The effective migration rate experienced by a neutral site linked to two loci at migration-selection balance

We assume two loci under selection ($\mathcal{A}$ and $\mathcal{B}$) and a third neutral locus $\mathcal{C}$, each with two alleles denoted by A_1 and A_2, B_1 and B_2, and C_1 and C_2, respectively. Let the frequencies of alleles A_1, A_2, B_1, B_2, C_1 and C_3 be p, $1-p$, q, $1-q$, n and $1-n$, respectively. We denote the recombination rate between locus i and j by r_{ij} and consider a continuous-time model, such that second-order terms of recombination rates can be ignored.

We first consider the case where the neutral locus lies between the selected loci, *i.e.* $\mathcal{A}$–$\mathcal{C}$–$\mathcal{B}$. We denote the frequencies of the eight haplotypes $A_1 C_1 B_1$, $A_1 C_1 B_2$, $A_2 C_1 B_1$, $A_2 C_1 B_2$, $A_1 C_2 B_1$, $A_1 C_2 B_2$, $A_2 C_2 B_1$, and $A_2 C_2 B_2$ by y_1, ..., y_8, respectively. In this case, the recombination rate between locus $\mathcal{A}$ and $\mathcal{C}$ is r_{AC} and the one between locus $\mathcal{C}$ and $\mathcal{B}$ is r_{CB}. As mentioned, we ignore interaction terms, so that $r_{AB} = r_{AC} + r_{CB}$.

Later, we consider the case where the neutral locus lies to the right ($\mathcal{A}$–$\mathcal{B}$–$\mathcal{C}$) or to the left ($\mathcal{C}$–$\mathcal{A}$–$\mathcal{B}$) of the two selected loci.

As outlined by Bürger and Akerman (2011), the recursion equations due to selection and migration are immediately obtained from the two-locus migration-selection model (eq. 2.2 in Bürger and Akerman, 2011), because locus $\mathcal{C}$ is neutral.

In this context, we point out that equations (4.25) and (4.26) in Bürger and Akerman (2011) are wrong and need to be replaced by

Order of loci: $\mathcal{A}$-$\mathcal{C}$-$\mathcal{B}$

Rules and assumptions

In[1]:=
```
recScale := {r1 → ρ1 ϵ, r2 → ρ2 ϵ}
recBackScale := {ρ1 → r1 / ϵ, ρ2 → r2 / ϵ}
```

In[3]:=
```
pDef := y1 + y2 + y5 + y6
qDef := y1 + y3 + y5 + y7
nDef := y1 + y2 + y3 + y4
DACDef := (y1 + y2) (y7 + y8) - (y5 + y6) (y3 + y4)
DCBDef := (y1 + y3) (y6 + y8) - (y2 + y4) (y5 + y7)
DABDef := (y1 + y5) (y4 + y8) - (y2 + y6) (y3 + y7)
```

In[9]:=
```
DACBDef := y1 - p q n - p DCB - q DAC - n DAB /.
   {p → pDef, q → qDef, n → nDef, DCB → DCBDef, DAC → DACDef, DAB → DABDef}
```

In[10]:=
```
gamToAllLD := {
   y1 → p q n + p DCB + q DAC + n DAB + DACB,
   y2 → p (1 - q) n  - p DCB + (1 - q) DAC - n DAB - DACB,
   y3 → (1 - p) q n  + (1 - p) DCB - q DAC - n DAB - DACB,
   y4 → (1 - p) (1 - q) n  - (1 - p) DCB - (1 - q) DAC + n DAB + DACB,
   y5 →  p q (1 - n)  - p DCB - q DAC + (1 - n) DAB - DACB,
   y6 → p (1 - q) (1 - n) + p DCB - (1 - q) DAC - (1 - n) DAB + DACB,
   y7 → (1 - p) q (1 - n)  - (1 - p) DCB + q DAC - (1 - n) DAB + DACB,
   y8 → (1 - p) (1 - q) (1 - n)  + (1 - p) DCB + (1 - q) DAC + (1 - n) DAB - DACB
  }
```

In[11]:=
```
allToGam := {
  p → y1 + y2 + y5 + y6,
  (*1-p→y3+y4+y7+y8,*)
  q → y1 + y3 + y5 + y7,
  (*1-q→y2+y4+y6+y8,*)
  n → y1 + y2 + y3 + y4
 }
```

Derivation of recursion equations

▪ Relationships between LD coefficients, allele and gamete frequencies.

Before deriving the recursion equations, we recall some relationships between gamete frequencies, allele frequencies and linkage disequilibria:

The gamete frequencies of marginal two-locus models in terms of the gamete frequencies of the three-locus model:

$$x_1 := [A_1\, B_1] = y_1 + y_5 \quad (1)$$

$$x_2 := [A_1\, B_2] = y_2 + y_6 \quad (2)$$

$$x_3 := [A_2\, B_1] = y_3 + y_7 \quad (3)$$

$$x_4 := [A_2\, B_2] = y_4 + y_8 \quad (4)$$

$$x_5 := [A_1\, C_1] = y_1 + y_2 \quad (5)$$

$$x_6 := [A_1\, C_2] = y_5 + y_6 \quad (6)$$

$$x_7 := [A_2\, C_1] = y_3 + y_4 \quad (7)$$

$$x_8 := [A_2\, C_2] = y_7 + y_8$$

$$x_9 := [C_1\, B_1] = y_1 + y_3$$

$$x_{10} := [C_1\, B_2] = y_2 + y_4$$

$$x_{11} := [C_2\, B_1] = y_5 + y_7$$

$$x_{12} := [C_2\, B_2] = y_6 + y_8$$

The allele frequencies in terms of the gamete frequencies:

$$p = x_1 + x_2 = y_1 + y_2 + y_5 + y_6 \quad (8)$$

$$1 - p = x_3 + x_4 = y_3 + y_4 + y_7 + y_8 \quad (9)$$

$$q = x_1 + x_3 = y_1 + y_3 + y_5 + y_7 \quad (10)$$

$$1 - q = x_2 + x_4 = y_2 + y_4 + y_6 + y_8 \quad (11)$$

$$n = y_1 + y_2 + y_3 + y_4 \quad (12)$$

$$1 - n = y_5 + y_6 + y_7 + y_8 \quad (13)$$

The two-way linkage disequilibria in terms of allele and gamete frequencies:

$$D_{AB} = x_1 - p\, q = (y_1 + y_5) - p\, q = x_1\, x_4 - x_2\, x_3 = (y_1 + y_5)\,(y_4 + y_8) - (y_2 + y_6)\,(y_3 + y_7) \quad (14)$$

$$D_{AC} = [A_1\, C_1] - p\, n = (y_1 + y_2) - p\, n = (y_1 + y_2) - [(y_1 + y_2) + (y_5 + y_6)][(y_1 + y_2) + (y_3 + y_4)] = (y_1 + y_2)\,(y_7 + y_8) - (y_5 + y_6)\,(y_3 + y_4) = x_5\, x_8 - x_6\, x_7 \quad (15)$$

$$D_{CB} = [C_1\, B_1] - q\, n = (y_1 + y_3) - q\, n = (y_1 + y_3) - [(y_1 + y_3) + (y_5 + y_7)][(y_1 + y_3) + (y_2 + y_4)] = (y_1 + y_3)\,(y_6 + y_8) - (y_2 + y_4)\,(y_5 + y_7) = x_9\, x_{12} - x_{10}\, x_{11} \quad (16)$$

The three-way linkage disequilibrium in terms of gamete frequencies, allele frequencies and two-way linkage disequilibria:

$$D_{ACB} = y_1 - p\, q\, n - p\, D_{CB} - q\, D_{AC} - n\, D_{AB} = y_1 - p\, q\, n - p\,[(y_1 + y_3) - q\, n] - q\,[(y_1 + y_2) - p\, n] - n\,[(y_1 + y_5) - p\, q] \quad (17)$$

Some marginal two-locus gamete frequencies in terms of allele frequencies and two-way linkage disequilibria:

$$x_1 = p\, q + D_{AB} \quad (18)$$

$$x_2 = p(1-q) - D_{AB} \quad (19)$$

$$x_3 = (1-p)\,q - D_{AB} \quad (20)$$

$$x_4 = (1-p)\,(1-q) + D_{AB} \quad (21)$$

$$x_5 = p\,n\ + D_{AC} \quad (22)$$

$$x_6 = p(1-n) - D_{AC} \quad (23)$$

$$x_7 = (1-p)\,n - D_{AC} \quad (24)$$

$$x_8 = (1-p)\,(1-n) + D_{AC} \quad (25)$$

$$x_9 = n\,q\ + D_{CB} \quad (26)$$

$$x_{10} = n(1-q) - D_{CB} \quad (27)$$

$$x_{11} = (1-n)\,q - D_{CB} \quad (28)$$

$$x_{12} = (1-n)\,(1-q) + D_{CB} \quad (29)$$

The three-locus gamete frequencies in terms of allele frequencies and two-way linkage disequilibria:

$$y_1 = p\,q\,n + p\,D_{CB} + q\,D_{AC} + n\,D_{AB} + D_{ACB} \quad (30)$$

$$y_2 = p\,(1-q)\,n\ - p\,D_{CB} + (1-q)\,D_{AC} - n\,D_{AB} - D_{ACB} \quad (31)$$

$$y_3 = (1-p)\,q\,n\ + (1-p)\,D_{CB} - q\,D_{AC} - n\,D_{AB} - D_{ACB} \quad (32)$$

$$y_4 = (1-p)\,(1-q)\,n\ - (1-p)\,D_{CB} - (1-q)\,D_{AC} + n\,D_{AB} + D_{ACB} \quad (33)$$

$$y_5 = p\,q\,(1-n)\ - p\,D_{CB} - q\,D_{AC} + (1-n)\,D_{AB} - D_{ACB} \quad (34)$$

$$y_6 = p\,(1-q)\,(1-n)\ + p\,D_{CB} - (1-q)\,D_{AC} - (1-n)\,D_{AB} + D_{ACB} \quad (35)$$

$$y_7 = (1-p)\,q\,(1-n)\ - (1-p)\,D_{CB} + q\,D_{AC} - (1-n)\,D_{AB} + D_{ACB} \quad (36)$$

$$y_8 = (1-p)\,(1-q)\,(1-n)\ + (1-p)\,D_{CB} + (1-q)\,D_{AC} + (1-n)\,D_{AB} - D_{ACB} \quad (37)$$

The recursion equation for y_1 (gamete $A_1\,N_1\,B_1$) in discrete time:

- **Automatisation of constrution of difference equations under recombination**

In[12]:=
```
gametesProduced[parent1_, parent2_, parentFreq1_, parentFreq2_] :=
 Module[{tup, factors, pf1, pf2, freqDist, δ},
  pf1 = parentFreq1;
  pf2 = parentFreq2;
  δ = 1;
  factors = {1/2 (1 - r1) (1 - r2), 1/2 (1 - r1) r2, 1/2 r1 r2, 1/2 r1 (1 - r2),
     1/2 r1 (1 - r2), 1/2 r1 r2, 1/2 (1 - r1) r2, 1/2 (1 - r1) (1 - r2)} * δ * pf1 * pf2;
  tup = Tuples[Partition[Riffle[parent1, parent2], 2]];
  freqDist = Table[Sum[factors[[i]], {i, Flatten[Position[tup, Union[tup][[i]]]]}],
    {i, Length[Union[tup]]}];
  Return[{Union[tup], freqDist}];

 ]
```

In[13]:=
```
gametes = {{A1, C1, B1}, {A1, C1, B2}, {A2, C1, B1},
  {A2, C1, B2}, {A1, C2, B1}, {A1, C2, B2}, {A2, C2, B1}, {A2, C2, B2}}
```

Out[13]=
```
{{A1, C1, B1}, {A1, C1, B2}, {A2, C1, B1}, {A2, C1, B2},
 {A1, C2, B1}, {A1, C2, B2}, {A2, C2, B1}, {A2, C2, B2}}
```

In[14]:=
```
matings = Flatten[
  Table[{gametes[[i]], gametes[[j]]}, {i, 1, Length[gametes]}, {j, 1, Length[gametes]}], 1]
Length[
 %]
```

Out[14]=
```
{{{A1, C1, B1}, {A1, C1, B1}}, {{A1, C1, B1}, {A1, C1, B2}},
 {{A1, C1, B1}, {A2, C1, B1}}, {{A1, C1, B1}, {A2, C1, B2}},
 {{A1, C1, B1}, {A1, C2, B1}}, {{A1, C1, B1}, {A1, C2, B2}}, {{A1, C1, B1}, {A2, C2, B1}},
 {{A1, C1, B1}, {A2, C2, B2}}, {{A1, C1, B2}, {A1, C1, B1}}, {{A1, C1, B2}, {A1, C1, B2}},
 {{A1, C1, B2}, {A2, C1, B1}}, {{A1, C1, B2}, {A2, C1, B2}}, {{A1, C1, B2}, {A1, C2, B1}},
 {{A1, C1, B2}, {A1, C2, B2}}, {{A1, C1, B2}, {A2, C2, B1}}, {{A1, C1, B2}, {A2, C2, B2}},
 {{A2, C1, B1}, {A1, C1, B1}}, {{A2, C1, B1}, {A1, C1, B2}}, {{A2, C1, B1}, {A2, C1, B1}},
 {{A2, C1, B1}, {A2, C1, B2}}, {{A2, C1, B1}, {A1, C2, B1}}, {{A2, C1, B1}, {A1, C2, B2}},
 {{A2, C1, B1}, {A2, C2, B1}}, {{A2, C1, B1}, {A2, C2, B2}}, {{A2, C1, B2}, {A1, C1, B1}},
 {{A2, C1, B2}, {A1, C1, B2}}, {{A2, C1, B2}, {A2, C1, B1}}, {{A2, C1, B2}, {A2, C1, B2}},
 {{A2, C1, B2}, {A1, C2, B1}}, {{A2, C1, B2}, {A1, C2, B2}}, {{A2, C1, B2}, {A2, C2, B1}},
 {{A2, C1, B2}, {A2, C2, B2}}, {{A1, C2, B1}, {A1, C1, B1}}, {{A1, C2, B1}, {A1, C1, B2}},
 {{A1, C2, B1}, {A2, C1, B1}}, {{A1, C2, B1}, {A2, C1, B2}}, {{A1, C2, B1}, {A1, C2, B1}},
 {{A1, C2, B1}, {A1, C2, B2}}, {{A1, C2, B1}, {A2, C2, B1}}, {{A1, C2, B1}, {A2, C2, B2}},
 {{A1, C2, B2}, {A1, C1, B1}}, {{A1, C2, B2}, {A1, C1, B2}}, {{A1, C2, B2}, {A2, C1, B1}},
 {{A1, C2, B2}, {A2, C1, B2}}, {{A1, C2, B2}, {A1, C2, B1}}, {{A1, C2, B2}, {A1, C2, B2}},
 {{A1, C2, B2}, {A2, C2, B1}}, {{A1, C2, B2}, {A2, C2, B2}}, {{A2, C2, B1}, {A1, C1, B1}},
 {{A2, C2, B1}, {A1, C1, B2}}, {{A2, C2, B1}, {A2, C1, B1}}, {{A2, C2, B1}, {A2, C1, B2}},
 {{A2, C2, B1}, {A1, C2, B1}}, {{A2, C2, B1}, {A1, C2, B2}}, {{A2, C2, B1}, {A2, C2, B1}},
 {{A2, C2, B1}, {A2, C2, B2}}, {{A2, C2, B2}, {A1, C1, B1}}, {{A2, C2, B2}, {A1, C1, B2}},
 {{A2, C2, B2}, {A2, C1, B1}}, {{A2, C2, B2}, {A2, C1, B2}}, {{A2, C2, B2}, {A1, C2, B1}},
 {{A2, C2, B2}, {A1, C2, B2}}, {{A2, C2, B2}, {A2, C2, B1}}, {{A2, C2, B2}, {A2, C2, B2}}}
```

Out[15]= 64

In[16]:=
```
gameteFreqs = {y1, y2, y3, y4, y5, y6, y7, y8}
```

Out[16]=
```
{y1, y2, y3, y4, y5, y6, y7, y8}
```

In[17]:=
```
freqPairs = Flatten[Table[{gameteFreqs[[i]], gameteFreqs[[j]]},
   {i, 1, Length[gameteFreqs]}, {j, 1, Length[gameteFreqs]}], 1]
% //
 Length
```

Out[17]=
```
{{y1, y1}, {y1, y2}, {y1, y3}, {y1, y4}, {y1, y5}, {y1, y6}, {y1, y7}, {y1, y8},
 {y2, y1}, {y2, y2}, {y2, y3}, {y2, y4}, {y2, y5}, {y2, y6}, {y2, y7}, {y2, y8},
 {y3, y1}, {y3, y2}, {y3, y3}, {y3, y4}, {y3, y5}, {y3, y6}, {y3, y7}, {y3, y8},
 {y4, y1}, {y4, y2}, {y4, y3}, {y4, y4}, {y4, y5}, {y4, y6}, {y4, y7}, {y4, y8},
 {y5, y1}, {y5, y2}, {y5, y3}, {y5, y4}, {y5, y5}, {y5, y6}, {y5, y7}, {y5, y8},
 {y6, y1}, {y6, y2}, {y6, y3}, {y6, y4}, {y6, y5}, {y6, y6}, {y6, y7}, {y6, y8},
 {y7, y1}, {y7, y2}, {y7, y3}, {y7, y4}, {y7, y5}, {y7, y6}, {y7, y7}, {y7, y8},
 {y8, y1}, {y8, y2}, {y8, y3}, {y8, y4}, {y8, y5}, {y8, y6}, {y8, y7}, {y8, y8}}
```

Out[18]= 64

```
MapThread[gametesProduced[#1[[1]], #1[[2]], #2[[1]], #2[[2]]] &,
  {matings[[1 ;; 2]], freqPairs[[1 ;; 2]]}] // TableForm
```

A1 C1 B1 $\quad (1-r1)(1-r2)\,y1^2 + r1(1-r2)\,y1^2 + (1-r1)\,r2\,y1^2 + r1\,r2\,y1^2$

A1 C1 B1 $\quad \frac{1}{2}(1-r1)(1-r2)\,y1\,y2 + \frac{1}{2}r1(1-r2)\,y1\,y2 + \frac{1}{2}(1-r1)\,r2\,y1\,y2 + \frac{1}{2}r1\,r2\,y1\,y2$
A1 C1 B2 $\quad \frac{1}{2}(1-r1)(1-r2)\,y1\,y2 + \frac{1}{2}r1(1-r2)\,y1\,y2 + \frac{1}{2}(1-r1)\,r2\,y1\,y2 + \frac{1}{2}r1\,r2\,y1\,y2$

In[19]:=
```
recSep = MapThread[gametesProduced[#1[[1]], #1[[2]], #2[[1]], #2[[2]]] &, {matings, freqPairs}];
TableForm[recSep]
```

Out[20]//TableForm=

A1 C1 B1 $\quad (1-r1)(1-r2)\,y1^2 + r1(1-r2)\,y1^2 + (1-r1)\,r2\,y1^2 + r1\,r2\,y1^2$

A1 C1 B1 $\quad \frac{1}{2}(1-r1)(1-r2)\,y1\,y2 + \frac{1}{2}r1(1-r2)\,y1\,y2 + \frac{1}{2}(1-r1)\,r2\,y1\,y2 + \frac{1}{2}r1\,r2\,y1\,y2$
A1 C1 B2 $\quad \frac{1}{2}(1-r1)(1-r2)\,y1\,y2 + \frac{1}{2}r1(1-r2)\,y1\,y2 + \frac{1}{2}(1-r1)\,r2\,y1\,y2 + \frac{1}{2}r1\,r2\,y1\,y2$

A1 C1 B1 $\quad \frac{1}{2}(1-r1)(1-r2)\,y1\,y3 + \frac{1}{2}r1(1-r2)\,y1\,y3 + \frac{1}{2}(1-r1)\,r2\,y1\,y3 + \frac{1}{2}r1\,r2\,y1\,y3$
A2 C1 B1 $\quad \frac{1}{2}(1-r1)(1-r2)\,y1\,y3 + \frac{1}{2}r1(1-r2)\,y1\,y3 + \frac{1}{2}(1-r1)\,r2\,y1\,y3 + \frac{1}{2}r1\,r2\,y1\,y3$

A1 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y4+\frac{1}{2}\,r1\,r2\,y1\,y4$
A1 C1 B2	$\frac{1}{2}\,r1\,(1-r2)\,y1\,y4+\frac{1}{2}(1-r1)\,r2\,y1\,y4$
A2 C1 B1	$\frac{1}{2}\,r1\,(1-r2)\,y1\,y4+\frac{1}{2}(1-r1)\,r2\,y1\,y4$
A2 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y4+\frac{1}{2}\,r1\,r2\,y1\,y4$
A1 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y5+\frac{1}{2}\,r1\,(1-r2)\,y1\,y5+\frac{1}{2}(1-r1)\,r2\,y1\,y5+\frac{1}{2}\,r1\,r2\,y1\,y5$
A1 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y5+\frac{1}{2}\,r1\,(1-r2)\,y1\,y5+\frac{1}{2}(1-r1)\,r2\,y1\,y5+\frac{1}{2}\,r1\,r2\,y1\,y5$
A1 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y6+\frac{1}{2}\,r1\,(1-r2)\,y1\,y6$
A1 C1 B2	$\frac{1}{2}(1-r1)\,r2\,y1\,y6+\frac{1}{2}\,r1\,r2\,y1\,y6$
A1 C2 B1	$\frac{1}{2}(1-r1)\,r2\,y1\,y6+\frac{1}{2}\,r1\,r2\,y1\,y6$
A1 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y6+\frac{1}{2}\,r1\,(1-r2)\,y1\,y6$
A1 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y7+\frac{1}{2}(1-r1)\,r2\,y1\,y7$
A1 C2 B1	$\frac{1}{2}\,r1\,(1-r2)\,y1\,y7+\frac{1}{2}\,r1\,r2\,y1\,y7$
A2 C1 B1	$\frac{1}{2}\,r1\,(1-r2)\,y1\,y7+\frac{1}{2}\,r1\,r2\,y1\,y7$
A2 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y7+\frac{1}{2}(1-r1)\,r2\,y1\,y7$
A1 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y8$
A1 C1 B2	$\frac{1}{2}(1-r1)\,r2\,y1\,y8$
A1 C2 B1	$\frac{1}{2}\,r1\,r2\,y1\,y8$
A1 C2 B2	$\frac{1}{2}\,r1\,(1-r2)\,y1\,y8$
A2 C1 B1	$\frac{1}{2}\,r1\,(1-r2)\,y1\,y8$
A2 C1 B2	$\frac{1}{2}\,r1\,r2\,y1\,y8$
A2 C2 B1	$\frac{1}{2}(1-r1)\,r2\,y1\,y8$
A2 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y8$
A1 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y2+\frac{1}{2}\,r1\,(1-r2)\,y1\,y2+\frac{1}{2}(1-r1)\,r2\,y1\,y2+\frac{1}{2}\,r1\,r2\,y1\,y2$
A1 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y2+\frac{1}{2}\,r1\,(1-r2)\,y1\,y2+\frac{1}{2}(1-r1)\,r2\,y1\,y2+\frac{1}{2}\,r1\,r2\,y1\,y2$
A1 C1 B2	$(1-r1)(1-r2)\,y2^2+r1\,(1-r2)\,y2^2+(1-r1)\,r2\,y2^2+r1\,r2\,y2^2$
A1 C1 B1	$\frac{1}{2}\,r1\,(1-r2)\,y2\,y3+\frac{1}{2}(1-r1)\,r2\,y2\,y3$
A1 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y3+\frac{1}{2}\,r1\,r2\,y2\,y3$
A2 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y3+\frac{1}{2}\,r1\,r2\,y2\,y3$
A2 C1 B2	$\frac{1}{2}\,r1\,(1-r2)\,y2\,y3+\frac{1}{2}(1-r1)\,r2\,y2\,y3$
A1 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y4+\frac{1}{2}\,r1\,(1-r2)\,y2\,y4+\frac{1}{2}(1-r1)\,r2\,y2\,y4+\frac{1}{2}\,r1\,r2\,y2\,y4$
A2 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y4+\frac{1}{2}\,r1\,(1-r2)\,y2\,y4+\frac{1}{2}(1-r1)\,r2\,y2\,y4+\frac{1}{2}\,r1\,r2\,y2\,y4$
A1 C1 B1	$\frac{1}{2}(1-r1)\,r2\,y2\,y5+\frac{1}{2}\,r1\,r2\,y2\,y5$
A1 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y5+\frac{1}{2}\,r1\,(1-r2)\,y2\,y5$
A1 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y5+\frac{1}{2}\,r1\,(1-r2)\,y2\,y5$
A1 C2 B2	$\frac{1}{2}(1-r1)\,r2\,y2\,y5+\frac{1}{2}\,r1\,r2\,y2\,y5$
A1 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y6+\frac{1}{2}\,r1\,(1-r2)\,y2\,y6+\frac{1}{2}(1-r1)\,r2\,y2\,y6+\frac{1}{2}\,r1\,r2\,y2\,y6$
A1 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y6+\frac{1}{2}\,r1\,(1-r2)\,y2\,y6+\frac{1}{2}(1-r1)\,r2\,y2\,y6+\frac{1}{2}\,r1\,r2\,y2\,y6$

A1 C1 B1	$\frac{1}{2}(1-r1)\,r2\,y2\,y7$
A1 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y7$
A1 C2 B1	$\frac{1}{2}r1\,(1-r2)\,y2\,y7$
A1 C2 B2	$\frac{1}{2}r1\,r2\,y2\,y7$
A2 C1 B1	$\frac{1}{2}r1\,r2\,y2\,y7$
A2 C1 B2	$\frac{1}{2}r1\,(1-r2)\,y2\,y7$
A2 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y7$
A2 C2 B2	$\frac{1}{2}(1-r1)\,r2\,y2\,y7$

