File S7

Proof of Result 7

As the transformation T of the population state is $T = T_2 \circ T_1$, where T_i corresponds to phase i, with selection of type i, for $i = 1, 2$, and as $\tilde{\mathbf{x}} = T_1\bar{\mathbf{x}}$, $\bar{\mathbf{x}} = T_2\tilde{\mathbf{x}}$, following the analysis for the case without cycles, the linear approximation matrix $\mathbf{L}_{\text{ex}}$ becomes

$$\mathbf{L}_{\text{ex}} = \mathbf{L}^2_{\text{ex}} \cdot \mathbf{L}^1_{\text{ex}}, \tag{S7.1}$$

where, as in (S4.1) and (S4.2), we have

$$\mathbf{L}^1_{\text{ex}} = \begin{pmatrix} (1-m)\bar{A} & (1-m)\bar{B} & m\bar{C} & m\bar{D} \\ (1-m)\bar{D} & (1-m)\bar{C} & m\bar{B} & m\bar{A} \\ m\bar{A} & m\bar{B} & (1-m)\bar{C} & (1-m)\bar{D} \\ m\bar{D} & m\bar{C} & (1-m)\bar{B} & (1-m)\bar{A} \end{pmatrix}, \tag{S7.2}$$

$$\mathbf{L}^2_{\text{ex}} = \begin{pmatrix} (1-m)\widetilde{A} & (1-m)\widetilde{B} & m\widetilde{C} & m\widetilde{D} \\ (1-m)\widetilde{D} & (1-m)\widetilde{C} & m\widetilde{B} & m\widetilde{A} \\ m\widetilde{A} & m\widetilde{B} & (1-m)\widetilde{C} & (1-m)\widetilde{D} \\ m\widetilde{D} & m\widetilde{C} & (1-m)\widetilde{B} & (1-m)\widetilde{A} \end{pmatrix}, \tag{S7.3}$$

and

$$\begin{aligned} (1+s\bar{x})\overline{A} &= (1+s)(1-\mu_b) + r(1-\bar{x})\big[(s+2)\mu_b - (s+1)\big] \\ (1+s\bar{x})\overline{B} &= (1+s)r\bar{x} + \mu_b\big[1-(s+2)r\bar{x}\big] \\ (1+s\bar{x})\overline{C} &= (1-\mu_b) + r\bar{x}\big[(s+2)\mu_b - 1\big] \\ (1+s\bar{x})\overline{D} &= (1+s)\mu_b + r(1-\bar{x})\big[1-(s+2)\mu_b\big], \end{aligned} \tag{S7.4}$$

$$\begin{aligned} \big[1+s(1-\tilde{x})\big]\widetilde{A} &= (1-\mu_b) + r(1-\tilde{x})\big[(2+s)\mu_b - 1\big] \\ \big[1+s(1-\tilde{x})\big]\widetilde{B} &= (1+s)\mu_b + r\tilde{x}\big[1-(s+2)\mu_b\big] \\ \big[1+s(1-\tilde{x})\big]\widetilde{C} &= (1+s)(1-\mu_b) + r\tilde{x}\big[(s+2)\mu_b - (s+1)\big] \\ \big[1+s(1-\tilde{x})\big]\widetilde{D} &= \mu_b + r(1-\tilde{x})\big[(s+1)-(s+2)\mu_b\big]. \end{aligned} \tag{S7.5}$$

When we multiply $\mathbf{L}^2_{\text{ex}}$ by $\mathbf{L}^1_{\text{ex}}$ we find that the product $\mathbf{L}_{\text{ex}}$ has the following structure:

$$\mathbf{L}_{\text{ex}} = \begin{pmatrix} a & e & h & d \\ b & f & g & c \\ c & g & f & b \\ d & h & e & a \end{pmatrix}, \tag{S7.6}$$

