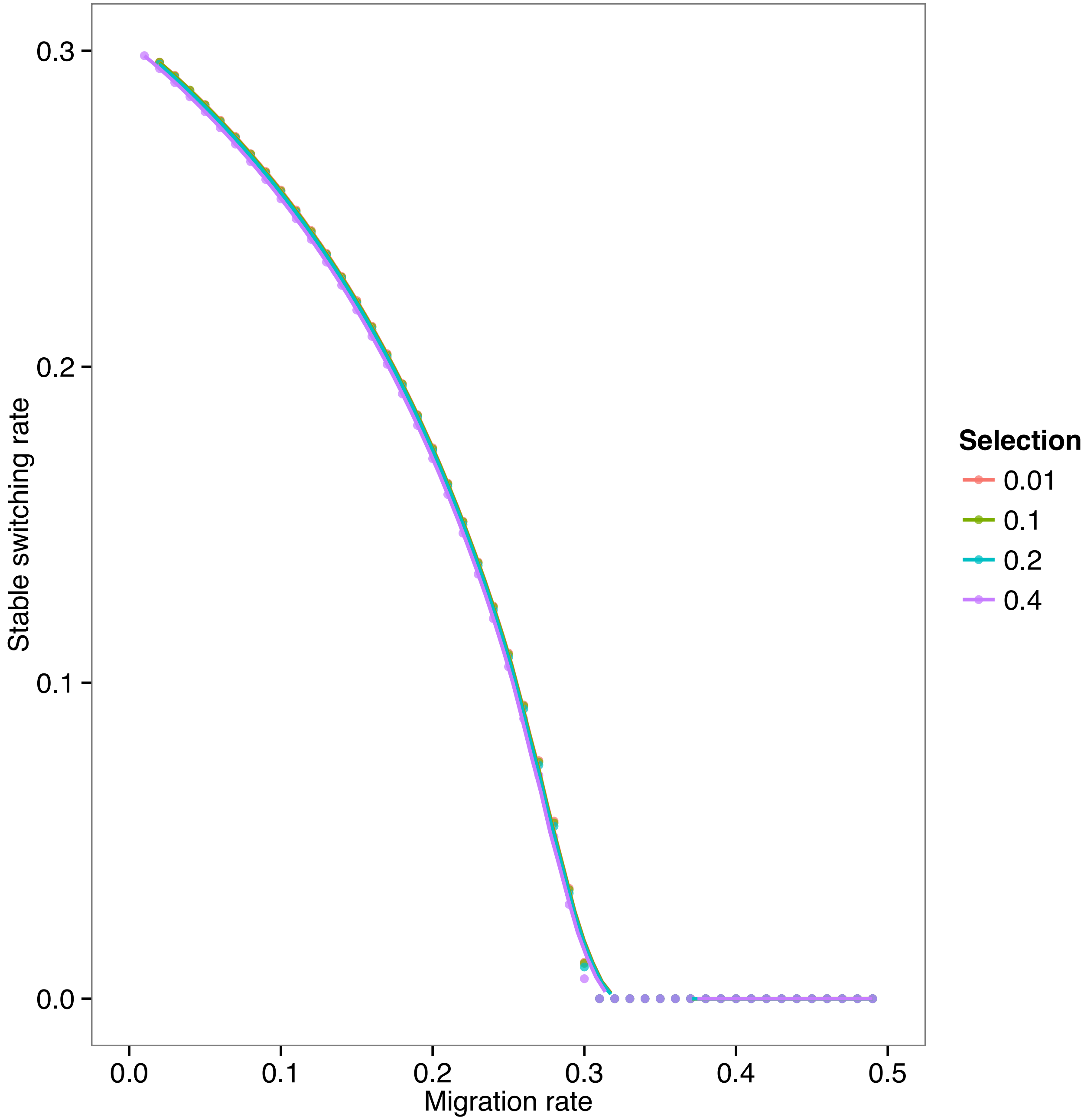

Stable switching rate
1.00
0.75
0.50
0.25
0.00
0.0
0.1
0.2
0.3
0.4
0.5
Migration rate
Recombination
0
0.1
0.2
0.3
0.4
0.5

Figure S1. The evolutionarily stable switching rate as function of migration and recombination rate, $n = 3$**.** The symmetric selection coefficient $s = 0.4$. Recombination rates shown in the legend. The stable switching rate for $n = 3$ is sensitive to the interplay of recombination and migration rates, with sudden possible discontinuities in the stable switching rate.



Stable switching rate
Migration rate
0.0
0.1
0.2
0.3
0.4
0.5
(A)
(B)
(C)
Environmental rate
4
5
7
10
11
15
20

Figure S2. The evolutionarily stable switching rate as function of migration and environmental rate of change, $n > 3$ for different recombination rates. The symmetric selection coefficient $s = 0.4$. The rate of environmental change n shown in the legend. The plotted curves represent a fit to the data using a generalized additive model with penalized cubic regression splines. In panel **A**, $r = 0$. In panel **B**, $r = 0.25$. In panel **C**, $r = 0.5$.



Stable switching rate
0.0
0.1
0.2
0.3
Migration rate
0.0
0.1
0.2
0.3
0.4
0.5
Selection
0.01
0.1
0.2
0.4

Figure S3. The evolutionarily stable switching rate as function of migration and symmetric selection coefficient s. Recombination rate is $r = 0$. The environment changes every $n = 4$ generations. The symmetric selection coefficient s shown in the legend. The plotted curves represent a fit to the data using a generalized additive model with penalized cubic regression splines. The stable switching rate is invariant to the strength of symmetric selection between the two demes.