A1 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y8+\frac{1}{2}(1-r1)\,r2\,y2\,y8$
A1 C2 B2	$\frac{1}{2}r1\,(1-r2)\,y2\,y8+\frac{1}{2}r1\,r2\,y2\,y8$
A2 C1 B2	$\frac{1}{2}r1\,(1-r2)\,y2\,y8+\frac{1}{2}r1\,r2\,y2\,y8$
A2 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y8+\frac{1}{2}(1-r1)\,r2\,y2\,y8$

A1 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y3+\frac{1}{2}r1\,(1-r2)\,y1\,y3+\frac{1}{2}(1-r1)\,r2\,y1\,y3+\frac{1}{2}r1\,r2\,y1\,y3$
A2 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y3+\frac{1}{2}r1\,(1-r2)\,y1\,y3+\frac{1}{2}(1-r1)\,r2\,y1\,y3+\frac{1}{2}r1\,r2\,y1\,y3$

A1 C1 B1	$\frac{1}{2}r1\,(1-r2)\,y2\,y3+\frac{1}{2}(1-r1)\,r2\,y2\,y3$
A1 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y3+\frac{1}{2}r1\,r2\,y2\,y3$
A2 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y3+\frac{1}{2}r1\,r2\,y2\,y3$
A2 C1 B2	$\frac{1}{2}r1\,(1-r2)\,y2\,y3+\frac{1}{2}(1-r1)\,r2\,y2\,y3$

A2 C1 B1	$(1-r1)(1-r2)\,y3^2+r1\,(1-r2)\,y3^2+(1-r1)\,r2\,y3^2+r1\,r2\,y3^2$

A2 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y4+\frac{1}{2}r1\,(1-r2)\,y3\,y4+\frac{1}{2}(1-r1)\,r2\,y3\,y4+\frac{1}{2}r1\,r2\,y3\,y4$
A2 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y4+\frac{1}{2}r1\,(1-r2)\,y3\,y4+\frac{1}{2}(1-r1)\,r2\,y3\,y4+\frac{1}{2}r1\,r2\,y3\,y4$

A1 C1 B1	$\frac{1}{2}r1\,(1-r2)\,y3\,y5+\frac{1}{2}r1\,r2\,y3\,y5$
A1 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y5+\frac{1}{2}(1-r1)\,r2\,y3\,y5$
A2 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y5+\frac{1}{2}(1-r1)\,r2\,y3\,y5$
A2 C2 B1	$\frac{1}{2}r1\,(1-r2)\,y3\,y5+\frac{1}{2}r1\,r2\,y3\,y5$

A1 C1 B1	$\frac{1}{2}r1\,(1-r2)\,y3\,y6$
A1 C1 B2	$\frac{1}{2}r1\,r2\,y3\,y6$
A1 C2 B1	$\frac{1}{2}(1-r1)\,r2\,y3\,y6$
A1 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y6$
A2 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y6$
A2 C1 B2	$\frac{1}{2}(1-r1)\,r2\,y3\,y6$
A2 C2 B1	$\frac{1}{2}r1\,r2\,y3\,y6$
A2 C2 B2	$\frac{1}{2}r1\,(1-r2)\,y3\,y6$

A2 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y7+\frac{1}{2}r1\,(1-r2)\,y3\,y7+\frac{1}{2}(1-r1)\,r2\,y3\,y7+\frac{1}{2}r1\,r2\,y3\,y7$
A2 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y7+\frac{1}{2}r1\,(1-r2)\,y3\,y7+\frac{1}{2}(1-r1)\,r2\,y3\,y7+\frac{1}{2}r1\,r2\,y3\,y7$

A2 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y8+\frac{1}{2}r1\,(1-r2)\,y3\,y8$
A2 C1 B2	$\frac{1}{2}(1-r1)\,r2\,y3\,y8+\frac{1}{2}r1\,r2\,y3\,y8$
A2 C2 B1	$\frac{1}{2}(1-r1)\,r2\,y3\,y8+\frac{1}{2}r1\,r2\,y3\,y8$
A2 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y8+\frac{1}{2}r1\,(1-r2)\,y3\,y8$

A1 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y4+\frac{1}{2}r1\,r2\,y1\,y4$
A1 C1 B2	$\frac{1}{2}r1\,(1-r2)\,y1\,y4+\frac{1}{2}(1-r1)\,r2\,y1\,y4$
A2 C1 B1	$\frac{1}{2}r1\,(1-r2)\,y1\,y4+\frac{1}{2}(1-r1)\,r2\,y1\,y4$
A2 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y4+\frac{1}{2}r1\,r2\,y1\,y4$

A1 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y4+\frac{1}{2}r1(1-r2)\,y2\,y4+\frac{1}{2}(1-r1)\,r2\,y2\,y4+\frac{1}{2}r1\,r2\,y2\,y4$
A2 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y4+\frac{1}{2}r1(1-r2)\,y2\,y4+\frac{1}{2}(1-r1)\,r2\,y2\,y4+\frac{1}{2}r1\,r2\,y2\,y4$
A2 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y4+\frac{1}{2}r1(1-r2)\,y3\,y4+\frac{1}{2}(1-r1)\,r2\,y3\,y4+\frac{1}{2}r1\,r2\,y3\,y4$
A2 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y4+\frac{1}{2}r1(1-r2)\,y3\,y4+\frac{1}{2}(1-r1)\,r2\,y3\,y4+\frac{1}{2}r1\,r2\,y3\,y4$
A2 C1 B2	$(1-r1)(1-r2)\,y4^2+r1(1-r2)\,y4^2+(1-r1)\,r2\,y4^2+r1\,r2\,y4^2$
A1 C1 B1	$\frac{1}{2}r1\,r2\,y4\,y5$
A1 C1 B2	$\frac{1}{2}r1(1-r2)\,y4\,y5$
A1 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y5$
A1 C2 B2	$\frac{1}{2}(1-r1)\,r2\,y4\,y5$
A2 C1 B1	$\frac{1}{2}(1-r1)\,r2\,y4\,y5$
A2 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y5$
A2 C2 B1	$\frac{1}{2}r1(1-r2)\,y4\,y5$
A2 C2 B2	$\frac{1}{2}r1\,r2\,y4\,y5$
A1 C1 B2	$\frac{1}{2}r1(1-r2)\,y4\,y6+\frac{1}{2}r1\,r2\,y4\,y6$
A1 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y6+\frac{1}{2}(1-r1)\,r2\,y4\,y6$
A2 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y6+\frac{1}{2}(1-r1)\,r2\,y4\,y6$
A2 C2 B2	$\frac{1}{2}r1(1-r2)\,y4\,y6+\frac{1}{2}r1\,r2\,y4\,y6$
A2 C1 B1	$\frac{1}{2}(1-r1)\,r2\,y4\,y7+\frac{1}{2}r1\,r2\,y4\,y7$
A2 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y7+\frac{1}{2}r1(1-r2)\,y4\,y7$
A2 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y7+\frac{1}{2}r1(1-r2)\,y4\,y7$
A2 C2 B2	$\frac{1}{2}(1-r1)\,r2\,y4\,y7+\frac{1}{2}r1\,r2\,y4\,y7$
A2 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y8+\frac{1}{2}r1(1-r2)\,y4\,y8+\frac{1}{2}(1-r1)\,r2\,y4\,y8+\frac{1}{2}r1\,r2\,y4\,y8$
A2 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y8+\frac{1}{2}r1(1-r2)\,y4\,y8+\frac{1}{2}(1-r1)\,r2\,y4\,y8+\frac{1}{2}r1\,r2\,y4\,y8$
A1 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y5+\frac{1}{2}r1(1-r2)\,y1\,y5+\frac{1}{2}(1-r1)\,r2\,y1\,y5+\frac{1}{2}r1\,r2\,y1\,y5$
A1 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y5+\frac{1}{2}r1(1-r2)\,y1\,y5+\frac{1}{2}(1-r1)\,r2\,y1\,y5+\frac{1}{2}r1\,r2\,y1\,y5$
A1 C1 B1	$\frac{1}{2}(1-r1)\,r2\,y2\,y5+\frac{1}{2}r1\,r2\,y2\,y5$
A1 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y5+\frac{1}{2}r1(1-r2)\,y2\,y5$
A1 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y5+\frac{1}{2}r1(1-r2)\,y2\,y5$
A1 C2 B2	$\frac{1}{2}(1-r1)\,r2\,y2\,y5+\frac{1}{2}r1\,r2\,y2\,y5$
A1 C1 B1	$\frac{1}{2}r1(1-r2)\,y3\,y5+\frac{1}{2}r1\,r2\,y3\,y5$
A1 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y5+\frac{1}{2}(1-r1)\,r2\,y3\,y5$
A2 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y5+\frac{1}{2}(1-r1)\,r2\,y3\,y5$
A2 C2 B1	$\frac{1}{2}r1(1-r2)\,y3\,y5+\frac{1}{2}r1\,r2\,y3\,y5$
A1 C1 B1	$\frac{1}{2}r1\,r2\,y4\,y5$
A1 C1 B2	$\frac{1}{2}r1(1-r2)\,y4\,y5$
A1 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y5$
A1 C2 B2	$\frac{1}{2}(1-r1)\,r2\,y4\,y5$
A2 C1 B1	$\frac{1}{2}(1-r1)\,r2\,y4\,y5$
A2 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y5$
A2 C2 B1	$\frac{1}{2}r1(1-r2)\,y4\,y5$
A2 C2 B2	$\frac{1}{2}r1\,r2\,y4\,y5$
A1 C2 B1	$(1-r1)(1-r2)\,y5^2+r1(1-r2)\,y5^2+(1-r1)\,r2\,y5^2+r1\,r2\,y5^2$
A1 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y5\,y6+\frac{1}{2}r1(1-r2)\,y5\,y6+\frac{1}{2}(1-r1)\,r2\,y5\,y6+\frac{1}{2}r1\,r2\,y5\,y6$
A1 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y5\,y6+\frac{1}{2}r1(1-r2)\,y5\,y6+\frac{1}{2}(1-r1)\,r2\,y5\,y6+\frac{1}{2}r1\,r2\,y5\,y6$

A1 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y5\,y7+\frac{1}{2}r1(1-r2)\,y5\,y7+\frac{1}{2}(1-r1)\,r2\,y5\,y7+\frac{1}{2}r1\,r2\,y5\,y7$
A2 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y5\,y7+\frac{1}{2}r1(1-r2)\,y5\,y7+\frac{1}{2}(1-r1)\,r2\,y5\,y7+\frac{1}{2}r1\,r2\,y5\,y7$
A1 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y5\,y8+\frac{1}{2}r1\,r2\,y5\,y8$
A1 C2 B2	$\frac{1}{2}r1(1-r2)\,y5\,y8+\frac{1}{2}(1-r1)\,r2\,y5\,y8$
A2 C2 B1	$\frac{1}{2}r1(1-r2)\,y5\,y8+\frac{1}{2}(1-r1)\,r2\,y5\,y8$
A2 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y5\,y8+\frac{1}{2}r1\,r2\,y5\,y8$
A1 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y6+\frac{1}{2}r1(1-r2)\,y1\,y6$
A1 C1 B2	$\frac{1}{2}(1-r1)\,r2\,y1\,y6+\frac{1}{2}r1\,r2\,y1\,y6$
A1 C2 B1	$\frac{1}{2}(1-r1)\,r2\,y1\,y6+\frac{1}{2}r1\,r2\,y1\,y6$
A1 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y6+\frac{1}{2}r1(1-r2)\,y1\,y6$
A1 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y6+\frac{1}{2}r1(1-r2)\,y2\,y6+\frac{1}{2}(1-r1)\,r2\,y2\,y6+\frac{1}{2}r1\,r2\,y2\,y6$
A1 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y6+\frac{1}{2}r1(1-r2)\,y2\,y6+\frac{1}{2}(1-r1)\,r2\,y2\,y6+\frac{1}{2}r1\,r2\,y2\,y6$
A1 C1 B1	$\frac{1}{2}r1(1-r2)\,y3\,y6$
A1 C1 B2	$\frac{1}{2}r1\,r2\,y3\,y6$
A1 C2 B1	$\frac{1}{2}(1-r1)\,r2\,y3\,y6$
A1 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y6$
A2 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y6$
A2 C1 B2	$\frac{1}{2}(1-r1)\,r2\,y3\,y6$
A2 C2 B1	$\frac{1}{2}r1\,r2\,y3\,y6$
A2 C2 B2	$\frac{1}{2}r1(1-r2)\,y3\,y6$
A1 C1 B2	$\frac{1}{2}r1(1-r2)\,y4\,y6+\frac{1}{2}r1\,r2\,y4\,y6$
A1 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y6+\frac{1}{2}(1-r1)\,r2\,y4\,y6$
A2 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y6+\frac{1}{2}(1-r1)\,r2\,y4\,y6$
A2 C2 B2	$\frac{1}{2}r1(1-r2)\,y4\,y6+\frac{1}{2}r1\,r2\,y4\,y6$
A1 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y5\,y6+\frac{1}{2}r1(1-r2)\,y5\,y6+\frac{1}{2}(1-r1)\,r2\,y5\,y6+\frac{1}{2}r1\,r2\,y5\,y6$
A1 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y5\,y6+\frac{1}{2}r1(1-r2)\,y5\,y6+\frac{1}{2}(1-r1)\,r2\,y5\,y6+\frac{1}{2}r1\,r2\,y5\,y6$
A1 C2 B2	$(1-r1)(1-r2)\,y6^2+r1(1-r2)\,y6^2+(1-r1)\,r2\,y6^2+r1\,r2\,y6^2$
A1 C2 B1	$\frac{1}{2}r1(1-r2)\,y6\,y7+\frac{1}{2}(1-r1)\,r2\,y6\,y7$
A1 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y6\,y7+\frac{1}{2}r1\,r2\,y6\,y7$
A2 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y6\,y7+\frac{1}{2}r1\,r2\,y6\,y7$
A2 C2 B2	$\frac{1}{2}r1(1-r2)\,y6\,y7+\frac{1}{2}(1-r1)\,r2\,y6\,y7$
A1 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y6\,y8+\frac{1}{2}r1(1-r2)\,y6\,y8+\frac{1}{2}(1-r1)\,r2\,y6\,y8+\frac{1}{2}r1\,r2\,y6\,y8$
A2 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y6\,y8+\frac{1}{2}r1(1-r2)\,y6\,y8+\frac{1}{2}(1-r1)\,r2\,y6\,y8+\frac{1}{2}r1\,r2\,y6\,y8$
A1 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y7+\frac{1}{2}(1-r1)\,r2\,y1\,y7$
A1 C2 B1	$\frac{1}{2}r1(1-r2)\,y1\,y7+\frac{1}{2}r1\,r2\,y1\,y7$
A2 C1 B1	$\frac{1}{2}r1(1-r2)\,y1\,y7+\frac{1}{2}r1\,r2\,y1\,y7$
A2 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y7+\frac{1}{2}(1-r1)\,r2\,y1\,y7$

A1 C1 B1	$\frac{1}{2}(1-r1)\,r2\,y2\,y7$
A1 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y7$
A1 C2 B1	$\frac{1}{2}r1\,(1-r2)\,y2\,y7$
A1 C2 B2	$\frac{1}{2}r1\,r2\,y2\,y7$
A2 C1 B1	$\frac{1}{2}r1\,r2\,y2\,y7$
A2 C1 B2	$\frac{1}{2}r1\,(1-r2)\,y2\,y7$
A2 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y7$
A2 C2 B2	$\frac{1}{2}(1-r1)\,r2\,y2\,y7$
A2 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y7+\frac{1}{2}r1\,(1-r2)\,y3\,y7+\frac{1}{2}(1-r1)\,r2\,y3\,y7+\frac{1}{2}r1\,r2\,y3\,y7$
A2 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y7+\frac{1}{2}r1\,(1-r2)\,y3\,y7+\frac{1}{2}(1-r1)\,r2\,y3\,y7+\frac{1}{2}r1\,r2\,y3\,y7$
A2 C1 B1	$\frac{1}{2}(1-r1)\,r2\,y4\,y7+\frac{1}{2}r1\,r2\,y4\,y7$
A2 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y7+\frac{1}{2}r1\,(1-r2)\,y4\,y7$
A2 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y7+\frac{1}{2}r1\,(1-r2)\,y4\,y7$
A2 C2 B2	$\frac{1}{2}(1-r1)\,r2\,y4\,y7+\frac{1}{2}r1\,r2\,y4\,y7$
A1 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y5\,y7+\frac{1}{2}r1\,(1-r2)\,y5\,y7+\frac{1}{2}(1-r1)\,r2\,y5\,y7+\frac{1}{2}r1\,r2\,y5\,y7$
A2 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y5\,y7+\frac{1}{2}r1\,(1-r2)\,y5\,y7+\frac{1}{2}(1-r1)\,r2\,y5\,y7+\frac{1}{2}r1\,r2\,y5\,y7$
A1 C2 B1	$\frac{1}{2}r1\,(1-r2)\,y6\,y7+\frac{1}{2}(1-r1)\,r2\,y6\,y7$
A1 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y6\,y7+\frac{1}{2}r1\,r2\,y6\,y7$
A2 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y6\,y7+\frac{1}{2}r1\,r2\,y6\,y7$
A2 C2 B2	$\frac{1}{2}r1\,(1-r2)\,y6\,y7+\frac{1}{2}(1-r1)\,r2\,y6\,y7$
A2 C2 B1	$(1-r1)(1-r2)\,y7^2+r1\,(1-r2)\,y7^2+(1-r1)\,r2\,y7^2+r1\,r2\,y7^2$
A2 C2 B1	$\frac{1}{2}(1-r1)(1-r2)\,y7\,y8+\frac{1}{2}r1\,(1-r2)\,y7\,y8+\frac{1}{2}(1-r1)\,r2\,y7\,y8+\frac{1}{2}r1\,r2\,y7\,y8$
A2 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y7\,y8+\frac{1}{2}r1\,(1-r2)\,y7\,y8+\frac{1}{2}(1-r1)\,r2\,y7\,y8+\frac{1}{2}r1\,r2\,y7\,y8$
A1 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y8$
A1 C1 B2	$\frac{1}{2}(1-r1)\,r2\,y1\,y8$
A1 C2 B1	$\frac{1}{2}r1\,r2\,y1\,y8$
A1 C2 B2	$\frac{1}{2}r1\,(1-r2)\,y1\,y8$
A2 C1 B1	$\frac{1}{2}r1\,(1-r2)\,y1\,y8$
A2 C1 B2	$\frac{1}{2}r1\,r2\,y1\,y8$
A2 C2 B1	$\frac{1}{2}(1-r1)\,r2\,y1\,y8$
A2 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y1\,y8$
A1 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y8+\frac{1}{2}(1-r1)\,r2\,y2\,y8$
A1 C2 B2	$\frac{1}{2}r1\,(1-r2)\,y2\,y8+\frac{1}{2}r1\,r2\,y2\,y8$
A2 C1 B2	$\frac{1}{2}r1\,(1-r2)\,y2\,y8+\frac{1}{2}r1\,r2\,y2\,y8$
A2 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y2\,y8+\frac{1}{2}(1-r1)\,r2\,y2\,y8$
A2 C1 B1	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y8+\frac{1}{2}r1\,(1-r2)\,y3\,y8$
A2 C1 B2	$\frac{1}{2}(1-r1)\,r2\,y3\,y8+\frac{1}{2}r1\,r2\,y3\,y8$
A2 C2 B1	$\frac{1}{2}(1-r1)\,r2\,y3\,y8+\frac{1}{2}r1\,r2\,y3\,y8$
A2 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y3\,y8+\frac{1}{2}r1\,(1-r2)\,y3\,y8$
A2 C1 B2	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y8+\frac{1}{2}r1\,(1-r2)\,y4\,y8+\frac{1}{2}(1-r1)\,r2\,y4\,y8+\frac{1}{2}r1\,r2\,y4\,y8$
A2 C2 B2	$\frac{1}{2}(1-r1)(1-r2)\,y4\,y8+\frac{1}{2}r1\,(1-r2)\,y4\,y8+\frac{1}{2}(1-r1)\,r2\,y4\,y8+\frac{1}{2}r1\,r2\,y4\,y8$

```
                  1/2 (1 - r1) (1 - r2) y5 y8 + 1/2 r1 r2 y5 y8
A1  C2  B1        1/2 r1 (1 - r2) y5 y8 + 1/2 (1 - r1) r2 y5 y8
A1  C2  B2        1/2 r1 (1 - r2) y5 y8 + 1/2 (1 - r1) r2 y5 y8
A2  C2  B1        1/2 (1 - r1) (1 - r2) y5 y8 + 1/2 r1 r2 y5 y8
A2  C2  B2

A1  C2  B2        1/2 (1 - r1) (1 - r2) y6 y8 + 1/2 r1 (1 - r2) y6 y8 + 1/2 (1 - r1) r2 y6 y8 + 1/2 r1 r2 y6 y8
A2  C2  B2        1/2 (1 - r1) (1 - r2) y6 y8 + 1/2 r1 (1 - r2) y6 y8 + 1/2 (1 - r1) r2 y6 y8 + 1/2 r1 r2 y6 y8

A2  C2  B1        1/2 (1 - r1) (1 - r2) y7 y8 + 1/2 r1 (1 - r2) y7 y8 + 1/2 (1 - r1) r2 y7 y8 + 1/2 r1 r2 y7 y8
A2  C2  B2        1/2 (1 - r1) (1 - r2) y7 y8 + 1/2 r1 (1 - r2) y7 y8 + 1/2 (1 - r1) r2 y7 y8 + 1/2 r1 r2 y7 y8

A2  C2  B2        (1 - r1) (1 - r2) y8^2 + r1 (1 - r2) y8^2 + (1 - r1) r2 y8^2 + r1 r2 y8^2
```

```
recSep[[1, 1]]
recSep[[1, 2]]
```

{{A1, C1, B1}}

$\{(1-r1)\,(1-r2)\,y1^2 + r1\,(1-r2)\,y1^2 + (1-r1)\,r2\,y1^2 + r1\,r2\,y1^2\}$

```
indices = Table[Position[recSep[[i, 1]], gametes[[2]]], {i, Length[recSep]}]
```

{{}, {{2}}, {}, {{2}}, {}, {{2}}, {}, {{2}}, {{2}}, {{1}}, {{2}}, {{1}}, {{2}}, {{1}}, {{2}}, {{1}}, {}, {{2}}, {}, {}, {}, {{2}}, {}, {}, {{2}}, {{1}}, {}, {}, {{2}}, {{1}}, {}, {}, {}, {{2}}, {}, {{2}}, {}, {}, {}, {}, {{2}}, {{1}}, {{2}}, {{1}}, {}, {}, {}, {}, {}, {{2}}, {}, {}, {}, {}, {}, {}, {{2}}, {{1}}, {}, {}, {}, {}, {}, {}}

```
Flatten[Table[Part[recSep, i, 2][[Flatten[indices[[i]]]]], {i, Length[indices]}]] //
  Total // FullSimplify
```

$y2^2 + r1\,(r2\,y3\,y6 + y4\,(y5 - r2\,y5 + y6)) + y2\,((1 - r2 + r1\,(-1 + 2\,r2))\,y3 + y4 + y5 - r2\,y5 + y6 + (-1 + r1)\,((-1 + r2)\,y7 - y8)) + y1\,(y2 + r2\,(y4 + y6 + y8) + r1\,(y4 - 2\,r2\,y4 - r2\,y8))$

In[21]:=

```
sumPerOffspringGamete[gametes_, gameteIndex_, separateRecursions_] :=
 Module[{indices, recSep, sum},
  recSep = separateRecursions;
  indices =
   Table[Position[recSep[[i, 1]], gametes[[gameteIndex]]], {i, Length[recSep]}];
  sum = Flatten[Table[Part[recSep, i, 2][[Flatten[indices[[i]]]]],
      {i, Length[indices]}]] // Total;
  Return[sum]
 ]
```

- **Differential equations under recombination**
- In terms of gamete frequencies

```
rec1 = sumPerOffspringGamete[gametes, 1, recSep] // FullSimplify
```

$y1^2 + r2\,y2\,(y3 + y5 + y7) + r1\,(r2\,y4\,y5 + y3\,(y5 + y6 - r2\,y6) + y2\,(y3 - 2\,r2\,y3 - r2\,y7)) + y1\,(y2 + y3 + (1 - r2 + r1\,(-1 + 2\,r2))\,y4 + y5 + y6 - r2\,y6 - (-1 + r1)\,(y7 + y8 - r2\,y8))$

The corresponding continuous-time differential equation:

```
y1D = FullSimplify[Series[rec1 - y1 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]
```

$y1^2 + r1\,y3\,(y2 + y5 + y6) + r2\,y2\,(y3 + y5 + y7) + y1\,(-1 + y2 + y3 - (-1 + r1 + r2)\,y4 + y5 + y6 + y7 + y8 - r2\,(y6 + y8) - r1\,(y7 + y8))$

```
Collect[y1D, {r1, r2}]
```

$-y1 + y1^2 + y1\,y2 + y1\,y3 + y1\,y4 + y1\,y5 + y1\,y6 + y1\,y7 + y1\,y8 + r2\,(-y1\,y4 + y2\,(y3 + y5 + y7) - y1\,(y6 + y8)) + r1\,(-y1\,y4 + y3\,(y2 + y5 + y6) - y1\,(y7 + y8))$

```
FullSimplify[(-y1 + y1^2 + y1 y2 + y1 y3 + y1 y4 + y1 y5 + y1 y6 + y1 y7 + y1 y8),
 Assumptions → {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]
```

0

We note that 'y1D' can be simplified to

In[22]:=
```
y1DotRec := -r1 (y1 (1 - p) - y3 p) - r2 (y1 (1 - q) - y2 q)
```

```
y1DotRec - y1D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7} //.
  {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify
```

```
0
```