where

$$\begin{aligned}
a &= (1-m)^2\widetilde{A}\overline{A} + (1-m)^2\widetilde{B}\overline{D} + m^2\widetilde{C}\overline{A} + m^2\widetilde{D}\overline{D} \\
b &= (1-m)^2\widetilde{D}\overline{A} + (1-m)^2\widetilde{C}\overline{D} + m^2\widetilde{B}\overline{A} + m^2\widetilde{A}\overline{D} \\
c &= m(1-m)\left[\widetilde{A}\overline{A} + \widetilde{B}\overline{D} + \widetilde{C}\overline{A} + \widetilde{D}\overline{D}\right] \\
d &= m(1-m)\left[\widetilde{D}\overline{A} + \widetilde{C}\overline{D} + \widetilde{B}\overline{A} + \widetilde{A}\overline{D}\right] \\
e &= (1-m)^2\widetilde{A}\overline{B} + (1-m)^2\widetilde{B}\overline{C} + m^2\widetilde{C}\overline{B} + m^2\widetilde{D}\overline{C} \\
f &= (1-m)^2\widetilde{D}\overline{B} + (1-m)^2\widetilde{C}\overline{C} + m^2\widetilde{B}\overline{B} + m^2\widetilde{A}\overline{C} \\
g &= m(1-m)\left[\widetilde{A}\overline{B} + \widetilde{B}\overline{C} + \widetilde{C}\overline{B} + \widetilde{D}\overline{C}\right] \\
h &= m(1-m)\left[\widetilde{D}\overline{B} + \widetilde{C}\overline{C} + \widetilde{B}\overline{B} + \widetilde{A}\overline{C}\right].
\end{aligned} \tag{S7.7}$$

Let $D(\lambda) = \det(\mathbf{L}_{\text{ex}} - \lambda\mathbf{I})$ be the characteristic polynomial of $\mathbf{L}_{\text{ex}}$. From (S7.6), $D(\lambda)$ factors into 2×2 determinants:

$$D(\lambda) = \begin{vmatrix} a+d-\lambda & e+h & 0 & 0 \\ b+c & f+g-\lambda & 0 & 0 \\ c & g & f-g-\lambda & b-c \\ d & h & e-h & a-d-\lambda \end{vmatrix}. \tag{S7.8}$$

Therefore $D(\lambda)$ can be written

$$D(\lambda) = D_1(\lambda)D_2(\lambda), \tag{S7.9}$$

where

$$\begin{aligned}
D_1(\lambda) &= \lambda^2 - (a+d+f+g)\lambda + (a+d)(f+g) - (b+c)(e+h) \\
D_2(\lambda) &= \lambda^2 - (a-d+f-g)\lambda + (a-d)(f-g) - (b-c)(e-h).
\end{aligned} \tag{S7.10}$$

As $0 < m < 1$ and $\overline{A}, \overline{B}, \overline{C}, \overline{D}$ and $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ are all positive, the matrix $\mathbf{L}_{\text{ex}}$ is a positive matrix and its largest eigenvalue in magnitude is positive. Observe that the discriminant of $D_1(\lambda)$ is

$$(a+d+f+g)^2 - 4\big[(a+d)(f+g) - (b+c)(e+h)\big], \tag{S7.11}$$

which is positive and equal to

$$[(a+d) - (f+g)]^2 + 4(b+c)(e+h). \tag{S7.12}$$

In addition, $(a+d+f+g)$ is positive. Therefore $D_1(\lambda)$ has real roots, and its largest root in magnitude is positive. Thus this positive root is less than 1 if $D_1(1) > 0$ and $D_1'(1) > 0$.

As the largest eigenvalue of $\mathbf{L}_{\rm ex}$ is positive, for stability of $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ we require that if the eigenvalues associated with $D_2(\lambda)$ are real and at least one is positive, they are both less than 1. Again the conditions for this are $D_2(\lambda) > 0$ and $D_2'(1) > 0$. Observe that

$$\begin{aligned} D_1'(1) &= 2 - (a + d + f + g) \\ D_2'(1) &= 2 - (a - d + f - g) = D_1'(1) + 2(d + g) > D_1'(1). \end{aligned} \tag{S7.13}$$

In view of (S7.13), for the largest eigenvalue of $\mathbf{L}_{\rm ex}$ to be less than one, we require

$$D_1(1) > 0, \qquad D_1'(1) > 0, \qquad D_2(1) > 0. \tag{S7.14}$$

We now compute the constant terms of $D_1(\lambda)$ and $D_2(\lambda)$. We already know, based on the properties of the matrices $\mathbf{L}^1_{\rm ex}$ and $\mathbf{L}^2_{\rm ex}$ that the constant terms of both $D_1(\lambda)$ and $D_2(\lambda)$ are the same and are equal to

$$(1 - 2m)^2 \left(\overline{A}\,\overline{C} - \overline{B}\,\overline{D}\right) \left(\widetilde{A}\widetilde{C} - \widetilde{B}\widetilde{D}\right). \tag{S7.15}$$

