File S4

Proof of Result 4

A straightforward computation shows that the 4×4 matrix $\mathbf{L}_{\text{ex}}$ can be written as

$$\mathbf{L}_{\text{ex}} = \begin{pmatrix} (1-m)A & (1-m)B & mC & mD \\ (1-m)D & (1-m)C & mB & mA \\ mA & mB & (1-m)C & (1-m)D \\ mD & mC & (1-m)B & (1-m)A \end{pmatrix}, \qquad (S4.1)$$

where

$$\begin{aligned} (1+sx^*)A &= (1+s)(1-\mu_b) + r(1-x^*)\big[(s+2)\mu_b - (s+1)\big] \\ (1+sx^*)B &= (1+s)rx^* + \mu_b\big[1-(s+2)rx^*\big] \\ (1+sx^*)C &= (1-\mu_b) + rx^*\big[(s+2)\mu_b - 1\big] \\ (1+sx^*)D &= (1+s)\mu_b + r(1-x^*)\big[1-(s+2)\mu_b\big]. \end{aligned} \qquad (S4.2)$$

Observe that "formally" A, B, C, D are linear in μ_b. Let A_0 be the value of A when $\mu_b = 0$ and A_1 be its value when $\mu_b = 1$. Similarly we have $B_0, B_1, C_0, C_1, D_0, D_1$. In fact,

$$\begin{aligned} (1+sx^*)A_0 &= (1+s)\big[1-r(1-x^*)\big] \\ (1+sx^*)A_1 &= r(1-x^*) \\ (1+sx^*)B_0 &= (1+s)rx^* \\ (1+sx^*)B_1 &= 1-rx^* \\ (1+sx^*)C_0 &= 1-rx^* \\ (1+sx^*)C_1 &= (1+s)rx^* \\ (1+sx^*)D_0 &= r(1-x^*) \\ (1+sx^*)D_1 &= (1+s)\big[1-r(1-x^*)\big]. \end{aligned} \qquad (S4.3)$$

As $0 < r < 1$, $0 < x^* < 1$ we have A_i, B_i, C_i, D_i positive for $i = 0, 1$. Hence, as A, B, C, D are linear in μ_b, A, B, C, D are all positive for $0 < \mu_b < 1$. Moreover we have

$$C_0 = B_1, \qquad C_1 = B_0, \qquad D_0 = A_1, \qquad D_1 = A_0. \qquad (S4.4)$$

Let $S(\lambda) = \det(\mathbf{L}_{\text{ex}} - \lambda \mathbf{I})$ be the characteristic polynomial of $\mathbf{L}_{\text{ex}}$. The structure of $\mathbf{L}_{\text{ex}}$ given in (S4.1) entails that $S(\lambda)$ factors into the product of two quadratic polynomials $S_1(\lambda)$ and $S_2(\lambda)$:

$$S(\lambda) = S_1(\lambda) S_2(\lambda), \qquad (S4.5)$$

where

$$
\begin{aligned}
S_1(\lambda) &= \lambda^2 - \lambda\big[(1-m)(A+C) + m(B+D)\big] + (1-2m)(AC-BD) \\
S_2(\lambda) &= \lambda^2 - \lambda\big[(1-m)(A+C) - m(B+D)\big] + (1-2m)(AC-BD).
\end{aligned}
\tag{S4.6}
$$

See Balkau and Feldman (1973) for analogous calculations with migration modification. Consider first the roots of $S_1(\lambda) = 0$. These are real since the discriminant of $S_1(\lambda) = 0$ is

$$
\begin{aligned}
&\big[(1-m)(A+C) + m(B+D)\big]^2 - 4(1-2m)(AC-BD) = \\
&= \big[(1-m)(A-C) + m(B-D)\big]^2 + 4m(1-m)(AD+BC) + 4(1-m)^2 BD + 4m^2 AC,
\end{aligned}
\tag{S4.7}
$$

which is positive since A, B, C, D are positive and $0 < m < 1$.