The following test term is from an independent derivation of Reinhard Bürger (personal communication).

```
testTerm1 := r2 (y[2] (y[3] + y[5] + y[7]) - y[1] (y[4] + y[6] + y[8])) +
    r1 (y[3] (y[2] + y[5] + y[6]) - y[1] (y[4] + y[7] + y[8])) /.
  {y[1] → y1, y[2] → y2, y[3] → y3, y[4] → y4, y[5] → y5, y[6] → y6, y[7] → y7, y[8] → y8}
```

```
r2 (-y1 y4 + y2 (y3 + y5 + y7) - y1 (y6 + y8)) +
  r1 (-y1 y4 + y3 (y2 + y5 + y6) - y1 (y7 + y8)) - testTerm1 // Simplify
```

```
0
```

```
rec2 = sumPerOffspringGamete[gametes, 2, recSep] // FullSimplify
```

$y2^2 + r1\,(r2\,y3\,y6 + y4\,(y5 - r2\,y5 + y6)) + y2\,((1 - r2 + r1\,(-1 + 2\,r2))\,y3 + y4 + y5 - r2\,y5 + y6 + (-1 + r1)\,((-1 + r2)\,y7 - y8)) + y1\,(y2 + r2\,(y4 + y6 + y8) + r1\,(y4 - 2\,r2\,y4 - r2\,y8))$

```
y2D = FullSimplify[Series[rec2 - y2 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]
```

$y2^2 + r1\,y4\,(y1 + y5 + y6) + r2\,y1\,(y4 + y6 + y8) + y2\,(-1 + y1 - (-1 + r1 + r2)\,y3 + y4 + y5 + y6 + y7 - r2\,(y5 + y7) + y8 - r1\,(y7 + y8))$

```
Collect[y2D, {r1, r2}]
```

$-y2 + y1\,y2 + y2^2 + y2\,y3 + y2\,y4 + y2\,y5 + y2\,y6 + y2\,y7 + y2\,y8 + r2\,(-y2\,y3 - y2\,(y5 + y7) + y1\,(y4 + y6 + y8)) + r1\,(-y2\,y3 + y4\,(y1 + y5 + y6) - y2\,(y7 + y8))$

```
FullSimplify[(-y2 + y1 y2 + y2^2 + y2 y3 + y2 y4 + y2 y5 + y2 y6 + y2 y7 + y2 y8),
 Assumptions → {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]
```

```
0
```

In[23]:=
```
y2DotRec := -r1 (y2 (1 - p) - y4 p) - r2 (y2 q - y1 (1 - q))
```

```
y2DotRec - y2D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7} //.
  {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify
```

```
0
```

The following test term is from an independent derivation of Reinhard Bürger (personal communication).

```
testTerm2 := r2 (-y[2] (y[3] + y[5] + y[7]) + y[1] (y[4] + y[6] + y[8])) +
    r1 (y[4] (y[1] + y[5] + y[6]) - y[2] (y[3] + y[7] + y[8])) /.
  {y[1] → y1, y[2] → y2, y[3] → y3, y[4] → y4, y[5] → y5, y[6] → y6, y[7] → y7, y[8] → y8}
```

```
r2 (-y2 y3 - y2 (y5 + y7) + y1 (y4 + y6 + y8)) +
  r1 (-y2 y3 + y4 (y1 + y5 + y6) - y2 (y7 + y8)) - testTerm2 // Simplify
```

```
0
```

```
rec3 = sumPerOffspringGamete[gametes, 3, recSep] // FullSimplify
```

$y3^2 + y3\,y4 + y3\,y5 - r1\,y3\,y5 + r2\,y4\,y5 - r1\,r2\,y4\,y5 + y3\,y6 - r1\,y3\,y6 - r2\,y3\,y6 + r1\,r2\,y3\,y6 + y3\,y7 + r2\,y4\,y7 + y2\,(y3 - r1\,y3 - r2\,y3 + 2\,r1\,r2\,y3 + r1\,r2\,y7) + y3\,y8 - r2\,y3\,y8 + y1\,(y3 + r2\,y4 + r1\,(y4 - 2\,r2\,y4 + y7 + y8 - r2\,y8))$

```
y3D = FullSimplify[Series[rec3 - y3 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]
```

$y3^2 + r2\,y4\,(y1 + y5 + y7) + r1\,y1\,(y4 + y7 + y8) + y3\,(-1 + y1 - (-1 + r1 + r2)\,y2 + y4 + y5 + y6 - r1\,(y5 + y6) + y7 + y8 - r2\,(y6 + y8))$

```
Collect[y3D, {r1, r2}]

-y3 + y1 y3 + y2 y3 + y3^2 + y3 y4 + y3 y5 + y3 y6 + y3 y7 + y3 y8 +
 r2 (-y2 y3 + y4 (y1 + y5 + y7) - y3 (y6 + y8)) + r1 (-y2 y3 - y3 (y5 + y6) + y1 (y4 + y7 + y8))

FullSimplify[(-y3 + y1 y3 + y2 y3 + y3^2 + y3 y4 + y3 y5 + y3 y6 + y3 y7 + y3 y8),
 Assumptions → {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]

0

In[24]:= y3DotRec := -r1 (y3 p - y1 (1 - p)) - r2 (y3 (1 - q) - y4 q)

y3DotRec - y3D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7} //.
  {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify

0
```

The following test term is from an independent derivation of Reinhard Bürger (personal communication).

```
testTerm3 := r2 (y[4] (y[1] + y[5] + y[7]) - y[3] (y[2] + y[6] + y[8])) +
    r1 (-y[3] (y[2] + y[5] + y[6]) + y[1] (y[4] + y[7] + y[8])) /.
  {y[1] → y1, y[2] → y2, y[3] → y3, y[4] → y4, y[5] → y5, y[6] → y6, y[7] → y7, y[8] → y8}

r2 (-y2 y3 + y4 (y1 + y5 + y7) - y3 (y6 + y8)) +
  r1 (-y2 y3 - y3 (y5 + y6) + y1 (y4 + y7 + y8)) - testTerm3 // Simplify

0

rec4 = sumPerOffspringGamete[gametes, 4, recSep] // FullSimplify

y4 (y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8) + r2 (-y4 (y1 + y5 + y7) + y3 (y2 + y6 + y8)) +
 r1 ((-1 + r2) y4 y5 - (r2 y3 + y4) y6 + y2 (y3 - 2 r2 y3 + y7 - r2 y7 + y8) + y1 (-y4 + 2 r2 y4 + r2 y8))

y4D = FullSimplify[Series[rec4 - y4 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]

y4 (-1 + y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8) +
 r2 (-y4 (y1 + y5 + y7) + y3 (y2 + y6 + y8)) + r1 (-y4 (y1 + y5 + y6) + y2 (y3 + y7 + y8))

Collect[y4D, {r1, r2}]

y4 (-1 + y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8) +
 r2 (-y4 (y1 + y5 + y7) + y3 (y2 + y6 + y8)) + r1 (-y4 (y1 + y5 + y6) + y2 (y3 + y7 + y8))

FullSimplify[(y4 (-1 + y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8)),
 Assumptions → {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]

0

In[25]:= y4DotRec := -r1 (y4 p - y2 (1 - p)) - r2 (y4 q - y3 (1 - q))

y4DotRec - y4D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7} //.
  {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify

0
```

The following test term is from an independent derivation of Reinhard Bürger (personal communication).

```
testTerm4 := r2 (-y[4] (y[1] + y[5] + y[7]) + y[3] (y[2] + y[6] + y[8])) +
    r1 (-y[4] (y[1] + y[5] + y[6]) + y[2] (y[3] + y[7] + y[8])) /.
  {y[1] → y1, y[2] → y2, y[3] → y3, y[4] → y4, y[5] → y5, y[6] → y6, y[7] → y7, y[8] → y8}

r2 (-y4 (y1 + y5 + y7) + y3 (y2 + y6 + y8)) +
  r1 (-y4 (y1 + y5 + y6) + y2 (y3 + y7 + y8)) - testTerm4 // Simplify

0

rec5 = sumPerOffspringGamete[gametes, 5, recSep] // FullSimplify

y3 y5 - r1 y3 y5 + y4 y5 - r1 y4 y5 - r2 y4 y5 + r1 r2 y4 y5 + y5^2 +
 r2 y3 y6 - r1 r2 y3 y6 + y5 y6 + y5 y7 + r1 y6 y7 + r2 y6 y7 - 2 r1 r2 y6 y7 -
 (-1 + r2) y2 (y5 + r1 y7) + (1 - r1 - r2 + 2 r1 r2) y5 y8 + y1 (y5 + r2 y6 + r1 y7 + r1 r2 y8)
```

```
y5D = FullSimplify[Series[rec5 - y5 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]
```

```
y5^2 + r1 (y1 + y2 + y6) y7 + r2 y6 (y1 + y3 + y7) +
 y5 (-1 + y1 + y2 - r2 y2 + y3 - r1 y3 + y4 + y6 + y7 + y8 - (r1 + r2) (y4 + y8))
```

```
Collect[y5D, {r1, r2}]
```

```
-y5 + y1 y5 + y2 y5 + y3 y5 + y4 y5 + y5^2 + y5 y6 + y5 y7 + y5 y8 +
 r1 (-y3 y5 + (y1 + y2 + y6) y7 - y5 (y4 + y8)) + r2 (-y2 y5 + y6 (y1 + y3 + y7) - y5 (y4 + y8))
```

```
FullSimplify[(-y5 + y1 y5 + y2 y5 + y3 y5 + y4 y5 + y5^2 + y5 y6 + y5 y7 + y5 y8),
 Assumptions → {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]
```

```
0
```

In[26]:=

```
y5DotRec := -r1 (y5 (1 - p) - y7 p) - r2 (y5 (1 - q) - y6 q)
```

```
y5DotRec - y5D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7} //.
  {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify
```

```
0
```

The following test term is from an independent derivation of Reinhard Bürger (personal communication).

```
testTerm5 := r2 (y[6] (y[1] + y[3] + y[7]) - y[5] (y[2] + y[4] + y[8])) +
   r1 ((y[1] + y[2] + y[6]) y[7] - y[5] (y[3] + y[4] + y[8])) /.
  {y[1] → y1, y[2] → y2, y[3] → y3, y[4] → y4, y[5] → y5, y[6] → y6, y[7] → y7, y[8] → y8}
```

```
r1 (-y3 y5 + (y1 + y2 + y6) y7 - y5 (y4 + y8)) +
  r2 (-y2 y5 + y6 (y1 + y3 + y7) - y5 (y4 + y8)) - testTerm5 // Simplify
```

```
0
```

```
rec6 = sumPerOffspringGamete[gametes, 6, recSep] // FullSimplify
```

```
y6 (y1 + y2 + y3 - r1 y3 + y4 + y5 + y6 + y7 - r1 (y4 + y7)) + (r1 (y1 + y2 + y5) + y6) y8 +
 r2 (y4 (y5 - r1 y5) - y6 (y1 + y3 - r1 y3 + y7 - 2 r1 y7) + y2 (y5 + r1 y7) + (y5 - r1 (y1 + 2 y5)) y8)
```

```
y6D = FullSimplify[Series[rec6 - y6 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]
```

```
y6 (-1 + y1 + y2 + y3 - r1 y3 + y4 + y5 + y6 + y7 - r1 (y4 + y7)) +
 (r1 (y1 + y2 + y5) + y6) y8 + r2 (-y6 (y1 + y3 + y7) + y5 (y2 + y4 + y8))
```

```
Collect[y6D, {r1, r2}]
```

```
-y6 + y1 y6 + y2 y6 + y3 y6 + y4 y6 + y5 y6 + y6^2 + y6 y7 + y6 y8 +
 r1 (-y3 y6 - y6 (y4 + y7) + (y1 + y2 + y5) y8) + r2 (-y6 (y1 + y3 + y7) + y5 (y2 + y4 + y8))
```

```
FullSimplify[-y6 + y1 y6 + y2 y6 + y3 y6 + y4 y6 + y5 y6 + y6^2 + y6 y7 + y6 y8,
 Assumptions →  {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]
```

```
0
```

In[27]:=

```
y6DotRec := -r1 (y6 (1 - p) - y8 p) - r2 (y6 q - y5 (1 - q))
```

```
y6DotRec - y6D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7} //.
  {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify
```

```
0
```

The following test term is from an independent derivation of Reinhard Bürger (personal communication).

```
testTerm6 := -r1 y[6] (y[3] + y[4] + y[7]) + r1 (y[1] + y[2] + y[5]) y[8] +
   r2 (-y[6] (y[1] + y[3] + y[7]) + y[5] (y[2] + y[4] + y[8])) /.
  {y[1] → y1, y[2] → y2, y[3] → y3, y[4] → y4, y[5] → y5, y[6] → y6, y[7] → y7, y[8] → y8}
```

```
r1 (-y3 y6 - y6 (y4 + y7) + (y1 + y2 + y5) y8) +
  r2 (-y6 (y1 + y3 + y7) + y5 (y2 + y4 + y8)) - testTerm6 // Simplify
```

```
0
```

```
rec7 = sumPerOffspringGamete[gametes, 7, recSep] // FullSimplify

y7 (y1 + y2 - r2 y2 + y3 + y4 + y5 + y6 - r2 (y4 + y6) + y7) + (r2 (y1 + y3 + y5) + y7) y8 +
 r1 (y4 (y5 - r2 y5) + y3 (y5 + r2 y6) - (y1 + y2 - r2 y2 + y6 - 2 r2 y6) y7 + (y5 - r2 (y1 + 2 y5)) y8)

y7D = FullSimplify[Series[rec7 - y7 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]

y7 (-1 + y1 + y2 - r2 y2 + y3 + y4 + y5 + y6 - r2 (y4 + y6) + y7) +
 (r2 (y1 + y3 + y5) + y7) y8 + r1 (- (y1 + y2 + y6) y7 + y5 (y3 + y4 + y8))

Collect[y7D, {r1, r2}]

-y7 + y1 y7 + y2 y7 + y3 y7 + y4 y7 + y5 y7 + y6 y7 + y7^2 + y7 y8 +
 r2 (-y2 y7 + (-y4 - y6) y7 + (y1 + y3 + y5) y8) + r1 (- (y1 + y2 + y6) y7 + y5 (y3 + y4 + y8))

FullSimplify[-y7 + y1 y7 + y2 y7 + y3 y7 + y4 y7 + y5 y7 + y6 y7 + y7^2 + y7 y8,
 Assumptions → {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]

0

In[28]:= y7DotRec := -r1 (y7 p - y5 (1 - p)) - r2 (y7 (1 - q) - y8 q)

y7DotRec - y7D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7} //.
  {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify

0
```

The following test term is from an independent derivation of Reinhard Bürger (personal communication).

```
testTerm7 := -r2 (y[2] + y[4] + y[6]) y[7] + r2 (y[1] + y[3] + y[5]) y[8] +
    r1 (- (y[1] + y[2] + y[6]) y[7] + y[5] (y[3] + y[4] + y[8])) /.
  {y[1] → y1, y[2] → y2, y[3] → y3, y[4] → y4, y[5] → y5, y[6] → y6, y[7] → y7, y[8] → y8}

r2 (-y2 y7 + (-y4 - y6) y7 + (y1 + y3 + y5) y8) +
  r1 (- (y1 + y2 + y6) y7 + y5 (y3 + y4 + y8)) - testTerm7 // Simplify

0

rec8 = sumPerOffspringGamete[gametes, 8, recSep] // FullSimplify

r2 (y2 + y4 + y6) y7 - r2 (y1 + y3 + y5) y8 + y8 (y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8) +
 r1 (y6 (y3 + y4 + y7) - (y1 + y2 + y5) y8 + r2 (-y2 y7 - y6 (y3 + 2 y7) + y1 y8 + y5 (y4 + 2 y8)))

y8D = FullSimplify[Series[rec8 - y8 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]

r2 (y2 + y4 + y6) y7 + r1 y6 (y3 + y4 + y7) - r1 (y1 + y2 + y5) y8 -
 r2 (y1 + y3 + y5) y8 + y8 (-1 + y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8)

Collect[y8D, {r1, r2}]

y8 (-1 + y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8) +
 r1 (y6 (y3 + y4 + y7) + (-y1 - y2 - y5) y8) + r2 ((y2 + y4 + y6) y7 + (-y1 - y3 - y5) y8)

FullSimplify[y8 (-1 + y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8),
 Assumptions → {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]

0

In[29]:= y8DotRec := -r1 (y8 p - y6 (1 - p)) - r2 (y8 q - y7 (1 - q))

y8DotRec - y8D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7} //.
  {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify

0
```

The following test term is from an independent derivation of Reinhard Bürger (personal communication).

```
testTerm8 := r2 (y[2] + y[4] + y[6]) y[7] + r1 y[6] (y[3] + y[4] + y[7]) -
    r1 (y[1] + y[2] + y[5]) y[8] - r2 (y[1] + y[3] + y[5]) y[8] /.
  {y[1] → y1, y[2] → y2, y[3] → y3, y[4] → y4, y[5] → y5, y[6] → y6, y[7] → y7, y[8] → y8}
```

```
r1 (y6 (y3 + y4 + y7) + (-y1 - y2 - y5) y8) +
   r2 ((y2 + y4 + y6) y7 + (-y1 - y3 - y5) y8) - testTerm8 // Simplify
```

```
0
```

Some further tests:

```
y1DotRec + y5DotRec // FullSimplify
```

```
r2 ((-1 + q) y1 - y5 + q (y2 + y5 + y6)) + r1 ((-1 + p) y1 - y5 + p (y3 + y5 + y7))
```

A bit of algebra confirms that this is equal to $(r_{AC} + r_{CB})\,(p\,q - y_1 - y_5) = -r_{AB}\,D_{AB}$, as expected from the marginal two-locus system.

```
y2DotRec + y6DotRec // FullSimplify
```

```
-r2 ((-1 + q) y1 - y5 + q (y2 + y5 + y6)) + r1 ((-1 + p) y2 - y6 + p (y4 + y6 + y8))
```

Again, some algebra confirms that this is equal to $(r_{AC} + r_{CB})\,(-p\,q + y_1 + y_5) = r_{AB}\,D_{AB}$, as expected from the marginal two-locus system. To see this, note that D_{AB} can also be defined as $-(y_2 + y_6) + p(1 - q)$.

```
y3DotRec + y7DotRec // FullSimplify
```

```
-r1 ((-1 + p) y1 - y5 + p (y3 + y5 + y7)) + r2 ((-1 + q) y3 - y7 + q (y4 + y7 + y8))
```

Noting that D_{AB} can also be defined as $-(y_3 + y_7) + (1 - p)\,q$, one can show that this is equal to $(r_{AC} + r_{CB})\,(-p\,q + y_1 + y_5) = r_{AB}\,D_{AB}$, as expected from the marginal two-locus system.

```
y4DotRec + y8DotRec // FullSimplify
```

```
-r1 ((-1 + p) y2 - y6 + p (y4 + y6 + y8)) - r2 ((-1 + q) y3 - y7 + q (y4 + y7 + y8))
```

which is equal to $(r_{AC} + r_{CB})\,(y_2 + y_6 - p(1 - q)) = -r_{AB}\,D_{AB}$, as expected from the marginal two-locus system.

- In terms of allele frequencies and LD

```
allToGam
```

```
{p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7, n → y1 + y2 + y3 + y4}
```

```
DACBDefAlt := y1 - (y1 + y2 + y5 + y6) (y1 + y3 + y5 + y7) (y1 + y2 + y3 + y4) -
   (y1 + y2 + y5 + y6) ((y1 + y3) (y6 + y8) - (y2 + y4) (y5 + y7)) -
   (y1 + y3 + y5 + y7) ((y1 + y2) (y7 + y8) - (y5 + y6) (y3 + y4)) -
   (y1 + y2 + y3 + y4) ((y1 + y5) (y4 + y8) - (y2 + y6) (y3 + y7))
```

```
DACBDefAlt - DACBDef
```

```
0
```

```
DACBDef // FullSimplify
```

```
y1 - (y1 + y2 + y3 + y4) (y1 + y2 + y5 + y6) (y1 + y3 + y5 + y7) -
  (y1 + y2 + y3 + y4) (-(y2 + y6) (y3 + y7) + (y1 + y5) (y4 + y8)) -
  (y1 + y2 + y5 + y6) (-(y2 + y4) (y5 + y7) + (y1 + y3) (y6 + y8)) -
  (y1 + y3 + y5 + y7) (-(y3 + y4) (y5 + y6) + (y1 + y2) (y7 + y8))
```

```
DANB[1] //. {x_[i_, j_] → x[i]} /.
  {y[1] → y1, y[2] → y2, y[3] → y3, y[4] → y4, y[5] → y5, y[6] → y6, y[7] → y7, y[8] → y8}
```

```
DANB[1]
```

In[30]:=
```
pDotRec := D[pDef /. {y1 → y1[t], y2 → y2[t], y5 → y5[t], y6 → y6[t]}, t] /.
     {y1'[t] → y1DotRec, y2'[t] → y2DotRec, y5'[t] → y5DotRec, y6'[t] → y6DotRec} /.
    gamToAllLD // FullSimplify
qDotRec := D[qDef /. {y1 → y1[t], y3 → y3[t], y5 → y5[t], y7 → y7[t]}, t] /.
     {y1'[t] → y1DotRec, y3'[t] → y3DotRec, y5'[t] → y5DotRec, y7'[t] → y7DotRec} /.
    gamToAllLD // FullSimplify
nDotRec := D[nDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t], y4 → y4[t]}, t] /.
     {y1'[t] → y1DotRec, y2'[t] → y2DotRec, y3'[t] → y3DotRec, y4'[t] → y4DotRec} /.
    gamToAllLD // FullSimplify
DACDotRec := D[DACDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t],
          y4 → y4[t], y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
      {y1'[t] → y1DotRec, y2'[t] → y2DotRec, y3'[t] → y3DotRec, y4'[t] → y4DotRec,
       y5'[t] → y5DotRec, y6'[t] → y6DotRec, y7'[t] → y7DotRec, y8'[t] → y8DotRec} /.
     {x_[t] → x} /. gamToAllLD // FullSimplify
DCBDotRec := D[DCBDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t], y4 → y4[t],
          y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
      {y1'[t] → y1DotRec, y2'[t] → y2DotRec, y3'[t] → y3DotRec, y4'[t] → y4DotRec,
       y5'[t] → y5DotRec, y6'[t] → y6DotRec, y7'[t] → y7DotRec, y8'[t] → y8DotRec} /.
     {x_[t] → x} /. gamToAllLD // FullSimplify
DABDotRec := D[DABDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t], y4 → y4[t],
          y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
      {y1'[t] → y1DotRec, y2'[t] → y2DotRec, y3'[t] → y3DotRec, y4'[t] → y4DotRec,
       y5'[t] → y5DotRec, y6'[t] → y6DotRec, y7'[t] → y7DotRec, y8'[t] → y8DotRec} /.
     {x_[t] → x} /. gamToAllLD // FullSimplify
DACBDotRec := D[DACBDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t],
          y4 → y4[t], y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
      {y1'[t] → y1DotRec, y2'[t] → y2DotRec, y3'[t] → y3DotRec, y4'[t] → y4DotRec,
       y5'[t] → y5DotRec, y6'[t] → y6DotRec, y7'[t] → y7DotRec, y8'[t] → y8DotRec} /.
     {x_[t] → x} /. gamToAllLD // FullSimplify
```

```
{pDotRec, qDotRec, nDotRec, DACDotRec, DCBDotRec, DABDotRec, DACBDotRec} // TableForm
```

```
0
0
0
-DAC r1
-DCB r2
-DAB (r1 + r2)
-DACB (r1 + r2)
```

An independently derived test term (Reinhard Bürger, personal communication):

```
testTerm := y1 - (y1 + y2 + y3 + y4) (y1 + y2 + y5 + y6) (y1 + y3 + y5 + y7) -
   (y1 + y2 + y3 + y4) (- (y2 + y6) (y3 + y7) + (y1 + y5) (y4 + y8)) -
   (y1 + y2 + y5 + y6) (- (y2 + y4) (y5 + y7) + (y1 + y3) (y6 + y8)) -
   (y1 + y3 + y5 + y7) (- (y3 + y4) (y5 + y6) + (y1 + y2) (y7 + y8))
```

```
DACBDef - testTerm // FullSimplify
```

```
0
```

```
testTerm
```

```
y1 - (y1 + y2 + y3 + y4) (y1 + y2 + y5 + y6) (y1 + y3 + y5 + y7) -
  (y1 + y2 + y3 + y4) ((-y2 - y6) (y3 + y7) + (y1 + y5) (y4 + y8)) -
  (y1 + y2 + y5 + y6) ((-y2 - y4) (y5 + y7) + (y1 + y3) (y6 + y8)) -
  (y1 + y3 + y5 + y7) ((-y3 - y4) (y5 + y6) + (y1 + y2) (y7 + y8))
```

In[37]:=
```
DACBRule := y1 - p q n - p (y1 + y3 - q n) - q (y1 + y2 - p n) - n (y1 + y5 - p q)
```

- **Differential equations under migration**
- In terms of gamete frequencies

In[38]:=
```
y1DotMig := -m y1
y2DotMig := -m y2
y3DotMig := -m y3
y4DotMig := m (nC - y4)
y5DotMig := -m y5
y6DotMig := -m y6
y7DotMig := -m y7
y8DotMig := m (1 - nC - y8)
```

```
FullSimplify[y1DotMig + y2DotMig + y3DotMig + y4DotMig + y5DotMig + y6DotMig +
  y7DotMig + y8DotMig, Assumptions → {y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]
```

```
0
```

```
FullSimplify[y1DotMig + y5DotMig /. {y2 + y4 + y6 + y8 → x2 + x4} /.
   {y3 + y4 + y7 + y8 → x3 + x4}] /. {y1 + y5 → x1}
```

```
-m x1
```

```
FullSimplify[y2DotMig + y6DotMig /. {y1 + y3 + y5 + y7 → x1 + x3, y3 + y4 + y7 + y8 → x3 + x4}] /.
 {y2 + y6 → x2}
```

```
-m x2
```

```
FullSimplify[y3DotMig + y7DotMig /. {y1 + y2 + y5 + y6 → x1 + x2, y2 + y4 + y6 + y8 → x2 + x4}] /.
 {y3 + y7 → x3}
```

```
-m x3
```

```
FullSimplify[y4DotMig + y8DotMig /. {y1 + y2 + y5 + y6 → x1 + x2, y1 + y3 + y5 + y7 → x1 + x3}] /.
 {y4 + y8 → x4}
```

```
-m (-1 + x4)
```