With the same technique used to compute (S4.10), we deduce that

$$(1 + s\overline{x})^2 \left(\overline{A}\,\overline{C} - \overline{B}\,\overline{D}\right) = (1 - 2\mu_b)\,(1 + s)(1 - r), \tag{S7.16}$$

and similarly

$$[1 + s\,(1 - \tilde{x})]^2 \left(\widetilde{A}\widetilde{C} - \widetilde{B}\widetilde{D}\right) = (1 - 2\mu_b)(1 + s)(1 - r). \tag{S7.17}$$

Therefore the constant terms of both $D_1(\lambda)$ and $D_2(\lambda)$ are the same and are equal to

$$(1 - 2m)^2\,(1 - 2\mu_b)^2\,(1 + s\overline{x})^{-2}\,[1 + s\,(1 - \tilde{x})]^{-2}\,(1 + s)^2\,(1 - r)^2\,, \tag{S7.18}$$

which is positive, and so $D_1(\lambda)$ has two positive roots. Also, as a, b, c, d are all positive,

$$\begin{aligned} D_2(1) &= 1 - (a - d + f - g) + (a - d)(f - g) - (b - c)(e - h) \\ &> 1 - (a + d + f + g) + (a + d)(f + g) - (b + c)(e + h) = D_1(1). \end{aligned} \tag{S7.19}$$

Hence for the symmetric equilibrium to be externally stable, we require that $D_1(1)$ and $D_1'(1)$ are both positive.

Now from (S6.5) we know that

$$(1+s\bar{x})\big[1+s(1-\tilde{x})\big] = (1+s) + sm_B\big[(s+2)\bar{x}-1\big]. \qquad (S7.20)$$

As $\bar{x} > \frac{1}{s+2}$ we have

$$(1+s\bar{x})\left[1+s\left(1-\tilde{x}\right)\right] > (1+s). \qquad (S7.21)$$

Thus the equal constant terms of $D_1(\lambda)$ and $D_2(\lambda)$ given in (S7.18) are positive and less than 1. As a result it is impossible for the two positive roots of $D_1(\lambda)$ to both be larger than 1, and they are both less than 1 provided $D_1(1) > 0$. Hence the external stability of the symmetric equilibrium requires that

$$D_1(1) = 1 - (a+d+f+g) + (a+d)(f+g) - (b+c)(e+h) \qquad (S7.22)$$

is positive (the last summand in (S7.22) is given in (S7.18)). We now compute $a+d+f+g$.

Computation of $(a+d+f+g)$

We have

$$\begin{aligned}(a+d+f+g) = &\left[(1-m)^2\,\widetilde{A} + m^2\widetilde{C} + m(1-m)\left(\widetilde{B}+\widetilde{D}\right)\right]\overline{A}\\ &+\left[(1-m)^2\,\widetilde{C} + m^2\widetilde{A} + m(1-m)\left(\widetilde{B}+\widetilde{D}\right)\right]\overline{C}\\ &+\left[(1-m)^2\,\widetilde{B} + m^2\widetilde{D} + m(1-m)\left(\widetilde{A}+\widetilde{C}\right)\right]\overline{D}\\ &+\left[(1-m)^2\,\widetilde{D} + m^2\widetilde{B} + m(1-m)\left(\widetilde{A}+\widetilde{C}\right)\right]\overline{B}.\end{aligned} \qquad (S7.23)$$

As $\overline{A},\overline{B},\overline{C},\overline{D}$ and also $\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D}$ given in (S7.4) and (S7.5), respectively, are all linear functions of μ_b, where $0 \le \mu_b \le 1$, we can represent them as $\overline{A} = (1-\mu_b)\overline{A}_0 + \mu_b\overline{A}_1$, etc. Hence

$$\begin{aligned}(1+s\bar{x})\overline{A}_0 &= (1+s)\left[1-r\left(1-\bar{x}\right)\right] = (1+s\bar{x})\overline{D}_1\\ (1+s\bar{x})\overline{B}_0 &= (1+s)r\bar{x} = (1+s\bar{x})\overline{C}_1\\ (1+s\bar{x})\overline{C}_0 &= 1-r\bar{x} = (1+s\bar{x})\overline{B}_1\\ (1+s\bar{x})\overline{D}_0 &= r(1-\bar{x}) = (1+s\bar{x})\overline{A}_1,\end{aligned} \qquad (S7.24)$$