In addition,

$$
\begin{aligned}
AC - BD = &\big[(1-\mu_b)A_0 + \mu_b A_1\big]\big[(1-\mu_b)(C_0 + \mu_b C_1\big] \\
&- \big[(1-\mu_b)B_0 + \mu_b B_1\big]\big[(1-\mu_b)D_0 + \mu_b D_1\big].
\end{aligned}
\tag{S4.8}
$$

Since $C_0 = B_1$, $C_1 = B_0$, $D_0 = A_1$, $D_1 = A_0$, (S4.8) reduces to

$$
AC - BD = (1-2\mu_b)(A_0 B_1 - A_1 B_0). \tag{S4.9}
$$

Substituting A_0, A_1, B_0, B_1 from (S4.3) we have

$$
(1+sx^*)^2(AC-BD) = (1-2\mu_b)(1+s)(1-r). \tag{S4.10}
$$

Since we assume $0 < m,\ \mu_b < \frac{1}{2}$, the two roots of $S_1(\lambda) = 0$ are positive. Both of these roots are less than 1 if and only if $S_1(1) > 0$ and $S_1'(1) > 0$.

$$
S_1'(1) = 2 - \big[(1-m)(A+C) + m(B+D)\big]. \tag{S4.11}
$$

As $C_0 = B_1, C_1 = B_0, D_0 = A_1, D_1 = A_0$, we have

$$
\begin{aligned}
(A+C) &= (1-\mu_b)(A_0 + C_0) + \mu_b(A_1 + C_1) \\
&= (1-\mu_b)(A_0 + C_0) + \mu_b(B_0 + D_0),
\end{aligned}
\tag{S4.12}
$$

$$
\begin{aligned}
(B+D) &= (1-\mu_b)(B_0 + D_0) + \mu_b(B_1 + D_1) \\
&= (1-\mu_b)(B_0 + D_0) + \mu_b(A_0 + C_0).
\end{aligned}
\tag{S4.13}
$$

Hence,

$$(1-m)(A+C)+m(B+D)=(1-m_b)(A_0+C_0)+m_b(B_0+D_0), \qquad (S4.14)$$

where

$$m_b = m+\mu_b-2m\mu_b. \qquad (S4.15)$$

Substituting for A_0, B_0, C_0, D_0, gives

$$S_1'(1)=(1+sx^*)^{-1}\big[r+rsx^*+s(1-r)(2x^*-1)+m_b(1-r)(s+2)\big]. \qquad (S4.16)$$

Now $s>0$, $0<r<1$, $m_b=m(1-\mu_b)+\mu_b(1-m)>0$, and $x^*>\frac{1}{2}$ if $0<m$, $\mu_B<\frac{1}{2}$. Therefore $S'(1)>0$ provided $0<m$, $\mu_B<\frac{1}{2}$. Using (S4.11) and (S4.16) it is easily seen that $S(1)>0$ if

$$(1+sx^*)^{-2}(1-r)\left\{(x^*)^2s^2+sx^*\big[-s+m_b(s+2)\big]-sm_b\right\}>0. \qquad (S4.17)$$

Using the equation $Q(x^*)=0$ from (14), we have

$$s(x^*)^2+\big[(s+2)m_B-s\big]x^*-m_B=0, \qquad (S4.18)$$

where

$$m_B=m+\mu_B-2m\mu_B. \qquad (S4.19)$$

Therefore (S4.17) is satisfied if and only if

$$(m_b-m_B)(1+sx^*)^{-2}(1-r)s\big[x^*(2+s)-1\big]>0. \qquad (S4.20)$$

As $x^*>\frac{1}{2}$, by Result 1, and $0<m<\frac{1}{2}$, (S4.20) holds if and only if $m_b>m_B$, which is true if and only if $\mu_b>\mu_B$.

It is not obvious that the roots of $S_2(\lambda)=0$ are real. However, as the matrix $\mathbf{L}^*_{\text{ex}}$ is positive, the Perron-Frobenius theory ensures that its largest eigenvalue in magnitude is positive. Therefore we just have to ensure that when both eigenvalues are real and positive they are less than 1; when they are real, both are positive or both are negative since $(1-2m)(1-2\mu_b)(AC-BD)$ is positive for $0<m,\ \mu_b<\frac{1}{2}$. The conditions for this are that both $S_2(1)$ and $S_2'(1)$ are positive. But

$$\begin{aligned} S_2(1) &= 1-\big[(1-m)(A+C)-m(B+D)\big]+(1-2m)(AC-BD) \\ &> 1-\big[(1-m)(A+C)+m(B+D)\big]+(1-2m)(AC-BD)=S_1(1), \end{aligned} \tag{S4.21}$$

and $S_1(1)>0$ when $0<m,\ \mu_b<\frac{1}{2}$ and $\mu_b>\mu_B$, so also $S_2(1)>0$. Similarly

$$\begin{aligned} S_2'(1) &= 2-\big[(1-m)(A+C)-m(B+D)\big] \\ &> 2-\big[(1-m)(A+C)+m(B+D)\big]=S_1'(1). \end{aligned} \tag{S4.22}$$

Thus, when $S_1'(1)>0$ also $S_2'(1)>0$.