- In terms of allele frequencies and LD

```
allToGam
```

```
{p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7, n → y1 + y2 + y3 + y4}
```

In[46]:=
```
pDotMig := D[pDef /. {y1 → y1[t], y2 → y2[t], y5 → y5[t], y6 → y6[t]}, t] /.
     {y1'[t] → y1DotMig, y2'[t] → y2DotMig, y5'[t] → y5DotMig, y6'[t] → y6DotMig} /.
    gamToAllLD // FullSimplify
qDotMig := D[qDef /. {y1 → y1[t], y3 → y3[t], y5 → y5[t], y7 → y7[t]}, t] /.
     {y1'[t] → y1DotMig, y3'[t] → y3DotMig, y5'[t] → y5DotMig, y7'[t] → y7DotMig} /.
    gamToAllLD // FullSimplify
nDotMig := D[nDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t], y4 → y4[t]}, t] /.
     {y1'[t] → y1DotMig, y2'[t] → y2DotMig, y3'[t] → y3DotMig, y4'[t] → y4DotMig} /.
    gamToAllLD // FullSimplify
DACDotMig := D[DACDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t],
          y4 → y4[t], y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
      {y1'[t] → y1DotMig, y2'[t] → y2DotMig, y3'[t] → y3DotMig, y4'[t] → y4DotMig,
       y5'[t] → y5DotMig, y6'[t] → y6DotMig, y7'[t] → y7DotMig, y8'[t] → y8DotMig} /.
     {x_[t] → x} /. gamToAllLD // FullSimplify
DCBDotMig := D[DCBDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t], y4 → y4[t],
          y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
      {y1'[t] → y1DotMig, y2'[t] → y2DotMig, y3'[t] → y3DotMig, y4'[t] → y4DotMig,
       y5'[t] → y5DotMig, y6'[t] → y6DotMig, y7'[t] → y7DotMig, y8'[t] → y8DotMig} /.
     {x_[t] → x} /. gamToAllLD // FullSimplify
DABDotMig := D[DABDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t], y4 → y4[t],
          y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
      {y1'[t] → y1DotMig, y2'[t] → y2DotMig, y3'[t] → y3DotMig, y4'[t] → y4DotMig,
       y5'[t] → y5DotMig, y6'[t] → y6DotMig, y7'[t] → y7DotMig, y8'[t] → y8DotMig} /.
     {x_[t] → x} /. gamToAllLD // FullSimplify
DACBDotMig := D[DACBDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t],
          y4 → y4[t], y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
      {y1'[t] → y1DotMig, y2'[t] → y2DotMig, y3'[t] → y3DotMig, y4'[t] → y4DotMig,
       y5'[t] → y5DotMig, y6'[t] → y6DotMig, y7'[t] → y7DotMig, y8'[t] → y8DotMig} /.
     {x_[t] → x} /. gamToAllLD // FullSimplify
```

Test of the pattern rule x_[t]->x:

```
D[DACBDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t],
        y4 → y4[t], y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
    {y1'[t] → y1DotMig, y2'[t] → y2DotMig, y3'[t] → y3DotMig, y4'[t] → y4DotMig,
     y5'[t] → y5DotMig, y6'[t] → y6DotMig, y7'[t] → y7DotMig, y8'[t] → y8DotMig} /.
   ruleRemoveDummy /. gamToAllLD // FullSimplify
```

m (- DACB + DCB p + DAC q + (n - nC) (DAB - p q))

```
% - DACBDotMig // FullSimplify
```

0

In[53]:=
```
ruleRemoveDummy := {y1[t] → y1, y2[t] → y2, y3[t] → y3,
  y4[t] → y4, y5[t] → y5, y6[t] → y6, y7[t] → y7, y8[t] → y8}
{pDotMig, qDotMig, nDotMig, DACDotMig, DCBDotMig, DABDotMig, DACBDotMig} // TableForm
```

- m p
- m q
m (- n + nC)
- m (DAC + (- n + nC) p)
- m (DCB + (- n + nC) q)
- DAB m + m p q
m (- DACB + DCB p + DAC q + (n - nC) (DAB - p q))

Differential equations under selection

In terms of gamete frequencies

In[54]:=
```
y1DotSel := y1 (a (y3 + y4 + y7 + y8) + b (y2 + y4 + y6 + y8))
y2DotSel := y2 (a (y3 + y4 + y7 + y8) - b (y1 + y3 + y5 + y7))
y3DotSel := y3 (-a (y1 + y2 + y5 + y6) + b (y2 + y4 + y6 + y8))
y4DotSel := y4 (-a (y1 + y2 + y5 + y6) - b (y1 + y3 + y5 + y7))

y5DotSel := y5 (a (y3 + y4 + y7 + y8) + b (y2 + y4 + y6 + y8))
y6DotSel := y6 (a (y3 + y4 + y7 + y8) - b (y1 + y3 + y5 + y7))
y7DotSel := y7 (-a (y1 + y2 + y5 + y6) + b (y2 + y4 + y6 + y8))
y8DotSel := y8 (-a (y1 + y2 + y5 + y6) - b (y1 + y3 + y5 + y7))
```

```
FullSimplify[y1DotSel + y5DotSel /. {y2 + y4 + y6 + y8 → x2 + x4} /.
   {y3 + y4 + y7 + y8 → x3 + x4}] /. {y1 + y5 → x1}
```

x1 (b (x2 + x4) + a (x3 + x4))

```
FullSimplify[y2DotSel + y6DotSel /. {y1 + y3 + y5 + y7 → x1 + x3, y3 + y4 + y7 + y8 → x3 + x4}] /.
 {y2 + y6 → x2}
```

x2 (- b (x1 + x3) + a (x3 + x4))

```
FullSimplify[y3DotSel + y7DotSel /. {y1 + y2 + y5 + y6 → x1 + x2, y2 + y4 + y6 + y8 → x2 + x4}] /.
 {y3 + y7 → x3}
```

x3 (- a (x1 + x2) + b (x2 + x4))

```
FullSimplify[y4DotSel + y8DotSel /. {y1 + y2 + y5 + y6 → x1 + x2, y1 + y3 + y5 + y7 → x1 + x3}] /.
 {y4 + y8 → x4}
```

- (a (x1 + x2) + b (x1 + x3)) x4

In terms of allele frequencies and LD

```
allToGam
```

{p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7, n → y1 + y2 + y3 + y4}

In[62]:=

```
pDotSel := D[pDef /. {y1 → y1[t], y2 → y2[t], y5 → y5[t], y6 → y6[t]}, t] /.
    {y1'[t] → y1DotSel, y2'[t] → y2DotSel, y5'[t] → y5DotSel, y6'[t] → y6DotSel} /.
   gamToAllLD // FullSimplify
qDotSel := D[qDef /. {y1 → y1[t], y3 → y3[t], y5 → y5[t], y7 → y7[t]}, t] /.
    {y1'[t] → y1DotSel, y3'[t] → y3DotSel, y5'[t] → y5DotSel, y7'[t] → y7DotSel} /.
   gamToAllLD // FullSimplify
nDotSel := D[nDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t], y4 → y4[t]}, t] /.
    {y1'[t] → y1DotSel, y2'[t] → y2DotSel, y3'[t] → y3DotSel, y4'[t] → y4DotSel} /.
   gamToAllLD // FullSimplify
DACDotSel := D[DACDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t],
        y4 → y4[t], y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
     {y1'[t] → y1DotSel, y2'[t] → y2DotSel, y3'[t] → y3DotSel, y4'[t] → y4DotSel,
      y5'[t] → y5DotSel, y6'[t] → y6DotSel, y7'[t] → y7DotSel, y8'[t] → y8DotSel} /.
    {x_[t] → x} /. gamToAllLD // FullSimplify
DCBDotSel := D[DCBDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t], y4 → y4[t],
        y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
     {y1'[t] → y1DotSel, y2'[t] → y2DotSel, y3'[t] → y3DotSel, y4'[t] → y4DotSel,
      y5'[t] → y5DotSel, y6'[t] → y6DotSel, y7'[t] → y7DotSel, y8'[t] → y8DotSel} /.
    {x_[t] → x} /. gamToAllLD // FullSimplify
DABDotSel := D[DABDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t], y4 → y4[t],
        y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
     {y1'[t] → y1DotSel, y2'[t] → y2DotSel, y3'[t] → y3DotSel, y4'[t] → y4DotSel,
      y5'[t] → y5DotSel, y6'[t] → y6DotSel, y7'[t] → y7DotSel, y8'[t] → y8DotSel} /.
    {x_[t] → x} /. gamToAllLD // FullSimplify
DACBDotSel := D[DACBDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t],
        y4 → y4[t], y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
     {y1'[t] → y1DotSel, y2'[t] → y2DotSel, y3'[t] → y3DotSel, y4'[t] → y4DotSel,
      y5'[t] → y5DotSel, y6'[t] → y6DotSel, y7'[t] → y7DotSel, y8'[t] → y8DotSel} /.
    {x_[t] → x} /. gamToAllLD // FullSimplify
```

```
{pDotSel, qDotSel, nDotSel, DACDotSel, DCBDotSel, DABDotSel, DACBDotSel} // TableForm
```

b DAB − a (−1 + p) p
a DAB − b (−1 + q) q
a DAC + b DCB
b DACB + a (DAC − 2 DAC p)
a DACB + b DCB (1 − 2 q)
DAB (a + b − 2 a p − 2 b q)
a (−2 DAB DAC + DACB − 2 DACB p) + b (DACB − 2 DAB DCB − 2 DACB q)

■ Differential equations under selection, migration and recombination

■ Remark

As we are assuming continuous-time dynamics, interactions of selection, migration and recombinations are ignored and the order of the processes is irrelevant. Therefore, the overall differential equations are obtained by adding those for the single processes.

■ In terms of gamete frequencies

In[69]:=

```
y1Dot := y1DotSel + y1DotMig + y1DotRec
y2Dot := y2DotSel + y2DotMig + y2DotRec
y3Dot := y3DotSel + y3DotMig + y3DotRec
y4Dot := y4DotSel + y4DotMig + y4DotRec

y5Dot := y5DotSel + y5DotMig + y5DotRec
y6Dot := y6DotSel + y6DotMig + y6DotRec
y7Dot := y7DotSel + y7DotMig + y7DotRec
y8Dot := y8DotSel + y8DotMig + y8DotRec
```

```
Simplify[y4DotSel + y4DotMig]
```

m (nC − y4) − y4 (a (y1 + y2 + y5 + y6) + b (y1 + y3 + y5 + y7))

```
{y1Dot, y2Dot, y3Dot, y4Dot, y5Dot, y6Dot, y7Dot, y8Dot} // Simplify // TableForm
```

```
-m y1 + r2 ((-1 + q) y1 + q y2) + r1 ((-1 + p) y1 + p y3) + y1 (b (y2 + y4 + y6 + y8) + a (y3 + y4 + y7 + y8)
-m y2 - r2 ((-1 + q) y1 + q y2) + r1 ((-1 + p) y2 + p y4) + y2 (-b (y1 + y3 + y5 + y7) + a (y3 + y4 + y7 + y8
-m y3 - r1 ((-1 + p) y1 + p y3) + r2 ((-1 + q) y3 + q y4) + y3 (-a (y1 + y2 + y5 + y6) + b (y2 + y4 + y6 + y8
m (nC - y4) - r1 ((-1 + p) y2 + p y4) - r2 ((-1 + q) y3 + q y4) + y4 (-a (y1 + y2 + y5 + y6) - b (y1 + y3 + y
-m y5 + r2 ((-1 + q) y5 + q y6) + r1 ((-1 + p) y5 + p y7) + y5 (b (y2 + y4 + y6 + y8) + a (y3 + y4 + y7 + y8)
-m y6 - r2 ((-1 + q) y5 + q y6) + r1 ((-1 + p) y6 + p y8) + y6 (-b (y1 + y3 + y5 + y7) + a (y3 + y4 + y7 + y8
-m y7 - r1 ((-1 + p) y5 + p y7) + r2 ((-1 + q) y7 + q y8) + y7 (-a (y1 + y2 + y5 + y6) + b (y2 + y4 + y6 + y8
(-a (y1 + y2 + y5 + y6) - b (y1 + y3 + y5 + y7)) y8 - m (-1 + nC + y8) - r1 ((-1 + p) y6 + p y8) - r2 ((-1 +
```

```
Map[Collect[#, {m, r1, r2}] &,
  {y1Dot, y2Dot, y3Dot, y4Dot, y5Dot, y6Dot, y7Dot, y8Dot}] // TableForm
```

```
-m y1 + r2 (-(1 - q) y1 + q y2) + r1 (-(1 - p) y1 + p y3) + y1 (b (y2 + y4 + y6 + y8) + a (y3 + y4 + y7 + y8)
-m y2 + r2 ((1 - q) y1 - q y2) + r1 (-(1 - p) y2 + p y4) + y2 (-b (y1 + y3 + y5 + y7) + a (y3 + y4 + y7 + y8)
-m y3 + r1 ((1 - p) y1 - p y3) + r2 (-(1 - q) y3 + q y4) + y3 (-a (y1 + y2 + y5 + y6) + b (y2 + y4 + y6 + y8)
m (nC - y4) + r1 ((1 - p) y2 - p y4) + r2 ((1 - q) y3 - q y4) + y4 (-a (y1 + y2 + y5 + y6) - b (y1 + y3 + y5 +
-m y5 + r2 (-(1 - q) y5 + q y6) + r1 (-(1 - p) y5 + p y7) + y5 (b (y2 + y4 + y6 + y8) + a (y3 + y4 + y7 + y8)
-m y6 + r2 ((1 - q) y5 - q y6) + r1 (-(1 - p) y6 + p y8) + y6 (-b (y1 + y3 + y5 + y7) + a (y3 + y4 + y7 + y8)
-m y7 + r1 ((1 - p) y5 - p y7) + r2 (-(1 - q) y7 + q y8) + y7 (-a (y1 + y2 + y5 + y6) + b (y2 + y4 + y6 + y8)
m (1 - nC - y8) + (-a (y1 + y2 + y5 + y6) - b (y1 + y3 + y5 + y7)) y8 + r1 ((1 - p) y6 - p y8) + r2 ((1 - q) y
```

- In terms of allele frequencies and LD

```
nDotMig
```

```
m (-n + nC)
```

```
nDotSel
```

```
a DAC + b DCB
```

In[77]:=

```
pDot := pDotSel + pDotMig + pDotRec
qDot := qDotSel + qDotMig + qDotRec
nDot := nDotSel + nDotMig + nDotRec
DACDot := DACDotSel + DACDotMig + DACDotRec
DCBDot := DCBDotSel + DCBDotMig + DCBDotRec
DABDot := DABDotSel + DABDotMig + DABDotRec
DACBDot := DACBDotSel + DACBDotMig + DACBDotRec
```

```
{pDot, qDot, nDot, DACDot, DCBDot, DABDot, DACBDot} // Simplify // TableForm
```

```
b DAB - (m + a (-1 + p)) p
a DAB - (m + b (-1 + q)) q
a DAC + b DCB + m (-n + nC)
b DACB + a (DAC - 2 DAC p) - m (DAC + (-n + nC) p) - DAC r1
a DACB + b DCB (1 - 2 q) - m (DCB + (-n + nC) q) - DCB r2
-DAB m + m p q + DAB (a + b - 2 a p - 2 b q) - DAB (r1 + r2)
a (-2 DAB DAC + DACB - 2 DACB p) + b (DACB - 2 DAB DCB - 2 DACB q) + m (-DACB + DCB p + DAC q + (n - nC) (D
```

```
Collect[DACBDot, {DAB, DAC, DCB, DACB}]
```

```
DAB (-2 a DAC - 2 b DCB + m (n - nC)) + DCB m p +
  DAC m q - m (n - nC) p q + DACB (a + b - m - 2 a p - 2 b q - r1 - r2)
```

```
FullSimplify[(a + b - m - 2 a p - 2 b q - r1 - r2)]
```

```
a + b - m - 2 a p - 2 b q - r1 - r2
```

```
Map[Collect[#, {m, r1, r2}] &,
  {pDot, qDot, nDot, DACDot, DCBDot, DABDot, DACBDot}] // TableForm
```

```
b DAB - m p - a (-1 + p) p
a DAB - m q - b (-1 + q) q
a DAC + b DCB + m (-n + nC)
b DACB + a (DAC - 2 DAC p) + m (-DAC - (-n + nC) p) - DAC r1
a DACB + b DCB (1 - 2 q) + m (-DCB - (-n + nC) q) - DCB r2
DAB (a + b - 2 a p - 2 b q) + m (-DAB + p q) - DAB r1 - DAB r2
a (-2 DAB DAC + DACB - 2 DACB p) + b (DACB - 2 DAB DCB - 2 DACB q) + m (-DACB + DCB p + DAC q + (n - nC) (D
```

These are the differential equations given by BA2011 for *n*, D_{AC}, D_{CB} and D_{ACB}, which are obviously different from those just above.

```
In[84]:= nDotTarget := m ((nC - n) (p + q - p q) - DAC (1 - q) - DCB (1 - p) + DACB) + a DAC + b DCB
DACDotTarget := - (r1 + m (1 - p + p q) - a (1 - 2 p)) DAC +
  m p (1 - p) DCB + m (nC - n) p DAB + (b - m p) DACB + m (nC - n) p (p + q - p q)
DCBDotTarget := - (r2 + m (1 - q + p q) - b (1 - 2 q)) DCB + m q (1 - q) DAC +
  m (nC - n) q DAB + (a - m q) DACB + m (nC - n) q (p + q - p q)
DACBDotTarget := m p q (nC - n) (p + q - p q) + m q (1 - (1 - q) p) DAC + m p (1 - (1 - p) q) DCB -
  m (nC - n) (p + q) DAB - (r1 + r2 + m (1 - p q) - a (1 - 2 p) - b (1 - 2 q)) DACB +
  m (nC - n) DAB^2 + (m (1 - q) - 2 a) DAB DAC + (m (1 - p) - 2 b) DAB DCB - m DAB DACB
```

Internal equilibrium

Coordinates

We recall the differential equations:

```
In[88]:= diffEqs =
  Map[Collect[#, {m, r1, r2}] &, {pDot, qDot, DABDot, nDot, DACDot, DCBDot, DACBDot}];
diffEqs // TableForm
```

```
Out[89]//TableForm=
b DAB - m p - a (-1 + p) p
a DAB - m q - b (-1 + q) q
DAB (a + b - 2 a p - 2 b q) + m (-DAB + p q) - DAB r1 - DAB r2
a DAC + b DCB + m (-n + nC)
b DACB + a (DAC - 2 DAC p) + m (-DAC - (-n + nC) p) - DAC r1
a DACB + b DCB (1 - 2 q) + m (-DCB - (-n + nC) q) - DCB r2
a (-2 DAB DAC + DACB - 2 DACB p) + b (DACB - 2 DAB DCB - 2 DACB q) + m (-DACB + DCB p + DAC q + (n - nC) (D
```

```
In[90]:= diffEqsTarget = Map[Collect[#, {m, r1, r2}] &,
   {pDot, qDot, DABDot, nDotTarget, DACDotTarget, DCBDotTarget, DACBDotTarget}];
diffEqsTarget // TableForm
```

```
Out[91]//TableForm=
b DAB - m p - a (-1 + p) p
a DAB - m q - b (-1 + q) q
DAB (a + b - 2 a p - 2 b q) + m (-DAB + p q) - DAB r1 - DAB r2
a DAC + b DCB + m (DACB - DCB (1 - p) - DAC (1 - q) + (-n + nC) (p + q - p q))
b DACB + a DAC (1 - 2 p) + m (-DACB p + DAB (-n + nC) p + DCB (1 - p) p + (-n + nC) p (p + q - p q) - DAC (1 -
a DACB + b DCB (1 - 2 q) + m (-DACB q + DAB (-n + nC) q + DAC (1 - q) q + (-n + nC) q (p + q - p q) - DCB (1 -
-2 a DAB DAC - 2 b DAB DCB + a DACB (1 - 2 p) + b DACB (1 - 2 q) + m (-DAB DACB + DAB^2 (-n + nC) + DAB DCB
```

For testing purposes, we introduce equations independently derived by Reinhard Bürger (*Mathematica* Notebook 'Three_Loci_gen1.nb', personal communication):

```
In[92]:= testEqs := {- (-1 + p) p α + DAB β - p μ, DAB α - (-1 + q) q β - q μ,
    - DAB r1 - DAB r2 + DAB (1 - 2 p) α + DAB (1 - 2 q) β - DAB μ + p q μ,
    DAN α + DNB β + (-n + nC) μ, - DAN r1 + DAN (1 - 2 p) α + DANB β + (- DAN + (n - nC) p) μ,
    - DNB r2 + DANB α + (DNB - 2 DNB q) β - DNB μ + n q μ - nC q μ,
    - DANB r1 - DANB r2 + (-2 DAB DAN + DANB - 2 DANB p) α + (DANB - 2 DAB DNB - 2 DANB q) β -
     DANB μ + DAB n μ - DAB nC μ + DNB p μ + DAN q μ - n p q μ + nC p q μ} /.
  {α → a, β → b, μ → m, DANB → DACB, DAN → DAC, DNB → DCB}

diffEqs - testEqs // FullSimplify
```

```
{0, 0, 0, 0, 0, 0, 0}
```

which confirms the equations.

BA2011 (eq. 3.15) showed the coordinates of the internal stable equilibrium to be

In[93]:=
```
R1 := Sqrt[(a + b + r)^2 - 8 m (r1 + r2)]
```

In[94]:=
```
pEqBA := 1/(8 a r) (b^2 - a^2 + 6 a r - r^2 - 4 m r + (a - b + r) R1) /. {r -> r1 + r2}
qEqBA := 1/(8 b r) (a^2 - b^2 + 6 b r - r^2 - 4 m r + (b - a + r) R1) /. {r -> r1 + r2}
DABEqBA := 1/(32 a b r^2) ((a - b - r) (a + b - r) (a - b + r) ((a + b + r) - R1) -
      4 m r (a^2 + b^2 + r^2 - 2 a b - 2 a r - 2 b r) - 8 m^2 r^2) /. {r -> r1 + r2}
nEqBA := nC
DACEqBA := 0
DCBEqBA := 0
DACBEqBA := 0
```

We have used the assumption that higher-order recombination terms can be ignored, and therefore $r = r_1 + r_2$.

In[101]:=
```
ruleApplyEq := {p -> pEqBA, q -> qEqBA, n -> nEqBA,
   DAB -> DABEqBA, DAC -> DACEqBA, DCB -> DCBEqBA, DACB -> DACBEqBA}
```

We have

```
diffEqs /. ruleApplyEq // FullSimplify
```

```
{0, 0, 0, 0, 0, 0, 0}
```

which confirms that this is indeed an equilibrium. We omit the proof that this equilibrium is asymptotically stable (*cf.* Bürger and Akerman 2011). Instead, we directly proceed to the computation of the Jacobian matrix.

```
diffEqsTarget /. ruleApplyEq // FullSimplify
```

```
{0, 0, 0, 0, 0, 0, 0}
```

The equations of BA2011 obviously also represent an equilibrium.

Jacobian matrix and effective migration rate

Generic

In[102]:=
```
J := Map[Table[D[#, {i}], {i, {p, q, DAB, n, DAC, DCB, DACB}}] &, diffEqs]
J // MatrixForm
```

Out[103]//MatrixForm=

-m - a (-1 + p) - a p	0	b	
0	-m - b (-1 + q) - b q	a	
-2 a DAB + m q	-2 b DAB + m p	a + b - m - 2 a p - 2 b q - r1 - r2	
0	0	0	
-2 a DAC + m (n - nC)	0	0	
0	-2 b DCB + m (n - nC)	0	
-2 a DACB + m (DCB - (n - nC) q)	-2 b DACB + m (DAC - (n - nC) p)	-2 a DAC - 2 b DCB + m (n - nC)	m (D

As an intermediate step, we set n, D_{AC}, D_{CB}, and D_{ACB} to their equilibrium values n_c, 0, 0, and 0, respectively.