$$\begin{aligned}\left[1+s\left(1-\tilde{x}\right)\right]\widetilde{A}_o &= 1-r\left(1-\tilde{x}\right) = \left[1+s\left(1-\tilde{x}\right)\right]\widetilde{D}_1\\ \left[1+s\left(1-\tilde{x}\right)\right]\widetilde{B}_o &= r\bar{x} = \left[1+s\left(1-\tilde{x}\right)\right]\widetilde{C}_1\\ \left[1+s\left(1-\tilde{x}\right)\right]\widetilde{C}_o &= (1+s)\left[1-r\tilde{x}\right] = \left[1+s\left(1-\tilde{x}\right)\right]\widetilde{B}_1\\ \left[1+s\left(1-\tilde{x}\right)\right]\widetilde{D}_o &= (1+s)r\left(1-\tilde{x}\right) = \left[1+s\left(1-\tilde{x}\right)\right]\widetilde{A}_1.\end{aligned} \qquad (S7.25)$$

Since

$$\begin{aligned} m_b &= m + \mu_b - 2m\mu_b = m(1-\mu_b) + \mu_b(1-m) \\ 1 - m_b &= 1 - m - \mu_b + 2m\mu_b = (1-m)(1-\mu_b), \end{aligned} \tag{S7.26}$$

we can write

$$\begin{aligned} (a+d+f+g) = &\left[(1-m)(1-m_b)\widetilde{A}_0 + (1-m)m_b\widetilde{D}_0 + m(1-m_b)\widetilde{B}_0 + m\cdot m_b\widetilde{C}_0\right]\overline{A} \\ &+ \left[(1-m)(1-m_b)\widetilde{C}_0 + (1-m)m_b\widetilde{B}_0 + m(1-m_b)\widetilde{D}_0 + m\cdot m_b\widetilde{A}_0\right]\overline{C} \\ &+ \left[(1-m)(1-m_b)\widetilde{B}_0 + (1-m)m_b\widetilde{C}_0 + m(1-m_b)\widetilde{A}_0 + m\cdot m_b\widetilde{D}_0\right]\overline{D} \\ &+ \left[(1-m)(1-m_b)\widetilde{D}_0 + (1-m)m_b\widetilde{A}_0 + m(1-m_b)\widetilde{C}_0 + m\cdot m_b\widetilde{B}_0\right]\overline{B}. \end{aligned} \tag{S7.27}$$

Substitute into (S7.27)

$$\begin{aligned} \overline{A} &= (1-\mu_b)\overline{A}_0 + \mu_b\overline{D}_0, \qquad \overline{B} = (1-\mu_b)\overline{B}_0 + \mu_b\overline{C}_0, \\ \overline{C} &= (1-\mu_b)\overline{C}_0 + \mu_b\overline{B}_0, \qquad \overline{D} = (1-\mu_b)\overline{D}_0 + \mu_b\overline{A}_0, \end{aligned} \tag{S7.38}$$

to obtain

$$\begin{aligned} (a+d+f+g) = &(1-m_b)^2\left[\widetilde{A}_0\overline{A}_0 + \widetilde{B}_0\overline{D}_0 + \widetilde{C}_0\overline{C}_0 + \widetilde{D}_0\overline{B}_0\right] \\ &+ m_b(1-m_b)\left[\left(\widetilde{B}_0 + \widetilde{D}_0\right)\left(\overline{A}_0 + \overline{C}_0\right) + \left(\widetilde{A}_0 + \widetilde{C}_0\right)\left(\overline{B}_0 + \overline{D}_0\right)\right] \\ &+ m_b^2\left[\widetilde{A}_0\overline{C}_0 + \widetilde{B}_0\overline{B}_0 + \widetilde{C}_0\overline{A}_0 + \widetilde{D}_0\overline{D}_0\right]. \end{aligned} \tag{S7.29}$$