In[104]:=
```
JPrep = J /. {n -> nC, DAC -> 0, DCB -> 0, DACB -> 0} // FullSimplify;
JPrep // MatrixForm
```

Out[105]//MatrixForm=

a - m - 2 a p	0	b	0	0	0
0	b - m - 2 b q	a	0	0	0
-2 a DAB + m q	-2 b DAB + m p	a + b - m - 2 a p - 2 b q - r1 - r2	0	0	0
0	0	0	-m	a	b
0	0	0	m p	a - m - 2 a p - r1	0
0	0	0	m q	0	b - m - 2
0	0	0	m (DAB - p q)	-2 a DAB + m q	-2 b DA

Now we plug in the equilibrium coordinates into the generic matrix of first-order partial derivatives:

ruleApplyEq

$$\left\{p \to \frac{1}{8\,a\,(r1+r2)}\left(-a^2+b^2+6\,a\,(r1+r2)-4\,m\,(r1+r2)-(r1+r2)^2+(a-b+r1+r2)\sqrt{-8\,m\,(r1+r2)+(a+b+r1+r2)^2}\right),\right.$$
$$q \to \frac{1}{8\,b\,(r1+r2)}\left(a^2-b^2+6\,b\,(r1+r2)-4\,m\,(r1+r2)-(r1+r2)^2+(-a+b+r1+r2)\sqrt{-8\,m\,(r1+r2)+(a+b+r1+r2)^2}\right),\ n \to nC,\ DAB \to \frac{1}{32\,a\,b\,(r1+r2)^2}$$
$$\left(-8\,m^2\,(r1+r2)^2-4\,m\,(r1+r2)\left(a^2-2\,a\,b+b^2-2\,a\,(r1+r2)-2\,b\,(r1+r2)+(r1+r2)^2\right)+(a-b-r1-r2)\,(a+b-r1-r2)\,(a-b+r1+r2)\left(a+b+r1+r2-\sqrt{-8\,m\,(r1+r2)+(a+b+r1+r2)^2}\right)\right),$$
$$\left. DAC \to 0,\ DCB \to 0,\ DACB \to 0\right\}$$

In[106]:= **JEqGeneric = JPrep /. ruleApplyEq // FullSimplify;**
JEqGeneric // MatrixForm

Out[107]//MatrixForm=

$$a - m - \frac{-a^2+b^2+6\,a\,(r1+r2)-4\,m\,(r1+r2)-(r1+r2)^2+(a-b+r1+r2 \ldots}{4\,(r1+r2)}$$

0

$$\frac{8\,m^2\,(r1+r2)^2+4\,m\,(r1+r2)\left(a^2+(-b+r1+r2)^2-2\,a\,(b+r1+r2)\right)-(a-b-r1-r2)\,(a+b-r1-r2)\,(a-b+r1+r2)\left(a+b+r1+r2-\sqrt{-8\,m\,(r1+r2)+(a+b+r1+\ldots}\right.}{16\,b\,(r1+r2)^2}$$

0

0

0

0

JEqGeneric⟦4 ;; 7, 4 ;; 7⟧ // MatrixForm

$$\frac{m\left(-a^2+b^2+6\,a\,(r\ldots\right.}{}$$

$$\frac{m\left(a^2-b^2+6\,b\,(r1\ldots\right.}{}$$

$$\frac{m\left(-\left(-a^2+b^2+6\,a\,(r1+r2)-4\,m\,(r1+r2)-(r1+r2)^2+(a-b+r1+r2)\sqrt{-8\,m\,(r1+r2)+(a+b+r1+r2)^2}\right)\left(a^2-b^2+6\,b\,(r1+r2)-4\,m\,(r1+r2)-(r1+r2)^2+(-\ldots\right.\right.}{}$$

```
JEqGeneric[[4 ;; 7, 4 ;; 7]]
```

$$
\begin{aligned}
&\Bigg\{\{-m, a, b, 0\}, \Bigg\{\frac{1}{8\, a\, (r1+r2)}\, m \Big(-a^2+b^2+6\, a\, (r1+r2)- \\
&\quad 4\, m\, (r1+r2)-(r1+r2)^2+(a-b+r1+r2)\sqrt{-8\, m\, (r1+r2)+(a+b+r1+r2)^2}\Big), \\
&\quad a-m-r1-\frac{1}{4\,(r1+r2)}\Big(-a^2+b^2+6\, a\, (r1+r2)-4\, m\, (r1+r2)-(r1+r2)^2+ \\
&\quad (a-b+r1+r2)\sqrt{-8\, m\, (r1+r2)+(a+b+r1+r2)^2}\Big), 0, b\Bigg\}, \\
&\quad \Bigg\{\frac{1}{8\, b\, (r1+r2)}\, m \Big(a^2-b^2+6\, b\, (r1+r2)-4\, m\, (r1+r2)-(r1+r2)^2+ \\
&\quad (-a+b+r1+r2)\sqrt{-8\, m\, (r1+r2)+(a+b+r1+r2)^2}\Big), 0, \\
&\quad b-m-r2-\frac{1}{4\,(r1+r2)}\Big(a^2-b^2+6\, b\, (r1+r2)-4\, m\, (r1+r2)-(r1+r2)^2+ \\
&\quad (-a+b+r1+r2)\sqrt{-8\, m\, (r1+r2)+(a+b+r1+r2)^2}\Big), a\Bigg\}, \\
&\quad \Bigg\{\frac{1}{64\, a\, b\, (r1+r2)^2}\, m \Bigg(-\Big(-a^2+b^2+6\, a\, (r1+r2)-4\, m\, (r1+r2)-(r1+r2)^2+ \\
&\quad (a-b+r1+r2)\sqrt{-8\, m\, (r1+r2)+(a+b+r1+r2)^2}\Big)\Big(a^2-b^2+6\, b\, (r1+r2)- \\
&\quad 4\, m\, (r1+r2)-(r1+r2)^2+(-a+b+r1+r2)\sqrt{-8\, m\, (r1+r2)+(a+b+r1+r2)^2}\Big)+ \\
&\quad 2\Big(-8\, m^2\, (r1+r2)^2-4\, m\, (r1+r2)\big(a^2+(-b+r1+r2)^2-2\, a\, (b+r1+r2)\big)+ \\
&\quad (a-b-r1-r2)\, (a+b-r1-r2)\, (a-b+r1+r2) \\
&\quad \Big(a+b+r1+r2-\sqrt{-8\, m\, (r1+r2)+(a+b+r1+r2)^2}\Big)\Big)\Bigg), \\
&\quad \frac{1}{16\, b\, (r1+r2)^2}\Big(8\, m^2\, (r1+r2)^2+4\, m\, (r1+r2)\big(a^2+(-b+r1+r2)^2-2\, a\, (b+r1+r2)\big)- \\
&\quad (a-b-r1-r2)\, (a+b-r1-r2)\, (a-b+r1+r2) \\
&\quad \Big(a+b+r1+r2-\sqrt{-8\, m\, (r1+r2)+(a+b+r1+r2)^2}\Big)+2\, m\, (r1+r2)\Big(a^2-b^2+6\, b\, (r1+r2)- \\
&\quad 4\, m\, (r1+r2)-(r1+r2)^2+(-a+b+r1+r2)\sqrt{-8\, m\, (r1+r2)+(a+b+r1+r2)^2}\Big)\Big), \\
&\quad \frac{1}{16\, a\, (r1+r2)^2}\Big(8\, m^2\, (r1+r2)^2+4\, m\, (r1+r2)\big(a^2+(-b+r1+r2)^2-2\, a\, (b+r1+r2)\big)- \\
&\quad (a-b-r1-r2)\, (a+b-r1-r2)\, (a-b+r1+r2) \\
&\quad \Big(a+b+r1+r2-\sqrt{-8\, m\, (r1+r2)+(a+b+r1+r2)^2}\Big)+ \\
&\quad 2\, m\, (r1+r2)\Big(-a^2+b^2+6\, a\, (r1+r2)-4\, m\, (r1+r2)-(r1+r2)^2+ \\
&\quad (a-b+r1+r2)\sqrt{-8\, m\, (r1+r2)+(a+b+r1+r2)^2}\Big)\Big), \\
&\quad \frac{1}{2}\Big(-a-b+2\, m-r1-r2-\sqrt{-8\, m\, (r1+r2)+(a+b+r1+r2)^2}\Big)\Bigg\}\Bigg\}
\end{aligned}
$$

Now we proceed analogous to the previous steps, but starting from eqs. (4.25) and (4.26) in BA2011 for the differentials of n, D_{AC}, D_{CB} and D_{ACB}.

In[108]:=
```
JTarget := Map[Table[D[#, i], {i, {p, q, DAB, n, DAC, DCB, DACB}}] &, diffEqsTarget]
JTarget // MatrixForm
```

Out[109]//MatrixForm=

```
-m - a (-1 + p) - a p
0
-2 a DAB + m q
m (DCB + (-n + nC) (1 - q))
-2 a DAC + m (-DACB + DAB (-n + nC) + DCB (1 - p) - DCB p + (-n + nC) p (1 - q) - DAC (-1 +
m (-DCB q + (-n + nC) (1 - q) q)
-2 a DACB + m (-DAB DCB - DAB (-n + nC) + DACB q + DCB p q + (-n + nC) p (1 - q) q + DAC (-1 + q) q + DCB
```

In[110]:=
```
JPrepTarget = JTarget /. {n → nC, DAC → 0, DCB → 0, DACB → 0} // FullSimplify;
JPrepTarget // TableForm
```

Out[111]//TableForm=

a - m - 2 a p	0	b	0
0	b - m - 2 b q	a	0
-2 a DAB + m q	-2 b DAB + m p	a + b - m - 2 a p - 2 b q - r1 - r2	0
0	0	0	m (p (-1 + q) - q)
0	0	0	-m p (DAB + p + q - p q)
0	0	0	-m q (DAB + p + q - p q)
0	0	0	-m (DAB + p (-1 + q) - q) (DAB - p

```
JPrep // TableForm
```

a - m - 2 a p	0	b	0	0	0
0	b - m - 2 b q	a	0	0	0
m q	m p	a + b - m - 2 a p - 2 b q - r1 - r2	0	0	0
0	0	0	-m	a	b
0	0	0	m p	a - m - 2 a p - r1	0
0	0	0	m q	0	b - m - 2
0	0	0	-m p q	m q	m p

```
JPrep - JPrepTarget // TableForm
```

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	-m - m (p (-1 + q) - q)	-m (-1 + q)	-m (-1 + p)
0	0	0	m p + m p (p + q - p q)	-m - m (-1 + p - p q)	m (-1 + p) p
0	0	0	m q + m q (p + q - p q)	m (-1 + q) q	-m - m (-1 + q - p q)
0	0	0	-m p q + m p q (p + q - p q)	m q - m (1 + p (-1 + q)) q	m p - m p (1 + (-1 + p) q)

We note that the two matrices differ. At least one must be wrong.

Now we plug in the equilibrium coordinates into the generic matrix of first-order partial derivatives:

JTarget /. ruleApplyEq // MatrixForm

```
( -m - (-a^2+b^2+6 a (r1+r2)-4 m (r1+r2)-(r1+r2)^2+(a-b+r1+r2) Sqrt[-8 m (r1+r2)+(a+b+r1+r2)^2])/(8 (r1+r2)) - a (-1 + ...
                                                  0
  (m (a^2-b^2+6 b (r1+r2)-4 m (r1+r2)-(r1+r2)^2+(-a+b+r1+r2) Sqrt[-8 m (r1+r2)+(a+b+r1+r2)^2]))/(8 b (r1+r2)) - (-8 m^2 (r1+r2)^2-4 m (r1+r2) (a^2-2 a b+b^2-2 a ...
                                                  0
                                                  0
                                                  0
                                                  0
```

- Using the Ansatz $m_e = -\lambda_N = m\, z$

JEqGeneric // MatrixForm

```
( 1/4 (-2 a + r1 + r2 + ((a-b) (a+b))/(r1+r2) - Sqrt[-8 m (r1 + r2) + (a + b + r1 + r2)^2] - a Sqrt[ ...
                                                  0
  (8 m^2 (r1+r2)^2+4 m (r1+r2) (a^2+(-b+r1+r2)^2-2 a (b+r1+r2))-(a-b-r1-r2) (a+b-r1-r2) (a-b+r1+r2) (a+b+r1+r2-Sqrt[-8 m (r1+r2)+(a+b+r1+ ...
  ---------------------------------------------------------------------------------------------------------------------------------
                                                  16 b (r1+r2)^2
                                                  0
                                                  0
                                                  0
                                                  0
```

Simplify[Series[Det[JTarget - x IdentityMatrix[7] /. x → -m z], {m, 0, 1}], Assumptions → {a ≥ 0, b ≥ 0, r1 ≥ 0, r2 ≥ 0}] // Normal

```
-a b m (a (-1 + 2 p) + b (-1 + 2 q) + r1 + r2) (a (-1 + 2 p) (2 DAB + (-1 + 2 p) (-1 + 2 q)) +
    (-1 + 2 q) (b (1 + 2 DAB - 2 p - 2 q + 4 p q) + (-1 + 2 p) (r1 + r2)))
  (r1 (r2 (p + q - p q - z) + b (3 q^2 + p (-1 + 4 q - 3 q^2) + q (-1 + DAB - 2 z) + z)) +
    a (r2 (-3 p^2 (-1 + q) - q + p (-1 + DAB + 4 q - 2 z) + z) + b (DAB^2 + q - 3 q^2 -
         3 p^2 (1 - 4 q + 3 q^2) + 2 DAB (p q - z) - z + 2 q z + p (1 - 7 q + 12 q^2 + 2 z - 4 q z))))
```

```
Simplify[Solve[% == 0, z]]
```

$$\{\{z \to (r1\,(-b\,(p + q - DAB\,q - 4\,p\,q - 3\,q^2 + 3\,p\,q^2) + (p + q - p\,q)\,r2) + a\,(b\,(DAB^2 + 2\,DAB\,p\,q - (-1 + 3\,p)\,(p\,(-1 + q) - q)\,(-1 + 3\,q)) - (3\,p^2\,(-1 + q) + q - p\,(-1 + DAB + 4\,q))\,r2))\,/\,(r1\,(b\,(-1 + 2\,q) + r2) + a\,(b\,(1 + 2\,DAB - 2\,p - 2\,q + 4\,p\,q) + (-1 + 2\,p)\,r2))\}\}$$

This is obviously not the same as in Bürger and Akerman (2011, Eq. 4.30).

Applying the same Ansatz to the Jacobian obtained from the differential equations developed in this *Mathematica* Notebook yields the correct solution:

```
Simplify[Series[Det[JEqGeneric - x IdentityMatrix[7] /. x → -m z], {m, 0, 1}],
  Assumptions → {a ≥ 0, b ≥ 0, r1 ≥ 0, r2 ≥ 0}] // Normal
```

$$a\,b\,m\,(a + b + r1 + r2)^2\,(a\,(b + r2)\,z + r1\,(r2\,(-1 + z) + b\,z))$$

```
Simplify[Solve[% == 0, z]]
```

$$\left\{\left\{z \to \frac{r1\,r2}{(a + r1)\,(b + r2)}\right\}\right\}$$

which is identical to eq. (4.30) in BA2011.

- Assuming weak migration

The coordinates of the internal equilibrium under the assumption of weak migration, up to and including first-order terms of *m*, can be obtained from eq. (4.1) in BA2011.

```
In[112]:= pEqBAWeakMig = FullSimplify[Series[pEqBA, {m, 0, 1}] // Normal,
    Assumptions → {0 < a < b, 0 < m, 0 < r1, 0 < r2}];
qEqBAWeakMig = FullSimplify[Series[qEqBA, {m, 0, 1}] // Normal,
    Assumptions → {0 < a < b, 0 < m, 0 < r1, 0 < r2}];
DABEqBAWeakMig = FullSimplify[Series[DABEqBA, {m, 0, 1}] // Normal,
    Assumptions → {0 < a < b, 0 < m, 0 < r1, 0 < r2}];
{pEqBAWeakMig, qEqBAWeakMig, DABEqBAWeakMig} // TableForm
```

$$1 - \frac{m\,(a+r1+r2)}{a\,(a+b+r1+r2)}$$
$$1 - \frac{m\,(b+r1+r2)}{b\,(a+b+r1+r2)}$$
$$\frac{m}{a+b+r1+r2}$$

Checking these against first-order terms w.r.t. *m* of Eq. (4.1) in BA2011:

$$1 - \frac{m}{a}\left(1 - \frac{b}{a + b + r}\right) - \text{pEqBAWeakMig} \;/.\; \{r \to r1 + r2\} \;//\; \text{FullSimplify}$$

0

$$1 - \frac{m}{b}\left(1 - \frac{a}{a + b + r}\right) - \text{qEqBAWeakMig} \;/.\; \{r \to r1 + r2\} \;//\; \text{FullSimplify}$$

0

$$\frac{m}{a + b + r} - \text{DABEqBAWeakMig} \;/.\; \{r \to r1 + r2\} \;//\; \text{FullSimplify}$$

0

```
In[115]:= ruleApplyEqWeakMig := {p → pEqBAWeakMig, q → qEqBAWeakMig,
  n → nC, DAB → DABEqBAWeakMig, DAC → 0, DCB → 0, DACB → 0}
ruleApplyEqWeakMig
```

$$\left\{p \to 1 - \frac{m\,(a + r1 + r2)}{a\,(a + b + r1 + r2)},\; q \to 1 - \frac{m\,(b + r1 + r2)}{b\,(a + b + r1 + r2)},\; n \to nC,\; DAB \to \frac{m}{a + b + r1 + r2},\; DAC \to 0,\; DCB \to 0,\; DACB \to 0\right\}$$

```
JPrep // MatrixForm
```

$$\begin{pmatrix} a-m-2\,a\,p & 0 & b & 0 & 0 & 0 \\ 0 & b-m-2\,b\,q & a & 0 & 0 & 0 \\ -2\,a\,DAB+m\,q & -2\,b\,DAB+m\,p & a+b-m-2\,a\,p-2\,b\,q-r1-r2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -m & a & b \\ 0 & 0 & 0 & m\,p & a-m-2\,a\,p-r1 & 0 \\ 0 & 0 & 0 & m\,q & 0 & b-m-2 \\ 0 & 0 & 0 & m\,(DAB-p\,q) & -2\,a\,DAB+m\,q & -2\,b\,DA \end{pmatrix}$$

Now we see the block structure claimed in eq. (4.27) of BA2011.

In[116]:=
```
JEq = JPrep /. ruleApplyEqWeakMig // FullSimplify;
JEq // MatrixForm
```

Out[117]//MatrixForm=

$$\begin{pmatrix} -a+m-\frac{2\,b\,m}{a+b+r1+r2} & 0 & b & (\\ 0 & -b+m-\frac{2\,a\,m}{a+b+r1+r2} & a & (\\ \frac{m\,(-a\,b+(b-m)\,(b+r1+r2))}{b\,(a+b+r1+r2)} & m\left(1-\frac{a\,(2\,b+m)+m\,(r1+r2)}{a\,(a+b+r1+r2)}\right) & -a-b+3\,m-r1-r2-\frac{2\,(a+b)\,m}{a+b+r1+r2} & (\\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & m\left(1-\frac{m\,(}{a\,(a}\right. \\ 0 & 0 & 0 & m\left(1-\frac{m\,(}{b\,(a}\right. \\ 0 & 0 & 0 & m\left(\frac{m}{a+b+r1+r2}-\left(1-\frac{m\,(a+r1}{a\,(a+b+r}\right.\right. \end{pmatrix}$$

The matrix above should correspond to the one given in Box 1 in BA2011, but it does not. For instance, the difference between the elements at the bottom right position is:

$$-a-b+3\,m-r1-r2-\frac{2\,(a+b)\,m}{a+b+r1+r2}-\left(-a-b-r+2\,m\,\frac{(a+b+2\,r)}{a+b+r}\right)\ \text{/. \{r → r1 + r2\} //}$$
```
FullSimplify
```

$-m$

In[118]:=
```
JEqTarget = JPrepTarget /. ruleApplyEqWeakMig // FullSimplify;
JEqTarget // MatrixForm
```

Out[119]//MatrixForm=

$$\begin{pmatrix} -a+m-\frac{2\,b\,m}{a+b+r1+r2} & 0 & b & \\ 0 & -b+m-\frac{2\,a\,m}{a+b+r1+r2} & a & \\ \frac{m\,(-a\,b+(b-m)\,(b+r1+r2))}{b\,(a+b+r1+r2)} & m\left(1-\frac{a\,(2\,b+m)+m\,(r1+r2)}{a\,(a+b+r1+r2)}\right) & -a-b+3\,m-r1-r2-\frac{2\,(a+b)\,m}{a+b+r1+r2} & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & -\frac{m\,\left(a^2-m\,(r1+r2)+a\,(b-m+r1+\right.}{} \\ 0 & 0 & 0 & -\frac{m\,(-a\,b-(b-m)\,(b+r1+r2)}{} \\ 0 & 0 & 0 & -m\left(-1+\frac{m\,\left(a^2\,b+a\,(b+m)\,(b+r\right.}{a\,b\,(}\right. \end{pmatrix}$$

But the matrix above is also different from the one presented in Box 1 in BA2011.

```
JEq // MatrixForm
```

$$\begin{pmatrix} -a+m-\frac{2\,b\,m}{a+b+r1+r2} & 0 & b & \cdots \\ 0 & -b+m-\frac{2\,a\,m}{a+b+r1+r2} & a & \cdots \\ \frac{m\,(-a\,b+(b-m)\,(b+r1+r2))}{b\,(a+b+r1+r2)} & m\left(1-\frac{a\,(2\,b+m)+m\,(r1+r2)}{a\,(a+b+r1+r2)}\right) & -a-b+3\,m-r1-r2-\frac{2\,(a+b)\,m}{a+b+r1+r2} & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & m\left(1-\frac{m\,\cdots}{a\,(\cdots}\right. \\ 0 & 0 & 0 & m\left(1-\frac{m\,\cdots}{b\,(\cdots}\right. \\ 0 & 0 & 0 & m\left(\frac{m}{a+b+r1+r2}-\left(1-\frac{m\,(a+r1\cdots}{a\,(a+b+r\cdots}\right.\right. \end{pmatrix}$$

In[120]:=
```
JEqN = JEq[[{4, 5, 6, 7}, {4, 5, 6, 7}]];
JEqN // MatrixForm
```

Out[121]//MatrixForm=

$$\begin{pmatrix} -m & a & b & \cdots \\ m\left(1-\frac{m\,(a+r1+r2)}{a\,(a+b+r1+r2)}\right) & -a+m-r1-\frac{2\,b\,m}{a+b+r1+r2} & 0 & \cdots \\ m\left(1-\frac{m\,(b+r1+r2)}{b\,(a+b+r1+r2)}\right) & 0 & -b+m-r2-\frac{2\,a\,m}{a+b+r1+r2} & \cdots \\ m\left(\frac{m}{a+b+r1+r2}-\left(1-\frac{m\,(a+r1+r2)}{a\,(a+b+r1+r2)}\right)\left(1-\frac{m\,(b+r1+r2)}{b\,(a+b+r1+r2)}\right)\right) & \frac{m\,(-a\,b+(b-m)\,(b+r1+r2))}{b\,(a+b+r1+r2)} & m\left(1-\frac{a\,(2\,b+m)+m\,(r1+r2)}{a\,(a+b+r1+r2)}\right) & -a-\cdots \end{pmatrix}$$

If we now assume that the migration rate *m* is weak, we obtain

In[122]:=
```
JEqmSmall := Simplify[Normal[Series[JEq, {m, 0, 1}]]]
JEqmSmall // MatrixForm
```

Out[123]//MatrixForm=

$$\begin{pmatrix} -a+m-\frac{2\,b\,m}{a+b+r1+r2} & 0 & b & 0 & 0 & \cdots \\ 0 & -b+m-\frac{2\,a\,m}{a+b+r1+r2} & a & 0 & 0 & \cdots \\ \frac{m\,(-a+b+r1+r2)}{a+b+r1+r2} & m-\frac{2\,b\,m}{a+b+r1+r2} & -a-b-r1-r2+\frac{m\,(a+b+3\,(r1+r2))}{a+b+r1+r2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & -m & a & \cdots \\ 0 & 0 & 0 & m & -a+m-r1-\frac{2\,b\,m}{a+b+r1+r2} & \cdots \\ 0 & 0 & 0 & m & 0 & -b+m\cdots \\ 0 & 0 & 0 & -m & \frac{m\,(-a+b+r1+r2)}{a+b+r1+r2} & m\cdots \end{pmatrix}$$

In[124]:=
```
JEqNmSmall := JEqmSmall[[4 ;; 7, 4 ;; 7]]
JEqNmSmall // MatrixForm
```

Out[125]//MatrixForm=

$$\begin{pmatrix} -m & a & b & 0 \\ m & -a+m-r1-\frac{2\,b\,m}{a+b+r1+r2} & 0 & b \\ m & 0 & -b+m-r2-\frac{2\,a\,m}{a+b+r1+r2} & a \\ -m & \frac{m\,(-a+b+r1+r2)}{a+b+r1+r2} & m-\frac{2\,b\,m}{a+b+r1+r2} & -a-b-r1-r2+\frac{m\,(a+b+3\,(r1+r2))}{a+b+r1+r2} \end{pmatrix}$$

This can alternativel be written as

$$\text{JEqNmSmallDispl} := \left\{\{-m, a, b, 0\}, \left\{m, -a-r1+\frac{m\,(a-b+r)}{a+b+r}, 0, b\right\}, \left\{m, 0, -b-r2+\frac{m\,(b-a+r)}{a+b+r}, a\right\}, \left\{-m, \frac{m\,(b-a+r)}{a+b+r}, \frac{m\,(a-b+r)}{a+b+r}, -a-b-r+\frac{m\,(a+b+3\,r)}{a+b+r}\right\}\right\}$$

```
JEqNmSmallDispl - JEqNmSmall /. {r → r1 + r2} // Simplify
```

```
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

```
JEqNmSmallDispl // MatrixForm
```

$$\begin{pmatrix} -m & a & b & 0 \\ m & -a+\frac{m(a-b+r)}{a+b+r}-r1 & 0 & b \\ m & 0 & -b+\frac{m(-a+b+r)}{a+b+r}-r2 & a \\ -m & \frac{m(-a+b+r)}{a+b+r} & \frac{m(a-b+r)}{a+b+r} & -a-b-r+\frac{m(a+b+3r)}{a+b+r} \end{pmatrix}$$

In[126]:=
```
JEqNTarget = JEqTarget[[{4, 5, 6, 7}, {4, 5, 6, 7}]];
JEqNTarget // MatrixForm
```

Out[127]//MatrixForm=

$$\left(\begin{array}{l} -m+\frac{m^3 (a+r1+r2)(b+r1+r2)}{a\,b\,(a+b+r1+r2)^2} \\ -\frac{m\left(a^2-m(r1+r2)+a(b-m+r1+r2)\right)\left(a^3 b-m^2(r1+r2)(b+r1+r2)+a^2 b(2b+m+2(r1+r2))+a(b+r1+r2)\left(b^2-m^2+b(m+r1+r2)\right)\right)}{a^2 b\,(a+b+r1+r2)^3} \\ -\frac{m(-a\,b-(b-m)(b+r1+r2))\left(-a^3 b+m^2(r1+r2)(b+r1+r2)-a^2 b(2b+m+2(r1+r2))-a(b+r1+r2)\left(b^2-m^2+b(m+r1+r2)\right)\right)}{a\,b^2(a+b+r1+r2)^3} \\ -m\left(-1+\frac{m\left(a^2 b+a(b+m)(b+r1+r2)+m(r1+r2)(b+r1+r2)\right)}{a\,b\,(a+b+r1+r2)^2}\right)\left(\frac{m}{a+b+r1+r2}-\left(1-\frac{m(a+r1+r2)}{a(a+b+r1+r2)}\right)\left(1-\frac{m(b+r1+r2)}{b(a+b+r1+r2)}\right)\right)\quad m\left(1-\right. \end{array}\right.$$

```
Eigenvalues[JEqN] // Simplify
```

A very large output was generated. Here is a sample of it:

```
{Root[a^3 b m^3 + 2 a^2 b^2 m^3 + <<133>> +
    (a^3 + 3 a^2 b + 3 a b^2 + b^3 + 3 a^2 r1 + 6 a b r1 + 3 b^2 r1 + 3 a r1^2 + 3 b r1^2 + r1^3 + 3 a^2 r2 + 6 a b r2 +
       3 b^2 r2 + 6 a r1 r2 + 6 b r1 r2 + 3 r1^2 r2 + 3 a r2^2 + 3 b r2^2 + 3 r1 r2^2 + r2^3) #1^4 &, 1],
 Root[a^3 b m^3 + 2 a^2 b^2 m^3 + <<133>> + (a^3 + 3 a^2 b + 3 a b^2 + b^3 + 3 a^2 r1 + 6 a b r1 +
       3 b^2 r1 + 3 a r1^2 + 3 b r1^2 + r1^3 + 3 a^2 r2 + 6 a b r2 + 3 b^2 r2 + 6 a r1 r2 +
       6 b r1 r2 + 3 r1^2 r2 + 3 a r2^2 + 3 b r2^2 + 3 r1 r2^2 + r2^3) #1^4 &, 2], Root[
  a^3 b m^3 + <<1>> + <<133>> + (a^3 + 3 a^2 b + 3 a b^2 + b^3 + <<13>> + 3 b r2^2 + 3 r1 r2^2 + r2^3) #1^4 &,
  3],
 Root[a^3 b m^3 + 2 a^2 b^2 m^3 + <<133>> +
    (a^3 + 3 a^2 b + 3 a b^2 + b^3 + 3 a^2 r1 + 6 a b r1 + 3 b^2 r1 + 3 a r1^2 + 3 b r1^2 + r1^3 + 3 a^2 r2 + 6 a b r2 +
       3 b^2 r2 + 6 a r1 r2 + 6 b r1 r2 + 3 r1^2 r2 + 3 a r2^2 + 3 b r2^2 + 3 r1 r2^2 + r2^3) #1^4 &, 4]}
```

Show Less | Show More | Show Full Output | Set Size Limit...