Equation (S7.29) can also be written as

$$\begin{aligned} (a+d+f+g) = &(1-2m_b)\left[\widetilde{A}_0\overline{A}_0 + \widetilde{B}_0\overline{D}_0 + \widetilde{C}_0\overline{C}_0 + \widetilde{D}_0\overline{B}_0\right] \\ &+ m_b\left[\left(\widetilde{B}_0 + \widetilde{D}_0\right)\left(\overline{A}_0 + \overline{C}_0\right) + \left(\widetilde{A}_0 + \widetilde{C}_0\right)\left(\overline{B}_0 + \overline{D}_0\right)\right] \\ &+ m_b^2\left[\left(\widetilde{A}_0 + \widetilde{C}_0\right)\left(\overline{A}_0 + \overline{C}_0\right) + \left(\widetilde{B}_0 + \widetilde{D}_0\right)\left(\overline{B}_0 + \overline{D}_0\right)\right. \\ &\qquad \left. - \left(\widetilde{A}_0 + \widetilde{C}_0\right)\left(\overline{B}_0 + \overline{D}_0\right) + \left(\widetilde{B}_0 + \widetilde{D}_0\right)\left(\overline{A}_0 + \overline{C}_0\right)\right], \end{aligned} \tag{S7.30}$$

or as

$$\begin{aligned} (a+d+f+g) = &(1-2m_b)\left[\widetilde{A}_0\overline{A}_0 + \widetilde{B}_0\overline{D}_0 + \widetilde{C}_0\overline{C}_0 + \widetilde{D}_0\overline{B}_0\right] \\ &+ m_b\left[\left(\widetilde{B}_0 + \widetilde{D}_0\right)\left(\overline{A}_0 + \overline{C}_0\right) + \left(\widetilde{A}_0 + \widetilde{C}_0\right)\left(\overline{B}_0 + \overline{D}_0\right)\right] \\ &+ m_b^2\left(\widetilde{A}_0 + \widetilde{C}_0 - \widetilde{B}_0 - \widetilde{D}_0\right)\left(\overline{A}_0 + \overline{C}_0 - \overline{B}_0 - \overline{D}_0\right). \end{aligned} \tag{S7.31}$$

From (S7.24) and (S7.25),

$$(1+s\bar{x})[1+s(1-\tilde{x})]\left[\widetilde{A}_0\overline{A}_0+\widetilde{B}_0\overline{D}_0+\widetilde{C}_0\overline{C}_0+\widetilde{D}_0\overline{D}_0\right]= \\ =(1+s)\left[2(1-r)+r^2\right]+r^2s(s+1)\bar{x}-r^2s\tilde{x}-r^2s^2\bar{x}\tilde{x}, \tag{S7.32}$$

$$(1+s\bar{x})[1+s(1-\tilde{x})]\left[\left(\widetilde{B}_0+\widetilde{D}_0\right)\left(\overline{A}_0+\overline{C}_0\right)+\left(\widetilde{A}_0+\widetilde{C}_0\right)\left(\overline{B}_0+\overline{D}_0\right)\right]= \\ =2r^2(1+s\bar{x})[1+s(1-\bar{x})]+r(1-r)(s+2)\left[(s+2)+s\left(\bar{x}-\tilde{x}\right)\right], \tag{S7.33}$$

$$(1+s\bar{x})[1+s(1-\tilde{x})]\left(\widetilde{A}_0+\widetilde{C}_0-\widetilde{B}_0-\widetilde{D}_0\right)\left(\overline{A}_0+\overline{C}_0-\overline{B}_0-\overline{D}_0\right)= \\ =(s+2)^2(1-r)^2. \tag{S7.34}$$

Remember that by (S7.18)

$$(1+s\bar{x})^2\left[1+(1-\tilde{x})\right]^2\left[(a+d)(f+g)-(b+c)(e+h)\right]= \\ =(1-2m)^2(1-2\mu_b)^2(s+1)^2(1-r)^2. \tag{S7.35}$$

But

$$(1-2m)(1-2\mu_b)=1-2(m+\mu_b-2m\mu_b)=1-2m_b. \tag{S7.36}$$

Therefore

$$(1+s\bar{x})^2\left[1+(1-\tilde{x})\right]^2\left[(a+d)(f+g)-(b+c)(e+h)\right]= \\ =(1-2m_b)^2(s+1)^2(1-r)^2. \tag{S7.37}$$

Combining all of this, we get that $D_1(1)=1-(a+d+f+g)+(a+d)(f+g)-(b+c)(e+h)$, which we compute as

$$\begin{aligned}&1-(1-2m_b)\frac{(1+s)\left[2(1-r)+r^2\right]+r^2s(s+1)\bar{x}-r^2s\tilde{x}-r^2s^2\bar{x}\tilde{x}}{(1+s\bar{x})[1+s(1-\tilde{x})]}\\ &-m_b\frac{2r^2(1+s\bar{x})[1+s(1-\tilde{x})]+r(1-r)(s+2)[(s+2)+s(\bar{x}-\tilde{x})]}{(1+s\bar{x})[1+s(1-\tilde{x})]}\\ &-m_b^2\frac{(s+2)^2(1-r)^2}{(1+s\bar{x})[1+s(1-\tilde{x})]}\\ &+\frac{(1-2m_b)^2(s+1)^2(1-r)^2}{(1+s\bar{x})^2\left[1+s\left(1-\tilde{x}\right)\right]^2}.\end{aligned} \tag{S7.38}$$

Observe that

$$r^2[(1+s)+s(s+1)\bar{x}-s\tilde{x}-s^2\bar{x}\tilde{x}]=r^2(1+s\bar{x})[1+s(1-\tilde{x})], \tag{S7.39}$$

so (S7.38) simplifies to

$$1-r^2-(1-2m_b)\frac{2(1+s)(1-r)}{(1+s\bar{x})[1+s(1-\tilde{x})]}-m_b\frac{r(1-r)(s+2)[(s+2)+s(\bar{x}-\tilde{x})]}{(1+s\bar{x})[1+s(1-\tilde{x})]}$$
$$-m_b^2\frac{(s+2)^2(1-r)^2}{(1+s\bar{x})[1+s(1-\tilde{x})]}+\frac{(1-2m_b)^2(s+1)^2(1-r)^2}{(1+s\bar{x})^2[1+s(1-\tilde{x})]^2}. \qquad (S7.40)$$

Clearly $D_1(1)$ of (S7.40) has a factor of $(1-r)$, and in fact

$$D_1(1)=(1-r)f(r), \qquad (S7.41)$$

where $f(r)$ is a linear function of r, for $0\le r\le 1$. Now

$$f(1)=2-(1-2m_b)\frac{2(1+s)}{(1+s\bar{x})[1+s(1-x)]}-m_b\frac{(s+2)[(s+2)+s(\bar{x}-\tilde{x})]}{(1+s\bar{x})[1+s(1-\tilde{x})]}. \qquad (S7.42)$$

Following (S6.7) we have

$$(1+s\bar{x})[1+s(1-\tilde{x})]=(1+s)+sm_B[(s+2)\bar{x}-1]. \qquad (S7.43)$$

We also have an equivalent expression for (S7.43) in terms of $\tilde{x}$, namely

$$(1+s\bar{x})[1+s(1-\tilde{x})]=(1+s)+sm_B[(s+1)-(s+2)\tilde{x}]. \qquad (S7.44)$$

Also, whereas $\bar{x}>\frac{1}{s+2}$, we have $\tilde{x}<\frac{s+1}{s+2}$. Applying all of this to (S7.42) and using the fact that

$$(1+s\bar{x})[1+s(1-\tilde{x})]=(1+s)+\frac{1}{2}sm_B[s+(s+2)(\bar{x}-\tilde{x})], \qquad (S7.45)$$

we get that

$$\begin{aligned}(1+s\bar{x})[1+s(1-\tilde{x})]f(1)&=2(s+1)+sm_B[s+(s+2)(\bar{x}-\tilde{x})]\\&\quad-2(1-2m_b)(s+1)-m_b(s+2)[(s+2)+s(\bar{x}-\tilde{x})]\\&=s^2m_B-m_b\left[(s+2)^2-4(s+1)\right]+s(s+2)(m_B-m_b)(\bar{x}-\tilde{x})\\&=s^2(m_B-m_b)+s(s+2)(m_B-m_b)(\bar{x}-\tilde{x}).\end{aligned} \qquad (S7.46)$$

Thus

$$(1+s\bar{x})[1+s(1-\tilde{x})]f(1)=s(m_B-m_b)[s+(s+2)(\bar{x}-\tilde{x})]. \qquad (S7.47)$$

But as $(s+2)\bar{x} > 1$, $(s+2)\tilde{x} < (s+1)$,

$$s + (s+2)(\bar{x} - \tilde{x}) = [(s+2)\bar{x} - 1] + [(s+1) - (s+2)\tilde{x}] > 0. \quad (S7.48)$$

It follows that the sign of $f(1)$ is the same as the sign of $(m_B - m_b)$.