```
Eigenvalues[JEqNTarget]
```

A very large output was generated. Here is a sample of it:

```
{Root[
  4 a^10 b^3 m + <<1731>> + (a^8 b^2 + 6 a^7 b^3 + 15 a^6 b^4 + 20 a^5 b^5 + 15 a^4 b^6 + 6 a^3 b^7 + a^2 b^8 + 6 a^7 b^2 r1 +
       30 a^6 b^3 r1 + 60 a^5 b^4 r1 + 60 a^4 b^5 r1 + 30 a^3 b^6 r1 + 6 a^2 b^7 r1 + 15 a^6 b^2 r1^2 + 60 a^5 b^3 r1^2 +
       90 a^4 b^4 r1^2 + 60 a^3 b^5 r1^2 + 15 a^2 b^6 r1^2 + 20 a^5 b^2 r1^3 + 60 a^4 b^3 r1^3 + 60 a^3 b^4 r1^3 +
       20 a^2 b^5 r1^3 + 15 a^4 b^2 r1^4 + <<38>> + 60 a^3 b^2 r1^3 r2^2 + 60 a^2 b^3 r1^3 r2^2 + 15 a^2 b^2 r1^4 r2^2 +
       20 a^5 b^2 r2^3 + 60 a^4 b^3 r2^3 + 60 a^3 b^4 r2^3 + 20 a^2 b^5 r2^3 + 60 a^4 b^2 r1 r2^3 + 120 a^3 b^3 r1 r2^3 +
       60 a^2 b^4 r1 r2^3 + 60 a^3 b^2 r1^2 r2^3 + 60 a^2 b^3 r1^2 r2^3 + 20 a^2 b^2 r1^3 r2^3 + 15 a^4 b^2 r2^4 +
       30 a^3 b^3 r2^4 + 15 a^2 b^4 r2^4 + 30 a^3 b^2 r1 r2^4 + 30 a^2 b^3 r1 r2^4 + 15 a^2 b^2 r1^2 r2^4 +
       6 a^3 b^2 r2^5 + 6 a^2 b^3 r2^5 + 6 a^2 b^2 r1 r2^5 + a^2 b^2 r2^6) #1^4 &, 1], <<2>>, Root[<<1>> &, 4]}
```

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- Using the Ansatz $m_e = -\lambda_N = m\,z$

JEq

$$\left\{\left\{-a+m-\frac{2\,b\,m}{a+b+r1+r2},\,0,\,b,\,0,\,0,\,0,\,0\right\},\,\left\{0,\,-b+m-\frac{2\,a\,m}{a+b+r1+r2},\,a,\,0,\,0,\,0,\,0\right\},\right.$$
$$\left\{\frac{m\,(-a\,b+(b-m)\,(b+r1+r2))}{b\,(a+b+r1+r2)},\,m\left(1-\frac{a\,(2\,b+m)+m\,(r1+r2)}{a\,(a+b+r1+r2)}\right),\right.$$
$$\left.-a-b+3\,m-r1-r2-\frac{2\,(a+b)\,m}{a+b+r1+r2},\,0,\,0,\,0,\,0\right\},\,\{0,\,0,\,0,\,-m,\,a,\,b,\,0\},$$
$$\left\{0,\,0,\,0,\,m\left(1-\frac{m\,(a+r1+r2)}{a\,(a+b+r1+r2)}\right),\,-a+m-r1-\frac{2\,b\,m}{a+b+r1+r2},\,0,\,b\right\},$$
$$\left\{0,\,0,\,0,\,m\left(1-\frac{m\,(b+r1+r2)}{b\,(a+b+r1+r2)}\right),\,0,\,-b+m-r2-\frac{2\,a\,m}{a+b+r1+r2},\,a\right\},$$
$$\left\{0,\,0,\,0,\,m\left(\frac{m}{a+b+r1+r2}-\left(1-\frac{m\,(a+r1+r2)}{a\,(a+b+r1+r2)}\right)\left(1-\frac{m\,(b+r1+r2)}{b\,(a+b+r1+r2)}\right)\right),\right.$$
$$\frac{m\,(-a\,b+(b-m)\,(b+r1+r2))}{b\,(a+b+r1+r2)},\,m\left(1-\frac{a\,(2\,b+m)+m\,(r1+r2)}{a\,(a+b+r1+r2)}\right),$$
$$\left.\left.-a-b+3\,m-r1-r2-\frac{2\,(a+b)\,m}{a+b+r1+r2}\right\}\right\}$$

Simplify[Series[Det[JEqTarget − x IdentityMatrix[7] /. x → −m $\frac{r1\,r2}{(r1+a)\,(r2+b)}$], {m, 0, 1}]]

$$a\,b\,(-a-b-r1-r2)\left(a^2\,b+a\,b^2+a\,b\,r1+a\,b\,r2+(a+b+r1+r2)\,(3\,a\,b+2\,b\,r1+2\,a\,r2)\right)m+O[m]^2$$

JEqTarget // TableForm

$-a+m-\frac{2\,b\,m}{a+b+r1+r2}$	0	b	0
0	$-b+m-\frac{2\,a\,m}{a+b+r1+r2}$	a	0
$\frac{m\,(-a\,b+(b-m)\,(b+r1+r2))}{b\,(a+b+r1+r2)}$	$m\left(1-\frac{a\,(2\,b+m)+m\,(r1+r2)}{a\,(a+b+r1+r2)}\right)$	$-a-b+3\,m-r1-r2-\frac{2\,(a+b)\,m}{a+b+r1+r2}$	0
0	0	0	$-m+\frac{m^3\,(a+r1+r2)\,(b+}{a\,b\,(a+b+r1+r}$
0	0	0	$-\frac{m\,(a^2-m\,(r1+r2)+a\,(b-}{}$
0	0	0	$-\frac{m\,(-a\,b-(b-m)\,(b+r1+}{}$
0	0	0	$-m\left(-1+\frac{m\,(a^2\,b+a\,(}{}\right.$

Dimensions[JEqTarget]

{7, 7}

Simplify[Series[Det[JEqTarget − x IdentityMatrix[7] /. x → −m z], {m, 0, 1}]] // Normal

$$a\,b\,m\,(a+b+r1+r2)^2\,(a\,(b\,(-4+z)+r2\,(-2+z))+r1\,(b\,(-2+z)+r2\,(-1+z)))$$

Simplify[Solve[% == 0, z]]

$$\left\{\left\{z\to\frac{(2\,a+r1)\,(2\,b+r2)}{(a+r1)\,(b+r2)}\right\}\right\}$$

This does not yield the same term as the one given in eq. (4.30) of BA2011, which suggests that the equations (4.25) and (4.26) in BA2011 are wrong.

Applying the same Ansatz to the Jacobian obtained from the differential equations developed in this *Mathematica* Notebook, however, yields the correct solution:

```
Simplify[Series[Det[(JEq /. ruleApplyEq) - x IdentityMatrix[7] /. x → -m z], {m, 0, 1}],
  Assumptions → {a ≥ 0, b ≥ 0, r1 ≥ 0, r2 ≥ 0}] // Normal
```

$$a\, b\, m\, (a + b + r1 + r2)^2\, (a\, (b + r2)\, z + r1\, (r2\, (-1 + z) + b\, z))$$

```
Simplify[Solve[% == 0, z]]
```

$$\left\{\left\{z \to \frac{r1\, r2}{(a + r1)\, (b + r2)}\right\}\right\}$$

which is identical to eq. (4.30) in BA2011.

We can also use the approximate J_N for small m:

```
Simplify[Series[Det[(JEqmSmall /. ruleApplyEq) - x IdentityMatrix[7] /. x → -m z],
   {m, 0, 1}], Assumptions → {a ≥ 0, b ≥ 0, r1 ≥ 0, r2 ≥ 0}] // Normal
```

$$a\, b\, m\, (a + b + r1 + r2)^2\, (a\, (b + r2)\, z + r1\, (r2\, (-1 + z) + b\, z))$$

```
Simplify[Solve[% == 0, z]]
```

$$\left\{\left\{z \to \frac{r1\, r2}{(a + r1)\, (b + r2)}\right\}\right\}$$

- Starting directly from eq. (4.28) in BA2011

Next, we directly start with J_N given in Box 1 (eq. 4.28) of BA2011 and compute the eigenvalues.

In[128]:=
```
JEqNBA2011 :=
 {{-m, a, b, m}, {m, -a - r1 + m (a - b + r)/(a + b + r), 0, b - m}, {m, 0, -b - r2 + m (b - a + r)/(a + b + r), a - m},
   {-m, m (b - a + r)/(a + b + r), m (a - b + r)/(a + b + r), -a - b - r + 2 m (a + b + 2 r)/(a + b + r)}} /. {r → r1 + r2}
JEqNBA2011 // MatrixForm
```

$$\begin{pmatrix} -m & a & b & m \\ m & -a - r1 + \frac{m\,(a-b+r1+r2)}{a+b+r1+r2} & 0 & b - m \\ m & 0 & -b - r2 + \frac{m\,(-a+b+r1+r2)}{a+b+r1+r2} & a - m \\ -m & \frac{m\,(-a+b+r1+r2)}{a+b+r1+r2} & \frac{m\,(a-b+r1+r2)}{a+b+r1+r2} & -a - b - r1 - r2 + \frac{2\,m\,(a+b+2\,(r1+r2))}{a+b+r1+r2} \end{pmatrix}$$

```
FullSimplify[Eigenvalues[JEqNBA2011], Assumptions → {0 < a < b, 0 < m, 0 < r}]
```

$$\begin{aligned}
&\{\mathrm{Root}[\\
&a^4 b m^2 + 3 a^3 b^2 m^2 + 3 a^2 b^3 m^2 + a b^4 m^2 + a^3 m^4 - a^2 b m^4 - a b^2 m^4 + b^3 m^4 + 2 a^3 b m^2 r1 + 3 a^2 b^2 m^2 r1 -\\
&\quad b^4 m^2 r1 - 2 a^2 b m^3 r1 + 2 b^3 m^3 r1 + 3 a^2 m^4 r1 - 6 a b m^4 r1 + 3 b^2 m^4 r1 - a^3 m^2 r1^2 - 3 a^2 b m^2 r1^2 -\\
&\quad 6 a b^2 m^2 r1^2 - 4 b^3 m^2 r1^2 + 4 a b m^3 r1^2 + 8 b^2 m^3 r1^2 - a m^4 r1^2 - b m^4 r1^2 - 3 a^2 m^2 r1^3 -\\
&\quad 8 a b m^2 r1^3 - 6 b^2 m^2 r1^3 + 4 a m^3 r1^3 + 10 b m^3 r1^3 - 3 m^4 r1^3 - 3 a m^2 r1^4 - 4 b m^2 r1^4 +\\
&\quad 4 m^3 r1^4 - m^2 r1^5 - a^4 m^2 r2 + 3 a^2 b^2 m^2 r2 + 2 a b^3 m^2 r2 + 2 a^3 m^3 r2 - 2 a b^2 m^3 r2 + 3 a^2 m^4 r2 -\\
&\quad 6 a b m^4 r2 + 3 b^2 m^4 r2 + a^4 m r1 r2 + 4 a^3 b m r1 r2 + 6 a^2 b^2 m r1 r2 + 4 a b^3 m r1 r2 + b^4 m r1 r2 -\\
&\quad 6 a^3 m^2 r1 r2 - 12 a^2 b m^2 r1 r2 - 12 a b^2 m^2 r1 r2 - 6 b^3 m^2 r1 r2 + 8 a^2 m^3 r1 r2 + 8 a b m^3 r1 r2 +\\
&\quad 8 b^2 m^3 r1 r2 - 2 a m^4 r1 r2 - 2 b m^4 r1 r2 + 4 a^3 m r1^2 r2 + 12 a^2 b m r1^2 r2 + 12 a b^2 m r1^2 r2 +\\
&\quad 4 b^3 m r1^2 r2 - 17 a^2 m^2 r1^2 r2 - 34 a b m^2 r1^2 r2 - 20 b^2 m^2 r1^2 r2 + 18 a m^3 r1^2 r2 + 24 b m^3 r1^2 r2 -\\
&\quad 9 m^4 r1^2 r2 + 6 a^2 m r1^3 r2 + 12 a b m r1^3 r2 + 6 b^2 m r1^3 r2 - 20 a m^2 r1^3 r2 - 22 b m^2 r1^3 r2 +\\
&\quad 16 m^3 r1^3 r2 + 4 a m r1^4 r2 + 4 b m r1^4 r2 - 8 m^2 r1^4 r2 + m r1^5 r2 - 4 a^3 m^2 r2^2 - 6 a^2 b m^2 r2^2 -\\
&\quad 3 a b^2 m^2 r2^2 - b^3 m^2 r2^2 + 8 a^2 m^3 r2^2 + 4 a b m^3 r2^2 - a m^4 r2^2 - b m^4 r2^2 + 4 a^3 m r1 r2^2 +\\
&\quad 12 a^2 b m r1 r2^2 + 12 a b^2 m r1 r2^2 + 4 b^3 m r1 r2^2 - 20 a^2 m^2 r1 r2^2 - 34 a b m^2 r1 r2^2 -\\
&\quad 17 b^2 m^2 r1 r2^2 + 24 a m^3 r1 r2^2 + 18 b m^3 r1 r2^2 - 9 m^4 r1 r2^2 + 12 a^2 m r1^2 r2^2 + 24 a b m r1^2 r2^2 +\\
&\quad 12 b^2 m r1^2 r2^2 - 35 a m^2 r1^2 r2^2 - 35 b m^2 r1^2 r2^2 + 24 m^3 r1^2 r2^2 + 12 a m r1^3 r2^2 + 12 b m r1^3 r2^2 -\\
&\quad 19 m^2 r1^3 r2^2 + 4 m r1^4 r2^2 - 6 a^2 m^2 r2^3 - 8 a b m^2 r2^3 - 3 b^2 m^2 r2^3 + 10 a m^3 r2^3 + 4 b m^3 r2^3 -\\
&\quad 3 m^4 r2^3 + 6 a^2 m r1 r2^3 + 12 a b m r1 r2^3 + 6 b^2 m r1 r2^3 - 22 a m^2 r1 r2^3 - 20 b m^2 r1 r2^3 +\\
&\quad 16 m^3 r1 r2^3 + 12 a m r1^2 r2^3 + 12 b m r1^2 r2^3 - 19 m^2 r1^2 r2^3 + 6 m r1^3 r2^3 - 4 a m^2 r2^4 -\\
&\quad 3 b m^2 r2^4 + 4 m^3 r2^4 + 4 a m r1 r2^4 + 4 b m r1 r2^4 - 8 m^2 r1 r2^4 + 4 m r1^2 r2^4 - m^2 r2^5 + m r1 r2^5 +
\end{aligned}$$