We now compute $f(0)$:

$$\begin{aligned} f(0) = 1 - (1-2m_b)&\frac{2(1+s)}{(1+s\bar{x})[1+s(1-\tilde{x})]} - m_b^2 \frac{(s+2)^2}{(1+s\bar{x})[1+s(1-\tilde{x})]} \\ &+ \frac{(1-2m_b)^2 (s+1)^2}{(1+s\bar{x})^2 [1+s(1-\tilde{x})]^2}. \end{aligned} \quad (S7.49)$$

Using the expression (S7.43) for the product of the two mean fitnesses, we get

$$\begin{aligned} (1+s\bar{x})^2 [1+s(1-\tilde{x})]^2 f(0) = &\{(1+s) + sm_B [(s+2)\bar{x} - 1]\}^2 \\ &- 2(1-2m_b)(s+1)\{(1+s) + sm_B [(s+2)\bar{x} - 1]\} \\ &- m_b^2 (s+2)\{(1+s) + sm_B [(s+2)\bar{x} - 1]\} \\ &+ (1-2m_b)^2 (s+1)^2 . \end{aligned} \quad (S7.50)$$

In (S7.50) we replace the $\bar{x}^2$ term using the equilibrium equation (36) to give

$$\begin{aligned} (1+s\bar{x})^2 [1+s(1-\tilde{x})]^2 f(0) = (m_B - m_b)s\Big\{&m_b s(s+1) \\ &- m_B^2 (s+2)^2 + m_B \Big[(s+1)(s+4) - m_b (s+2)^2\Big] \\ &+ m_B(s+2)\bar{x}\Big[m_B (s+2)^2 + m_b (s+2)^2 - 4(s+1)\Big]\Big\}. \end{aligned} \quad (S7.51)$$

The right-hand side of (S7.51) is $(m_B - m_b)s$ multiplied by

$$m_b s(s+1) + m_B (s+2)^2 (m_B + m_b)[\bar{x}(s+2) - 1] + m_B(s+1)[(s+4) - 4\bar{x}(s+2)]. \quad (S7.52)$$

We will show that (S7.52) is always positive. In fact, (S7.52) is equal to

$$\begin{aligned} m_b s(s+1) + m_B \cdot m_b (s+2)^2 \big[\bar{x}(s+2) - 1\big] + m_B^2 (s+2)^2 \big[\bar{x}(s+2) - 1\big] \\ + m_B(s+1)\big[(s+4) - 4\bar{x}(s+2)\big]. \end{aligned} \quad (S7.53)$$

From the equilibrium equation (36) we get that

$$m_B\big[(s+2)\bar{x} - 1\big] = s\bar{x}^2 + 2\bar{x} - 1. \quad (S7.54)$$

Hence (S7.53) is equal to

$$m_b s(s+1) + m_B \cdot m_b\,(s+2)^2\left[\bar{x}(s+2)-1\right]+ \\ + m_B\,(s+2)^2\left[s\bar{x}^2+2\bar{x}-1\right]+m_B(s+1)\big[(s+4)-4\bar{x}(s+2)\big]. \tag{S7.55}$$

The last two terms have a factor m_B that multiplies

$$\begin{aligned}(s+1)(s+4)-(s+2)^2+(s+2)^2\,\bar{x}(2+s\bar{x})-4\bar{x}(s+1)(s+2) &= \\ = s+(s+2)\bar{x}\big[(s+2)(2+s\bar{x})-4(s+1)\big] & \\ = s+(s+2)\bar{x}\big[(s+2)s\bar{x}-2s\big] & \\ = s\left[(s+2)^2\,\bar{x}^2-2(s+2)\bar{x}+1\right]=s\big[(s+2)\bar{x}-1\big]^2, &\end{aligned} \tag{S7.56}$$

which is positive. To sum up, $f(0)$ also has the same sign of $(m_B - m_b)$, and so

$$D_1(1) = (1-r)s(m_B - m_b)\Delta(r), \tag{S7.57}$$

where $\Delta(r)$ is a linear function of r that is positive for all $0 \le r \le 1$. As $(m_B - m_b) = (1-2m)(\mu_B - \mu_b)$, this proves the following result.