```
(a^5 b + 4 a^4 b^2 + 6 a^3 b^3 + 4 a^2 b^4 + a b^5 - a^4 b m - 3 a^3 b^2 m - 3 a^2 b^3 m - a b^4 m - 2 a^4 m^2 + 4 a^2 b^2 m^2 -
   2 b^4 m^2 - a^3 m^3 + a^2 b m^3 + a b^2 m^3 - b^3 m^3 + 5 a^4 b r1 + 16 a^3 b^2 r1 + 18 a^2 b^3 r1 + 8 a b^4 r1 + b^5 r1 -
   8 a^3 b m r1 - 19 a^2 b^2 m r1 - 14 a b^3 m r1 - 3 b^4 m r1 - 6 a^3 m^2 r1 + 2 a^2 b m^2 r1 + 2 a b^2 m^2 r1 -
   6 b^3 m^2 r1 + 3 a^2 m^3 r1 + 2 a b m^3 r1 + 3 b^2 m^3 r1 + 10 a^3 b r1^2 + 24 a^2 b^2 r1^2 + 18 a b^3 r1^2 +
   4 b^4 r1^2 - 18 a^2 b m r1^2 - 29 a b^2 m r1^2 - 11 b^3 m r1^2 - 6 a^2 m^2 r1^2 + 4 a b m^2 r1^2 - 2 b^2 m^2 r1^2 +
   5 a m^3 r1^2 + 5 b m^3 r1^2 + 10 a^2 b r1^3 + 16 a b^2 r1^3 + 6 b^3 r1^3 - 16 a b m r1^3 - 13 b^2 m r1^3 -
   2 a m^2 r1^3 + 2 b m^2 r1^3 + m^3 r1^3 + 5 a b r1^4 + 4 b^2 r1^4 - 5 b m r1^4 + b r1^5 + a^5 r2 + 8 a^4 b r2 +
   18 a^3 b^2 r2 + 16 a^2 b^3 r2 + 5 a b^4 r2 - 3 a^4 m r2 - 14 a^3 b m r2 - 19 a^2 b^2 m r2 - 8 a b^3 m r2 -
   6 a^3 m^2 r2 + 2 a^2 b m^2 r2 + 2 a b^2 m^2 r2 - 6 b^3 m^2 r2 + 3 a^2 m^3 r2 + 2 a b m^3 r2 + 3 b^2 m^3 r2 +
   5 a^4 r1 r2 + 32 a^3 b r1 r2 + 54 a^2 b^2 r1 r2 + 32 a b^3 r1 r2 + 5 b^4 r1 r2 - 12 a^3 m r1 r2 -
   50 a^2 b m r1 r2 - 50 a b^2 m r1 r2 - 12 b^3 m r1 r2 - 8 a^2 m^2 r1 r2 + 8 a b m^2 r1 r2 - 8 b^2 m^2 r1 r2 +
   10 a m^3 r1 r2 + 10 b m^3 r1 r2 + 10 a^3 r1^2 r2 + 48 a^2 b r1^2 r2 + 54 a b^2 r1^2 r2 + 16 b^3 r1^2 r2 -
   18 a^2 m r1^2 r2 - 58 a b m r1^2 r2 - 31 b^2 m r1^2 r2 - 2 a m^2 r1^2 r2 + 2 b m^2 r1^2 r2 + 3 m^3 r1^2 r2 +
   10 a^2 r1^3 r2 + 32 a b r1^3 r2 + 18 b^2 r1^3 r2 - 12 a m r1^3 r2 - 22 b m r1^3 r2 + 5 a r1^4 r2 +
   8 b r1^4 r2 - 3 m r1^4 r2 + r1^5 r2 + 4 a^4 r2^2 + 18 a^3 b r2^2 + 24 a^2 b^2 r2^2 + 10 a b^3 r2^2 -
   11 a^3 m r2^2 - 29 a^2 b m r2^2 - 18 a b^2 m r2^2 - 2 a^2 m^2 r2^2 + 4 a b m^2 r2^2 - 6 b^2 m^2 r2^2 +
   5 a m^3 r2^2 + 5 b m^3 r2^2 + 16 a^3 r1 r2^2 + 54 a^2 b r1 r2^2 + 48 a b^2 r1 r2^2 + 10 b^3 r1 r2^2 -
   31 a^2 m r1 r2^2 - 58 a b m r1 r2^2 - 18 b^2 m r1 r2^2 + 2 a m^2 r1 r2^2 - 2 b m^2 r1 r2^2 + 3 m^3 r1 r2^2 +
   24 a^2 r1^2 r2^2 + 54 a b r1^2 r2^2 + 24 b^2 r1^2 r2^2 - 29 a m r1^2 r2^2 - 29 b m r1^2 r2^2 + 16 a r1^3 r2^2 +
   18 b r1^3 r2^2 - 9 m r1^3 r2^2 + 4 r1^4 r2^2 + 6 a^3 r2^3 + 16 a^2 b r2^3 + 10 a b^2 r2^3 - 13 a^2 m r2^3 -
   16 a b m r2^3 + 2 a m^2 r2^3 - 2 b m^2 r2^3 + m^3 r2^3 + 18 a^2 r1 r2^3 + 32 a b r1 r2^3 + 10 b^2 r1 r2^3 -
   22 a m r1 r2^3 - 12 b m r1 r2^3 + 18 a r1^2 r2^3 + 16 b r1^2 r2^3 - 9 m r1^2 r2^3 + 6 r1^3 r2^3 + 4 a^2 r2^4 +
   5 a b r2^4 - 5 a m r2^4 + 8 a r1 r2^4 + 5 b r1 r2^4 - 3 m r1 r2^4 + 4 r1^2 r2^4 + a r2^5 + r1 r2^5) #1 +
(a^5 + 6 a^4 b + 13 a^3 b^2 + 13 a^2 b^3 + 6 a b^4 + b^5 - a^4 m - 4 a^3 b m - 6 a^2 b^2 m - 4 a b^3 m - b^4 m -
   2 a^3 m^2 - 2 a^2 b m^2 - 2 a b^2 m^2 - 2 b^3 m^2 + 5 a^4 r1 + 24 a^3 b r1 + 39 a^2 b^2 r1 + 26 a b^3 r1 +
   6 b^4 r1 - 8 a^3 m r1 - 26 a^2 b m r1 - 28 a b^2 m r1 - 10 b^3 m r1 - 2 a^2 m^2 r1 - 2 b^2 m^2 r1 +
   10 a^3 r1^2 + 36 a^2 b r1^2 + 39 a b^2 r1^2 + 13 b^3 r1^2 - 18 a^2 m r1^2 - 40 a b m r1^2 - 22 b^2 m r1^2 +
   6 a m^2 r1^2 + 6 b m^2 r1^2 + 10 a^2 r1^3 + 24 a b r1^3 + 13 b^2 r1^3 - 16 a m r1^3 - 18 b m r1^3 +
   6 m^2 r1^3 + 5 a r1^4 + 6 b r1^4 - 5 m r1^4 + r1^5 + 6 a^4 r2 + 26 a^3 b r2 + 39 a^2 b^2 r2 + 24 a b^3 r2 +
   5 b^4 r2 - 10 a^3 m r2 - 28 a^2 b m r2 - 26 a b^2 m r2 - 8 b^3 m r2 - 2 a^2 m^2 r2 - 2 b^2 m^2 r2 +
   24 a^3 r1 r2 + 78 a^2 b r1 r2 + 78 a b^2 r1 r2 + 24 b^3 r1 r2 - 40 a^2 m r1 r2 - 80 a b m r1 r2 -
   40 b^2 m r1 r2 + 12 a m^2 r1 r2 + 12 b m^2 r1 r2 + 36 a^2 r1^2 r2 + 78 a b r1^2 r2 + 39 b^2 r1^2 r2 -
   50 a m r1^2 r2 - 52 b m r1^2 r2 + 18 m^2 r1^2 r2 + 24 a r1^3 r2 + 26 b r1^3 r2 - 20 m r1^3 r2 +
   6 r1^4 r2 + 13 a^3 r2^2 + 39 a^2 b r2^2 + 36 a b^2 r2^2 + 10 b^3 r2^2 - 22 a^2 m r2^2 - 40 a b m r2^2 -
   18 b^2 m r2^2 + 6 a m^2 r2^2 + 6 b m^2 r2^2 + 39 a^2 r1 r2^2 + 78 a b r1 r2^2 + 36 b^2 r1 r2^2 - 52 a m r1 r2^2 -
   50 b m r1 r2^2 + 18 m^2 r1 r2^2 + 39 a r1^2 r2^2 + 39 b r1^2 r2^2 - 30 m r1^2 r2^2 + 13 r1^3 r2^2 +
   13 a^2 r2^3 + 24 a b r2^3 + 10 b^2 r2^3 - 18 a m r2^3 - 16 b m r2^3 + 6 m^2 r2^3 + 26 a r1 r2^3 +
   24 b r1 r2^3 - 20 m r1 r2^3 + 13 r1^2 r2^3 + 6 a r2^4 + 5 b r2^4 - 5 m r2^4 + 6 r1 r2^4 + r2^5) #1^2 +
(2 a^4 + 8 a^3 b + 12 a^2 b^2 + 8 a b^3 + 2 b^4 - a^3 m - 3 a^2 b m - 3 a b^2 m - b^3 m + 8 a^3 r1 + 24 a^2 b r1 +
   24 a b^2 r1 + 8 b^3 r1 - 7 a^2 m r1 - 14 a b m r1 - 7 b^2 m r1 + 12 a^2 r1^2 + 24 a b r1^2 + 12 b^2 r1^2 -
   11 a m r1^2 - 11 b m r1^2 + 8 a r1^3 + 8 b r1^3 - 5 m r1^3 + 2 r1^4 + 8 a^3 r2 + 24 a^2 b r2 +
   24 a b^2 r2 + 8 b^3 r2 - 7 a^2 m r2 - 14 a b m r2 - 7 b^2 m r2 + 24 a^2 r1 r2 + 48 a b r1 r2 +
   24 b^2 r1 r2 - 22 a m r1 r2 - 22 b m r1 r2 + 24 a r1^2 r2 + 24 b r1^2 r2 - 15 m r1^2 r2 +
   8 r1^3 r2 + 12 a^2 r2^2 + 24 a b r2^2 + 12 b^2 r2^2 - 11 a m r2^2 - 11 b m r2^2 + 24 a r1 r2^2 +
   24 b r1 r2^2 - 15 m r1 r2^2 + 12 r1^2 r2^2 + 8 a r2^3 + 8 b r2^3 - 5 m r2^3 + 8 r1 r2^3 + 2 r2^4) #1^3 +
(a^3 + 3 a^2 b + 3 a b^2 + b^3 + 3 a^2 r1 + 6 a b r1 + 3 b^2 r1 + 3 a r1^2 + 3 b r1^2 + r1^3 + 3 a^2 r2 + 6 a b r2 +
   3 b^2 r2 + 6 a r1 r2 + 6 b r1 r2 + 3 r1^2 r2 + 3 a r2^2 + 3 b r2^2 + 3 r1 r2^2 + r2^3) #1^4 &, 1],
Root[a^4 b m^2 + 3 a^3 b^2 m^2 + 3 a^2 b^3 m^2 + a b^4 m^2 + a^3 m^4 - a^2 b m^4 - a b^2 m^4 + b^3 m^4 +
   2 a^3 b m^2 r1 + 3 a^2 b^2 m^2 r1 - b^4 m^2 r1 - 2 a^2 b m^3 r1 +
   2 b^3 m^3 r1 + 3 a^2 m^4 r1 - 6 a b m^4 r1 + 3 b^2 m^4 r1 - a^3 m^2 r1^2 -
   3 a^2 b m^2 r1^2 - 6 a b^2 m^2 r1^2 - 4 b^3 m^2 r1^2 + 4 a b m^3 r1^2 +
   8 b^2 m^3 r1^2 - a m^4 r1^2 - b m^4 r1^2 - 3 a^2 m^2 r1^3 - 8 a b m^2 r1^3 -
   6 b^2 m^2 r1^3 + 4 a m^3 r1^3 + 10 b m^3 r1^3 - 3 m^4 r1^3 - 3 a m^2 r1^4 -
   4 b m^2 r1^4 + 4 m^3 r1^4 - m^2 r1^5 - a^4 m^2 r2 + 3 a^2 b^2 m^2 r2 +
   2 a b^3 m^2 r2 + 2 a^3 m^3 r2 - 2 a b^2 m^3 r2 + 3 a^2 m^4 r2 - 6 a b m^4 r2 +
   3 b^2 m^4 r2 + a^4 m r1 r2 + 4 a^3 b m r1 r2 + 6 a^2 b^2 m r1 r2 +
```

$4\,a\,b^3\,m\,r1\,r2+b^4\,m\,r1\,r2-6\,a^3\,m^2\,r1\,r2-12\,a^2\,b\,m^2\,r1\,r2-$
$12\,a\,b^2\,m^2\,r1\,r2-6\,b^3\,m^2\,r1\,r2+8\,a^2\,m^3\,r1\,r2+8\,a\,b\,m^3\,r1\,r2+$
$8\,b^2\,m^3\,r1\,r2-2\,a\,m^4\,r1\,r2-2\,b\,m^4\,r1\,r2+4\,a^3\,m\,r1^2\,r2+$
$12\,a^2\,b\,m\,r1^2\,r2+12\,a\,b^2\,m\,r1^2\,r2+4\,b^3\,m\,r1^2\,r2-17\,a^2\,m^2\,r1^2\,r2-$
$34\,a\,b\,m^2\,r1^2\,r2-20\,b^2\,m^2\,r1^2\,r2+18\,a\,m^3\,r1^2\,r2+24\,b\,m^3\,r1^2\,r2-$
$9\,m^4\,r1^2\,r2+6\,a^2\,m\,r1^3\,r2+12\,a\,b\,m\,r1^3\,r2+6\,b^2\,m\,r1^3\,r2-$
$20\,a\,m^2\,r1^3\,r2-22\,b\,m^2\,r1^3\,r2+16\,m^3\,r1^3\,r2+4\,a\,m\,r1^4\,r2+$
$4\,b\,m\,r1^4\,r2-8\,m^2\,r1^4\,r2+m\,r1^5\,r2-4\,a^3\,m^2\,r2^2-6\,a^2\,b\,m^2\,r2^2-$
$3\,a\,b^2\,m^2\,r2^2-b^3\,m^2\,r2^2+8\,a^2\,m^3\,r2^2+4\,a\,b\,m^3\,r2^2-a\,m^4\,r2^2-$
$b\,m^4\,r2^2+4\,a^3\,m\,r1\,r2^2+12\,a^2\,b\,m\,r1\,r2^2+12\,a\,b^2\,m\,r1\,r2^2+$
$4\,b^3\,m\,r1\,r2^2-20\,a^2\,m^2\,r1\,r2^2-34\,a\,b\,m^2\,r1\,r2^2-17\,b^2\,m^2\,r1\,r2^2+$
$24\,a\,m^3\,r1\,r2^2+18\,b\,m^3\,r1\,r2^2-9\,m^4\,r1\,r2^2+12\,a^2\,m\,r1^2\,r2^2+$
$24\,a\,b\,m\,r1^2\,r2^2+12\,b^2\,m\,r1^2\,r2^2-35\,a\,m^2\,r1^2\,r2^2-$
$35\,b\,m^2\,r1^2\,r2^2+24\,m^3\,r1^2\,r2^2+12\,a\,m\,r1^3\,r2^2+12\,b\,m\,r1^3\,r2^2-$
$19\,m^2\,r1^3\,r2^2+4\,m\,r1^4\,r2^2-6\,a^2\,m^2\,r2^3-8\,a\,b\,m^2\,r2^3-$
$3\,b^2\,m^2\,r2^3+10\,a\,m^3\,r2^3+4\,b\,m^3\,r2^3-3\,m^4\,r2^3+6\,a^2\,m\,r1\,r2^3+$
$12\,a\,b\,m\,r1\,r2^3+6\,b^2\,m\,r1\,r2^3-22\,a\,m^2\,r1\,r2^3-20\,b\,m^2\,r1\,r2^3+$
$16\,m^3\,r1\,r2^3+12\,a\,m\,r1^2\,r2^3+12\,b\,m\,r1^2\,r2^3-19\,m^2\,r1^2\,r2^3+$
$6\,m\,r1^3\,r2^3-4\,a\,m^2\,r2^4-3\,b\,m^2\,r2^4+4\,m^3\,r2^4+4\,a\,m\,r1\,r2^4+$
$4\,b\,m\,r1\,r2^4-8\,m^2\,r1\,r2^4+4\,m\,r1^2\,r2^4-m^2\,r2^5+m\,r1\,r2^5+$
$\big(a^5\,b+4\,a^4\,b^2+6\,a^3\,b^3+4\,a^2\,b^4+a\,b^5-a^4\,b\,m-3\,a^3\,b^2\,m-3\,a^2\,b^3\,m-a\,b^4\,m-2\,a^4\,m^2+4\,a^2\,b^2\,m^2-$
$2\,b^4\,m^2-a^3\,m^3+a^2\,b\,m^3+a\,b^2\,m^3-b^3\,m^3+5\,a^4\,b\,r1+16\,a^3\,b^2\,r1+18\,a^2\,b^3\,r1+8\,a\,b^4\,r1+b^5\,r1-$
$8\,a^3\,b\,m\,r1-19\,a^2\,b^2\,m\,r1-14\,a\,b^3\,m\,r1-3\,b^4\,m\,r1-6\,a^3\,m^2\,r1+2\,a^2\,b\,m^2\,r1+2\,a\,b^2\,m^2\,r1-$
$6\,b^3\,m^2\,r1+3\,a^2\,m^3\,r1+2\,a\,b\,m^3\,r1+3\,b^2\,m^3\,r1+10\,a^3\,b\,r1^2+24\,a^2\,b^2\,r1^2+18\,a\,b^3\,r1^2+$
$4\,b^4\,r1^2-18\,a^2\,b\,m\,r1^2-29\,a\,b^2\,m\,r1^2-11\,b^3\,m\,r1^2-6\,a^2\,m^2\,r1^2+4\,a\,b\,m^2\,r1^2-2\,b^2\,m^2\,r1^2+$
$5\,a\,m^3\,r1^2+5\,b\,m^3\,r1^2+10\,a^2\,b\,r1^3+16\,a\,b^2\,r1^3+6\,b^3\,r1^3-16\,a\,b\,m\,r1^3-13\,b^2\,m\,r1^3-$
$2\,a\,m^2\,r1^3+2\,b\,m^2\,r1^3+m^3\,r1^3+5\,a\,b\,r1^4+4\,b^2\,r1^4-5\,b\,m\,r1^4+b\,r1^5+a^5\,r2+8\,a^4\,b\,r2+$
$18\,a^3\,b^2\,r2+16\,a^2\,b^3\,r2+5\,a\,b^4\,r2-3\,a^4\,m\,r2-14\,a^3\,b\,m\,r2-19\,a^2\,b^2\,m\,r2-8\,a\,b^3\,m\,r2-$
$6\,a^3\,m^2\,r2+2\,a^2\,b\,m^2\,r2+2\,a\,b^2\,m^2\,r2-6\,b^3\,m^2\,r2+3\,a^2\,m^3\,r2+2\,a\,b\,m^3\,r2+3\,b^2\,m^3\,r2+$
$5\,a^4\,r1\,r2+32\,a^3\,b\,r1\,r2+54\,a^2\,b^2\,r1\,r2+32\,a\,b^3\,r1\,r2+5\,b^4\,r1\,r2-12\,a^3\,m\,r1\,r2-$
$50\,a^2\,b\,m\,r1\,r2-50\,a\,b^2\,m\,r1\,r2-12\,b^3\,m\,r1\,r2-8\,a^2\,m^2\,r1\,r2+8\,a\,b\,m^2\,r1\,r2-8\,b^2\,m^2\,r1\,r2+$
$10\,a\,m^3\,r1\,r2+10\,b\,m^3\,r1\,r2+10\,a^3\,r1^2\,r2+48\,a^2\,b\,r1^2\,r2+54\,a\,b^2\,r1^2\,r2+16\,b^3\,r1^2\,r2-$
$18\,a^2\,m\,r1^2\,r2-58\,a\,b\,m\,r1^2\,r2-31\,b^2\,m\,r1^2\,r2-2\,a\,m^2\,r1^2\,r2+2\,b\,m^2\,r1^2\,r2+3\,m^3\,r1^2\,r2+$
$10\,a^2\,r1^3\,r2+32\,a\,b\,r1^3\,r2+18\,b^2\,r1^3\,r2-12\,a\,m\,r1^3\,r2-22\,b\,m\,r1^3\,r2+5\,a\,r1^4\,r2+$
$8\,b\,r1^4\,r2-3\,m\,r1^4\,r2+r1^5\,r2+4\,a^4\,r2^2+18\,a^3\,b\,r2^2+24\,a^2\,b^2\,r2^2+10\,a\,b^3\,r2^2-$
$11\,a^3\,m\,r2^2-29\,a^2\,b\,m\,r2^2-18\,a\,b^2\,m\,r2^2-2\,a^2\,m^2\,r2^2+4\,a\,b\,m^2\,r2^2-6\,b^2\,m^2\,r2^2+$
$5\,a\,m^3\,r2^2+5\,b\,m^3\,r2^2+16\,a^3\,r1\,r2^2+54\,a^2\,b\,r1\,r2^2+48\,a\,b^2\,r1\,r2^2+10\,b^3\,r1\,r2^2-$
$31\,a^2\,m\,r1\,r2^2-58\,a\,b\,m\,r1\,r2^2-18\,b^2\,m\,r1\,r2^2+2\,a\,m^2\,r1\,r2^2-2\,b\,m^2\,r1\,r2^2+3\,m^3\,r1\,r2^2+$
$24\,a^2\,r1^2\,r2^2+54\,a\,b\,r1^2\,r2^2+24\,b^2\,r1^2\,r2^2-29\,a\,m\,r1^2\,r2^2-29\,b\,m\,r1^2\,r2^2+16\,a\,r1^3\,r2^2+$
$18\,b\,r1^3\,r2^2-9\,m\,r1^3\,r2^2+4\,r1^4\,r2^2+6\,a^3\,r2^3+16\,a^2\,b\,r2^3+10\,a\,b^2\,r2^3-13\,a^2\,m\,r2^3-$
$16\,a\,b\,m\,r2^3+2\,a\,m^2\,r2^3-2\,b\,m^2\,r2^3+m^3\,r2^3+18\,a^2\,r1\,r2^3+32\,a\,b\,r1\,r2^3+10\,b^2\,r1\,r2^3-$
$22\,a\,m\,r1\,r2^3-12\,b\,m\,r1\,r2^3+18\,a\,r1^2\,r2^3+16\,b\,r1^2\,r2^3-9\,m\,r1^2\,r2^3+6\,r1^3\,r2^3+4\,a^2\,r2^4+$
$5\,a\,b\,r2^4-5\,a\,m\,r2^4+8\,a\,r1\,r2^4+5\,b\,r1\,r2^4-3\,m\,r1\,r2^4+4\,r1^2\,r2^4+a\,r2^5+r1\,r2^5\big)\,\#1+$
$\big(a^5+6\,a^4\,b+13\,a^3\,b^2+13\,a^2\,b^3+6\,a\,b^4+b^5-a^4\,m-4\,a^3\,b\,m-6\,a^2\,b^2\,m-4\,a\,b^3\,m-b^4\,m-$
$2\,a^3\,m^2-2\,a^2\,b\,m^2-2\,a\,b^2\,m^2-2\,b^3\,m^2+5\,a^4\,r1+24\,a^3\,b\,r1+39\,a^2\,b^2\,r1+26\,a\,b^3\,r1+$
$6\,b^4\,r1-8\,a^3\,m\,r1-26\,a^2\,b\,m\,r1-28\,a\,b^2\,m\,r1-10\,b^3\,m\,r1-2\,a^2\,m^2\,r1-2\,b^2\,m^2\,r1+$
$10\,a^3\,r1^2+36\,a^2\,b\,r1^2+39\,a\,b^2\,r1^2+13\,b^3\,r1^2-18\,a^2\,m\,r1^2-40\,a\,b\,m\,r1^2-22\,b^2\,m\,r1^2+$
$6\,a\,m^2\,r1^2+6\,b\,m^2\,r1^2+10\,a^2\,r1^3+24\,a\,b\,r1^3+13\,b^2\,r1^3-16\,a\,m\,r1^3-18\,b\,m\,r1^3+$
$6\,m^2\,r1^3+5\,a\,r1^4+6\,b\,r1^4-5\,m\,r1^4+r1^5+6\,a^4\,r2+26\,a^3\,b\,r2+39\,a^2\,b^2\,r2+24\,a\,b^3\,r2+$
$5\,b^4\,r2-10\,a^3\,m\,r2-28\,a^2\,b\,m\,r2-26\,a\,b^2\,m\,r2-8\,b^3\,m\,r2-2\,a^2\,m^2\,r2-2\,b^2\,m^2\,r2+$
$24\,a^3\,r1\,r2+78\,a^2\,b\,r1\,r2+78\,a\,b^2\,r1\,r2+24\,b^3\,r1\,r2-40\,a^2\,m\,r1\,r2-80\,a\,b\,m\,r1\,r2-$
$40\,b^2\,m\,r1\,r2+12\,a\,m^2\,r1\,r2+12\,b\,m^2\,r1\,r2+36\,a^2\,r1^2\,r2+78\,a\,b\,r1^2\,r2+39\,b^2\,r1^2\,r2-$
$50\,a\,m\,r1^2\,r2-52\,b\,m\,r1^2\,r2+18\,m^2\,r1^2\,r2+24\,a\,r1^3\,r2+26\,b\,r1^3\,r2-20\,m\,r1^3\,r2+$
$6\,r1^4\,r2+13\,a^3\,r2^2+39\,a^2\,b\,r2^2+36\,a\,b^2\,r2^2+10\,b^3\,r2^2-22\,a^2\,m\,r2^2-40\,a\,b\,m\,r2^2-$
$18\,b^2\,m\,r2^2+6\,a\,m^2\,r2^2+6\,b\,m^2\,r2^2+39\,a^2\,r1\,r2^2+78\,a\,b\,r1\,r2^2+36\,b^2\,r1\,r2^2-52\,a\,m\,r1\,r2^2-$
$50\,b\,m\,r1\,r2^2+18\,m^2\,r1\,r2^2+39\,a\,r1^2\,r2^2+39\,b\,r1^2\,r2^2-30\,m\,r1^2\,r2^2+13\,r1^3\,r2^2+$
$13\,a^2\,r2^3+24\,a\,b\,r2^3+10\,b^2\,r2^3-18\,a\,m\,r2^3-16\,b\,m\,r2^3+6\,m^2\,r2^3+26\,a\,r1\,r2^3+$

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      24 b r1 r2^3 - 20 m r1 r2^3 + 13 r1^2 r2^3 + 6 a r2^4 + 5 b r2^4 - 5 m r2^4 + 6 r1 r2^4 + r2^5) #1^2 +
    (2 a^4 + 8 a^3 b + 12 a^2 b^2 + 8 a b^3 + 2 b^4 - a^3 m - 3 a^2 b m - 3 a b^2 m - b^3 m + 8 a^3 r1 + 24 a^2 b r1 +
      24 a b^2 r1 + 8 b^3 r1 - 7 a^2 m r1 - 14 a b m r1 - 7 b^2 m r1 + 12 a^2 r1^2 + 24 a b r1^2 + 12 b^2 r1^2 -
      11 a m r1^2 - 11 b m r1^2 + 8 a r1^3 + 8 b r1^3 - 5 m r1^3 + 2 r1^4 + 8 a^3 r2 + 24 a^2 b r2 +
      24 a b^2 r2 + 8 b^3 r2 - 7 a^2 m r2 - 14 a b m r2 - 7 b^2 m r2 + 24 a^2 r1 r2 + 48 a b r1 r2 +
      24 b^2 r1 r2 - 22 a m r1 r2 - 22 b m r1 r2 + 24 a r1^2 r2 + 24 b r1^2 r2 - 15 m r1^2 r2 +
      8 r1^3 r2 + 12 a^2 r2^2 + 24 a b r2^2 + 12 b^2 r2^2 - 11 a m r2^2 - 11 b m r2^2 + 24 a r1 r2^2 +
      24 b r1 r2^2 - 15 m r1 r2^2 + 12 r1^2 r2^2 + 8 a r2^3 + 8 b r2^3 - 5 m r2^3 + 8 r1 r2^3 + 2 r2^4) #1^3 +
    (a^3 + 3 a^2 b + 3 a b^2 + b^3 + 3 a^2 r1 + 6 a b r1 + 3 b^2 r1 + 3 a r1^2 + 3 b r1^2 + r1^3 + 3 a^2 r2 + 6 a b r2 +
      3 b^2 r2 + 6 a r1 r2 + 6 b r1 r2 + 3 r1^2 r2 + 3 a r2^2 + 3 b r2^2 + 3 r1 r2^2 + r2^3) #1^4 &, 2],
Root[a^4 b m^2 + 3 a^3 b^2 m^2 + 3 a^2 b^3 m^2 + a b^4 m^2 + a^3 m^4 - a^2 b m^4 - a b^2 m^4 + b^3 m^4 +
    2 a^3 b m^2 r1 + 3 a^2 b^2 m^2 r1 - b^4 m^2 r1 - 2 a^2 b m^3 r1 +
    2 b^3 m^3 r1 + 3 a^2 m^4 r1 - 6 a b m^4 r1 + 3 b^2 m^4 r1 - a^3 m^2 r1^2 -
    3 a^2 b m^2 r1^2 - 6 a b^2 m^2 r1^2 - 4 b^3 m^2 r1^2 + 4 a b m^3 r1^2 +
    8 b^2 m^3 r1^2 - a m^4 r1^2 - b m^4 r1^2 - 3 a^2 m^2 r1^3 - 8 a b m^2 r1^3 -
    6 b^2 m^2 r1^3 + 4 a m^3 r1^3 + 10 b m^3 r1^3 - 3 m^4 r1^3 -
    3 a m^2 r1^4 - 4 b m^2 r1^4 + 4 m^3 r1^4 - m^2 r1^5 - a^4 m^2 r2 +
    3 a^2 b^2 m^2 r2 + 2 a b^3 m^2 r2 + 2 a^3 m^3 r2 - 2 a b^2 m^3 r2 +
    3 a^2 m^4 r2 - 6 a b m^4 r2 + 3 b^2 m^4 r2 + a^4 m r1 r2 +
    4 a^3 b m r1 r2 + 6 a^2 b^2 m r1 r2 + 4 a b^3 m r1 r2 + b^4 m r1 r2 -
    6 a^3 m^2 r1 r2 - 12 a^2 b m^2 r1 r2 - 12 a b^2 m^2 r1 r2 -
    6 b^3 m^2 r1 r2 + 8 a^2 m^3 r1 r2 + 8 a b m^3 r1 r2 + 8 b^2 m^3 r1 r2 -
    2 a m^4 r1 r2 - 2 b m^4 r1 r2 + 4 a^3 m r1^2 r2 + 12 a^2 b m r1^2 r2 +
    12 a b^2 m r1^2 r2 + 4 b^3 m r1^2 r2 - 17 a^2 m^2 r1^2 r2 -
    34 a b m^2 r1^2 r2 - 20 b^2 m^2 r1^2 r2 + 18 a m^3 r1^2 r2 +
    24 b m^3 r1^2 r2 - 9 m^4 r1^2 r2 + 6 a^2 m r1^3 r2 + 12 a b m r1^3 r2 +
    6 b^2 m r1^3 r2 - 20 a m^2 r1^3 r2 - 22 b m^2 r1^3 r2 + 16 m^3 r1^3 r2 +
    4 a m r1^4 r2 + 4 b m r1^4 r2 - 8 m^2 r1^4 r2 + m r1^5 r2 -
    4 a^3 m^2 r2^2 - 6 a^2 b m^2 r2^2 - 3 a b^2 m^2 r2^2 - b^3 m^2 r2^2 +
    8 a^2 m^3 r2^2 + 4 a b m^3 r2^2 - a m^4 r2^2 - b m^4 r2^2 + 4 a^3 m r1 r2^2 +
    12 a^2 b m r1 r2^2 + 12 a b^2 m r1 r2^2 + 4 b^3 m r1 r2^2 -
    20 a^2 m^2 r1 r2^2 - 34 a b m^2 r1 r2^2 - 17 b^2 m^2 r1 r2^2 +
    24 a m^3 r1 r2^2 + 18 b m^3 r1 r2^2 - 9 m^4 r1 r2^2 + 12 a^2 m r1^2 r2^2 +
    24 a b m r1^2 r2^2 + 12 b^2 m r1^2 r2^2 - 35 a m^2 r1^2 r2^2 -
    35 b m^2 r1^2 r2^2 + 24 m^3 r1^2 r2^2 + 12 a m r1^3 r2^2 + 12 b m r1^3 r2^2 -
    19 m^2 r1^3 r2^2 + 4 m r1^4 r2^2 - 6 a^2 m^2 r2^3 - 8 a b m^2 r2^3 -
    3 b^2 m^2 r2^3 + 10 a m^3 r2^3 + 4 b m^3 r2^3 - 3 m^4 r2^3 + 6 a^2 m r1 r2^3 +
    12 a b m r1 r2^3 + 6 b^2 m r1 r2^3 - 22 a m^2 r1 r2^3 - 20 b m^2 r1 r2^3 +
    16 m^3 r1 r2^3 + 12 a m r1^2 r2^3 + 12 b m r1^2 r2^3 - 19 m^2 r1^2 r2^3 +
    6 m r1^3 r2^3 - 4 a m^2 r2^4 - 3 b m^2 r2^4 + 4 m^3 r2^4 + 4 a m r1 r2^4 +
    4 b m r1 r2^4 - 8 m^2 r1 r2^4 + 4 m r1^2 r2^4 - m^2 r2^5 + m r1 r2^5 +
    (a^5 b + 4 a^4 b^2 + 6 a^3 b^3 + 4 a^2 b^4 + a b^5 - a^4 b m - 3 a^3 b^2 m - 3 a^2 b^3 m - a b^4 m - 2 a^4 m^2 + 4 a^2 b^2 m^2 -
      2 b^4 m^2 - a^3 m^3 + a^2 b m^3 + a b^2 m^3 - b^3 m^3 + 5 a^4 b r1 + 16 a^3 b^2 r1 + 18 a^2 b^3 r1 + 8 a b^4 r1 + b^5 r1 -
      8 a^3 b m r1 - 19 a^2 b^2 m r1 - 14 a b^3 m r1 - 3 b^4 m r1 - 6 a^3 m^2 r1 + 2 a^2 b m^2 r1 + 2 a b^2 m^2 r1 -
      6 b^3 m^2 r1 + 3 a^2 m^3 r1 + 2 a b m^3 r1 + 3 b^2 m^3 r1 + 10 a^3 b r1^2 + 24 a^2 b^2 r1^2 + 18 a b^3 r1^2 +
      4 b^4 r1^2 - 18 a^2 b m r1^2 - 29 a b^2 m r1^2 - 11 b^3 m r1^2 - 6 a^2 m^2 r1^2 + 4 a b m^2 r1^2 - 2 b^2 m^2 r1^2 +
      5 a m^3 r1^2 + 5 b m^3 r1^2 + 10 a^2 b r1^3 + 16 a b^2 r1^3 + 6 b^3 r1^3 - 16 a b m r1^3 - 13 b^2 m r1^3 -
      2 a m^2 r1^3 + 2 b m^2 r1^3 + m^3 r1^3 + 5 a b r1^4 + 4 b^2 r1^4 - 5 b m r1^4 + b r1^5 + a^5 r2 + 8 a^4 b r2 +
      18 a^3 b^2 r2 + 16 a^2 b^3 r2 + 5 a b^4 r2 - 3 a^4 m r2 - 14 a^3 b m r2 - 19 a^2 b^2 m r2 - 8 a b^3 m r2 -
      6 a^3 m^2 r2 + 2 a^2 b m^2 r2 + 2 a b^2 m^2 r2 - 6 b^3 m^2 r2 + 3 a^2 m^3 r2 + 2 a b m^3 r2 + 3 b^2 m^3 r2 +
      5 a^4 r1 r2 + 32 a^3 b r1 r2 + 54 a^2 b^2 r1 r2 + 32 a b^3 r1 r2 + 5 b^4 r1 r2 - 12 a^3 m r1 r2 -
      50 a^2 b m r1 r2 - 50 a b^2 m r1 r2 - 12 b^3 m r1 r2 - 8 a^2 m^2 r1 r2 + 8 a b m^2 r1 r2 - 8 b^2 m^2 r1 r2 +
      10 a m^3 r1 r2 + 10 b m^3 r1 r2 + 10 a^3 r1^2 r2 + 48 a^2 b r1^2 r2 + 54 a b^2 r1^2 r2 + 16 b^3 r1^2 r2 -
      18 a^2 m r1^2 r2 - 58 a b m r1^2 r2 - 31 b^2 m r1^2 r2 - 2 a m^2 r1^2 r2 + 2 b m^2 r1^2 r2 + 3 m^3 r1^2 r2 +
      10 a^2 r1^3 r2 + 32 a b r1^3 r2 + 18 b^2 r1^3 r2 - 12 a m r1^3 r2 - 22 b m r1^3 r2 + 5 a r1^4 r2 +
      8 b r1^4 r2 - 3 m r1^4 r2 + r1^5 r2 + 4 a^4 r2^2 + 18 a^3 b r2^2 + 24 a^2 b^2 r2^2 + 10 a b^3 r2^2 -
      11 a^3 m r2^2 - 29 a^2 b m r2^2 - 18 a b^2 m r2^2 - 2 a^2 m^2 r2^2 + 4 a b m^2 r2^2 - 6 b^2 m^2 r2^2 +
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$5\,a\,m^3\,r2^2 + 5\,b\,m^3\,r2^2 + 16\,a^3\,r1\,r2^2 + 54\,a^2\,b\,r1\,r2^2 + 48\,a\,b^2\,r1\,r2^2 + 10\,b^3\,r1\,r2^2 -$
$31\,a^2\,m\,r1\,r2^2 - 58\,a\,b\,m\,r1\,r2^2 - 18\,b^2\,m\,r1\,r2^2 + 2\,a\,m^2\,r1\,r2^2 - 2\,b\,m^2\,r1\,r2^2 + 3\,m^3\,r1\,r2^2 +$
$24\,a^2\,r1^2\,r2^2 + 54\,a\,b\,r1^2\,r2^2 + 24\,b^2\,r1^2\,r2^2 - 29\,a\,m\,r1^2\,r2^2 - 29\,b\,m\,r1^2\,r2^2 + 16\,a\,r1^3\,r2^2 +$
$18\,b\,r1^3\,r2^2 - 9\,m\,r1^3\,r2^2 + 4\,r1^4\,r2^2 + 6\,a^3\,r2^3 + 16\,a^2\,b\,r2^3 + 10\,a\,b^2\,r2^3 - 13\,a^2\,m\,r2^3 -$
$16\,a\,b\,m\,r2^3 + 2\,a\,m^2\,r2^3 - 2\,b\,m^2\,r2^3 + m^3\,r2^3 + 18\,a^2\,r1\,r2^3 + 32\,a\,b\,r1\,r2^3 + 10\,b^2\,r1\,r2^3 -$
$22\,a\,m\,r1\,r2^3 - 12\,b\,m\,r1\,r2^3 + 18\,a\,r1^2\,r2^3 + 16\,b\,r1^2\,r2^3 - 9\,m\,r1^2\,r2^3 + 6\,r1^3\,r2^3 + 4\,a^2\,r2^4 +$
$5\,a\,b\,r2^4 - 5\,a\,m\,r2^4 + 8\,a\,r1\,r2^4 + 5\,b\,r1\,r2^4 - 3\,m\,r1\,r2^4 + 4\,r1^2\,r2^4 + a\,r2^5 + r1\,r2^5\big)\,\#1 +$
$\big(a^5 + 6\,a^4\,b + 13\,a^3\,b^2 + 13\,a^2\,b^3 + 6\,a\,b^4 + b^5 - a^4\,m - 4\,a^3\,b\,m - 6\,a^2\,b^2\,m - 4\,a\,b^3\,m - b^4\,m -$
$2\,a^3\,m^2 - 2\,a^2\,b\,m^2 - 2\,a\,b^2\,m^2 - 2\,b^3\,m^2 + 5\,a^4\,r1 + 24\,a^3\,b\,r1 + 39\,a^2\,b^2\,r1 + 26\,a\,b^3\,r1 +$
$6\,b^4\,r1 - 8\,a^3\,m\,r1 - 26\,a^2\,b\,m\,r1 - 28\,a\,b^2\,m\,r1 - 10\,b^3\,m\,r1 - 2\,a^2\,m^2\,r1 - 2\,b^2\,m^2\,r1 +$
$10\,a^3\,r1^2 + 36\,a^2\,b\,r1^2 + 39\,a\,b^2\,r1^2 + 13\,b^3\,r1^2 - 18\,a^2\,m\,r1^2 - 40\,a\,b\,m\,r1^2 - 22\,b^2\,m\,r1^2 +$
$6\,a\,m^2\,r1^2 + 6\,b\,m^2\,r1^2 + 10\,a^2\,r1^3 + 24\,a\,b\,r1^3 + 13\,b^2\,r1^3 - 16\,a\,m\,r1^3 - 18\,b\,m\,r1^3 +$
$6\,m^2\,r1^3 + 5\,a\,r1^4 + 6\,b\,r1^4 - 5\,m\,r1^4 + r1^5 + 6\,a^4\,r2 + 26\,a^3\,b\,r2 + 39\,a^2\,b^2\,r2 + 24\,a\,b^3\,r2 +$
$5\,b^4\,r2 - 10\,a^3\,m\,r2 - 28\,a^2\,b\,m\,r2 - 26\,a\,b^2\,m\,r2 - 8\,b^3\,m\,r2 - 2\,a^2\,m^2\,r2 - 2\,b^2\,m^2\,r2 +$
$24\,a^3\,r1\,r2 + 78\,a^2\,b\,r1\,r2 + 78\,a\,b^2\,r1\,r2 + 24\,b^3\,r1\,r2 - 40\,a^2\,m\,r1\,r2 - 80\,a\,b\,m\,r1\,r2 -$
$40\,b^2\,m\,r1\,r2 + 12\,a\,m^2\,r1\,r2 + 12\,b\,m^2\,r1\,r2 + 36\,a^2\,r1^2\,r2 + 78\,a\,b\,r1^2\,r2 + 39\,b^2\,r1^2\,r2 -$
$50\,a\,m\,r1^2\,r2 - 52\,b\,m\,r1^2\,r2 + 18\,m^2\,r1^2\,r2 + 24\,a\,r1^3\,r2 + 26\,b\,r1^3\,r2 - 20\,m\,r1^3\,r2 +$
$6\,r1^4\,r2 + 13\,a^3\,r2^2 + 39\,a^2\,b\,r2^2 + 36\,a\,b^2\,r2^2 + 10\,b^3\,r2^2 - 22\,a^2\,m\,r2^2 - 40\,a\,b\,m\,r2^2 -$
$18\,b^2\,m\,r2^2 + 6\,a\,m^2\,r2^2 + 6\,b\,m^2\,r2^2 + 39\,a^2\,r1\,r2^2 + 78\,a\,b\,r1\,r2^2 + 36\,b^2\,r1\,r2^2 - 52\,a\,m\,r1\,r2^2 -$
$50\,b\,m\,r1\,r2^2 + 18\,m^2\,r1\,r2^2 + 39\,a\,r1^2\,r2^2 + 39\,b\,r1^2\,r2^2 - 30\,m\,r1^2\,r2^2 + 13\,r1^3\,r2^2 +$
$13\,a^2\,r2^3 + 24\,a\,b\,r2^3 + 10\,b^2\,r2^3 - 18\,a\,m\,r2^3 - 16\,b\,m\,r2^3 + 6\,m^2\,r2^3 + 26\,a\,r1\,r2^3 +$
$24\,b\,r1\,r2^3 - 20\,m\,r1\,r2^3 + 13\,r1^2\,r2^3 + 6\,a\,r2^4 + 5\,b\,r2^4 - 5\,m\,r2^4 + 6\,r1\,r2^4 + r2^5\big)\,\#1^2 +$
$\big(2\,a^4 + 8\,a^3\,b + 12\,a^2\,b^2 + 8\,a\,b^3 + 2\,b^4 - a^3\,m - 3\,a^2\,b\,m - 3\,a\,b^2\,m - b^3\,m + 8\,a^3\,r1 + 24\,a^2\,b\,r1 +$
$24\,a\,b^2\,r1 + 8\,b^3\,r1 - 7\,a^2\,m\,r1 - 14\,a\,b\,m\,r1 - 7\,b^2\,m\,r1 + 12\,a^2\,r1^2 + 24\,a\,b\,r1^2 + 12\,b^2\,r1^2 -$
$11\,a\,m\,r1^2 - 11\,b\,m\,r1^2 + 8\,a\,r1^3 + 8\,b\,r1^3 - 5\,m\,r1^3 + 2\,r1^4 + 8\,a^3\,r2 + 24\,a^2\,b\,r2 +$
$24\,a\,b^2\,r2 + 8\,b^3\,r2 - 7\,a^2\,m\,r2 - 14\,a\,b\,m\,r2 - 7\,b^2\,m\,r2 + 24\,a^2\,r1\,r2 + 48\,a\,b\,r1\,r2 +$
$24\,b^2\,r1\,r2 - 22\,a\,m\,r1\,r2 - 22\,b\,m\,r1\,r2 + 24\,a\,r1^2\,r2 + 24\,b\,r1^2\,r2 - 15\,m\,r1^2\,r2 +$
$8\,r1^3\,r2 + 12\,a^2\,r2^2 + 24\,a\,b\,r2^2 + 12\,b^2\,r2^2 - 11\,a\,m\,r2^2 - 11\,b\,m\,r2^2 + 24\,a\,r1\,r2^2 +$
$24\,b\,r1\,r2^2 - 15\,m\,r1\,r2^2 + 12\,r1^2\,r2^2 + 8\,a\,r2^3 + 8\,b\,r2^3 - 5\,m\,r2^3 + 8\,r1\,r2^3 + 2\,r2^4\big)\,\#1^3 +$
$\big(a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3 + 3\,a^2\,r1 + 6\,a\,b\,r1 + 3\,b^2\,r1 + 3\,a\,r1^2 + 3\,b\,r1^2 + r1^3 + 3\,a^2\,r2 + 6\,a\,b\,r2 +$
$3\,b^2\,r2 + 6\,a\,r1\,r2 + 6\,b\,r1\,r2 + 3\,r1^2\,r2 + 3\,a\,r2^2 + 3\,b\,r2^2 + 3\,r1\,r2^2 + r2^3\big)\,\#1^4\,\&,\,3\big],$
$\text{Root}\big[a^4\,b\,m^2 + 3\,a^3\,b^2\,m^2 + 3\,a^2\,b^3\,m^2 + a\,b^4\,m^2 + a^3\,m^4 - a^2\,b\,m^4 - a\,b^2\,m^4 + b^3\,m^4 +$
$2\,a^3\,b\,m^2\,r1 + 3\,a^2\,b^2\,m^2\,r1 - b^4\,m^2\,r1 - 2\,a^2\,b\,m^3\,r1 +$
$2\,b^3\,m^3\,r1 + 3\,a^2\,m^4\,r1 - 6\,a\,b\,m^4\,r1 + 3\,b^2\,m^4\,r1 -$
$a^3\,m^2\,r1^2 - 3\,a^2\,b\,m^2\,r1^2 - 6\,a\,b^2\,m^2\,r1^2 - 4\,b^3\,m^2\,r1^2 +$
$4\,a\,b\,m^3\,r1^2 + 8\,b^2\,m^3\,r1^2 - a\,m^4\,r1^2 - b\,m^4\,r1^2 -$
$3\,a^2\,m^2\,r1^3 - 8\,a\,b\,m^2\,r1^3 - 6\,b^2\,m^2\,r1^3 + 4\,a\,m^3\,r1^3 +$
$10\,b\,m^3\,r1^3 - 3\,m^4\,r1^3 - 3\,a\,m^2\,r1^4 - 4\,b\,m^2\,r1^4 +$
$4\,m^3\,r1^4 - m^2\,r1^5 - a^4\,m^2\,r2 + 3\,a^2\,b^2\,m^2\,r2 + 2\,a\,b^3\,m^2\,r2 +$
$2\,a^3\,m^3\,r2 - 2\,a\,b^2\,m^3\,r2 + 3\,a^2\,m^4\,r2 - 6\,a\,b\,m^4\,r2 +$
$3\,b^2\,m^4\,r2 + a^4\,m\,r1\,r2 + 4\,a^3\,b\,m\,r1\,r2 + 6\,a^2\,b^2\,m\,r1\,r2 +$
$4\,a\,b^3\,m\,r1\,r2 + b^4\,m\,r1\,r2 - 6\,a^3\,m^2\,r1\,r2 - 12\,a^2\,b\,m^2\,r1\,r2 -$
$12\,a\,b^2\,m^2\,r1\,r2 - 6\,b^3\,m^2\,r1\,r2 + 8\,a^2\,m^3\,r1\,r2 +$
$8\,a\,b\,m^3\,r1\,r2 + 8\,b^2\,m^3\,r1\,r2 - 2\,a\,m^4\,r1\,r2 - 2\,b\,m^4\,r1\,r2 +$
$4\,a^3\,m\,r1^2\,r2 + 12\,a^2\,b\,m\,r1^2\,r2 + 12\,a\,b^2\,m\,r1^2\,r2 +$
$4\,b^3\,m\,r1^2\,r2 - 17\,a^2\,m^2\,r1^2\,r2 - 34\,a\,b\,m^2\,r1^2\,r2 -$
$20\,b^2\,m^2\,r1^2\,r2 + 18\,a\,m^3\,r1^2\,r2 + 24\,b\,m^3\,r1^2\,r2 -$
$9\,m^4\,r1^2\,r2 + 6\,a^2\,m\,r1^3\,r2 + 12\,a\,b\,m\,r1^3\,r2 + 6\,b^2\,m\,r1^3\,r2 -$
$20\,a\,m^2\,r1^3\,r2 - 22\,b\,m^2\,r1^3\,r2 + 16\,m^3\,r1^3\,r2 + 4\,a\,m\,r1^4\,r2 +$
$4\,b\,m\,r1^4\,r2 - 8\,m^2\,r1^4\,r2 + m\,r1^5\,r2 - 4\,a^3\,m^2\,r2^2 -$
$6\,a^2\,b\,m^2\,r2^2 - 3\,a\,b^2\,m^2\,r2^2 - b^3\,m^2\,r2^2 + 8\,a^2\,m^3\,r2^2 +$
$4\,a\,b\,m^3\,r2^2 - a\,m^4\,r2^2 - b\,m^4\,r2^2 + 4\,a^3\,m\,r1\,r2^2 +$
$12\,a^2\,b\,m\,r1\,r2^2 + 12\,a\,b^2\,m\,r1\,r2^2 + 4\,b^3\,m\,r1\,r2^2 -$
$20\,a^2\,m^2\,r1\,r2^2 - 34\,a\,b\,m^2\,r1\,r2^2 - 17\,b^2\,m^2\,r1\,r2^2 +$
$24\,a\,m^3\,r1\,r2^2 + 18\,b\,m^3\,r1\,r2^2 - 9\,m^4\,r1\,r2^2 + 12\,a^2\,m\,r1^2\,r2^2 +$
$24\,a\,b\,m\,r1^2\,r2^2 + 12\,b^2\,m\,r1^2\,r2^2 - 35\,a\,m^2\,r1^2\,r2^2 -$

```
35 b m^2 r1^2 r2^2 + 24 m^3 r1^2 r2^2 + 12 a m r1^3 r2^2 +
12 b m r1^3 r2^2 - 19 m^2 r1^3 r2^2 + 4 m r1^4 r2^2 - 6 a^2 m^2 r2^3 -
8 a b m^2 r2^3 - 3 b^2 m^2 r2^3 + 10 a m^3 r2^3 + 4 b m^3 r2^3 -
3 m^4 r2^3 + 6 a^2 m r1 r2^3 + 12 a b m r1 r2^3 + 6 b^2 m r1 r2^3 -
22 a m^2 r1 r2^3 - 20 b m^2 r1 r2^3 + 16 m^3 r1 r2^3 +
12 a m r1^2 r2^3 + 12 b m r1^2 r2^3 - 19 m^2 r1^2 r2^3 + 6 m r1^3 r2^3 -
4 a m^2 r2^4 - 3 b m^2 r2^4 + 4 m^3 r2^4 + 4 a m r1 r2^4 +
4 b m r1 r2^4 - 8 m^2 r1 r2^4 + 4 m r1^2 r2^4 - m^2 r2^5 + m r1 r2^5 +
(a^5 b + 4 a^4 b^2 + 6 a^3 b^3 + 4 a^2 b^4 + a b^5 - a^4 b m - 3 a^3 b^2 m - 3 a^2 b^3 m - a b^4 m - 2 a^4 m^2 + 4 a^2 b^2 m^2 -
   2 b^4 m^2 - a^3 m^3 + a^2 b m^3 + a b^2 m^3 - b^3 m^3 + 5 a^4 b r1 + 16 a^3 b^2 r1 + 18 a^2 b^3 r1 + 8 a b^4 r1 + b^5 r1 -
   8 a^3 b m r1 - 19 a^2 b^2 m r1 - 14 a b^3 m r1 - 3 b^4 m r1 - 6 a^3 m^2 r1 + 2 a^2 b m^2 r1 + 2 a b^2 m^2 r1 -
   6 b^3 m^2 r1 + 3 a^2 m^3 r1 + 2 a b m^3 r1 + 3 b^2 m^3 r1 + 10 a^3 b r1^2 + 24 a^2 b^2 r1^2 + 18 a b^3 r1^2 +
   4 b^4 r1^2 - 18 a^2 b m r1^2 - 29 a b^2 m r1^2 - 11 b^3 m r1^2 - 6 a^2 m^2 r1^2 + 4 a b m^2 r1^2 - 2 b^2 m^2 r1^2 +
   5 a m^3 r1^2 + 5 b m^3 r1^2 + 10 a^2 b r1^3 + 16 a b^2 r1^3 + 6 b^3 r1^3 - 16 a b m r1^3 - 13 b^2 m r1^3 -
   2 a m^2 r1^3 + 2 b m^2 r1^3 + m^3 r1^3 + 5 a b r1^4 + 4 b^2 r1^4 - 5 b m r1^4 + b r1^5 + a^5 r2 + 8 a^4 b r2 +
   18 a^3 b^2 r2 + 16 a^2 b^3 r2 + 5 a b^4 r2 - 3 a^4 m r2 - 14 a^3 b m r2 - 19 a^2 b^2 m r2 - 8 a b^3 m r2 -
   6 a^3 m^2 r2 + 2 a^2 b m^2 r2 + 2 a b^2 m^2 r2 - 6 b^3 m^2 r2 + 3 a^2 m^3 r2 + 2 a b m^3 r2 + 3 b^2 m^3 r2 +
   5 a^4 r1 r2 + 32 a^3 b r1 r2 + 54 a^2 b^2 r1 r2 + 32 a b^3 r1 r2 + 5 b^4 r1 r2 - 12 a^3 m r1 r2 -
   50 a^2 b m r1 r2 - 50 a b^2 m r1 r2 - 12 b^3 m r1 r2 - 8 a^2 m^2 r1 r2 + 8 a b m^2 r1 r2 - 8 b^2 m^2 r1 r2 +
   10 a m^3 r1 r2 + 10 b m^3 r1 r2 + 10 a^3 r1^2 r2 + 48 a^2 b r1^2 r2 + 54 a b^2 r1^2 r2 + 16 b^3 r1^2 r2 -
   18 a^2 m r1^2 r2 - 58 a b m r1^2 r2 - 31 b^2 m r1^2 r2 - 2 a m^2 r1^2 r2 + 2 b m^2 r1^2 r2 + 3 m^3 r1^2 r2 +
   10 a^2 r1^3 r2 + 32 a b r1^3 r2 + 18 b^2 r1^3 r2 - 12 a m r1^3 r2 - 22 b m r1^3 r2 + 5 a r1^4 r2 +
   8 b r1^4 r2 - 3 m r1^4 r2 + r1^5 r2 + 4 a^4 r2^2 + 18 a^3 b r2^2 + 24 a^2 b^2 r2^2 + 10 a b^3 r2^2 -
   11 a^3 m r2^2 - 29 a^2 b m r2^2 - 18 a b^2 m r2^2 - 2 a^2 m^2 r2^2 + 4 a b m^2 r2^2 - 6 b^2 m^2 r2^2 +
   5 a m^3 r2^2 + 5 b m^3 r2^2 + 16 a^3 r1 r2^2 + 54 a^2 b r1 r2^2 + 48 a b^2 r1 r2^2 + 10 b^3 r1 r2^2 -
   31 a^2 m r1 r2^2 - 58 a b m r1 r2^2 - 18 b^2 m r1 r2^2 + 2 a m^2 r1 r2^2 - 2 b m^2 r1 r2^2 + 3 m^3 r1 r2^2 +
   24 a^2 r1^2 r2^2 + 54 a b r1^2 r2^2 + 24 b^2 r1^2 r2^2 - 29 a m r1^2 r2^2 - 29 b m r1^2 r2^2 + 16 a r1^3 r2^2 +
   18 b r1^3 r2^2 - 9 m r1^3 r2^2 + 4 r1^4 r2^2 + 6 a^3 r2^3 + 16 a^2 b r2^3 + 10 a b^2 r2^3 - 13 a^2 m r2^3 -
   16 a b m r2^3 + 2 a m^2 r2^3 - 2 b m^2 r2^3 + m^3 r2^3 + 18 a^2 r1 r2^3 + 32 a b r1 r2^3 + 10 b^2 r1 r2^3 -
   22 a m r1 r2^3 - 12 b m r1 r2^3 + 18 a r1^2 r2^3 + 16 b r1^2 r2^3 - 9 m r1^2 r2^3 + 6 r1^3 r2^3 + 4 a^2 r2^4 +
   5 a b r2^4 - 5 a m r2^4 + 8 a r1 r2^4 + 5 b r1 r2^4 - 3 m r1 r2^4 + 4 r1^2 r2^4 + a r2^5 + r1 r2^5) #1 +
(a^5 + 6 a^4 b + 13 a^3 b^2 + 13 a^2 b^3 + 6 a b^4 + b^5 - a^4 m - 4 a^3 b m - 6 a^2 b^2 m - 4 a b^3 m - b^4 m -
   2 a^3 m^2 - 2 a^2 b m^2 - 2 a b^2 m^2 - 2 b^3 m^2 + 5 a^4 r1 + 24 a^3 b r1 + 39 a^2 b^2 r1 + 26 a b^3 r1 +
   6 b^4 r1 - 8 a^3 m r1 - 26 a^2 b m r1 - 28 a b^2 m r1 - 10 b^3 m r1 - 2 a^2 m^2 r1 - 2 b^2 m^2 r1 +
   10 a^3 r1^2 + 36 a^2 b r1^2 + 39 a b^2 r1^2 + 13 b^3 r1^2 - 18 a^2 m r1^2 - 40 a b m r1^2 - 22 b^2 m r1^2 +
   6 a m^2 r1^2 + 6 b m^2 r1^2 + 10 a^2 r1^3 + 24 a b r1^3 + 13 b^2 r1^3 - 16 a m r1^3 - 18 b m r1^3 +
   6 m^2 r1^3 + 5 a r1^4 + 6 b r1^4 - 5 m r1^4 + r1^5 + 6 a^4 r2 + 26 a^3 b r2 + 39 a^2 b^2 r2 + 24 a b^3 r2 +
   5 b^4 r2 - 10 a^3 m r2 - 28 a^2 b m r2 - 26 a b^2 m r2 - 8 b^3 m r2 - 2 a^2 m^2 r2 - 2 b^2 m^2 r2 +
   24 a^3 r1 r2 + 78 a^2 b r1 r2 + 78 a b^2 r1 r2 + 24 b^3 r1 r2 - 40 a^2 m r1 r2 - 80 a b m r1 r2 -
   40 b^2 m r1 r2 + 12 a m^2 r1 r2 + 12 b m^2 r1 r2 + 36 a^2 r1^2 r2 + 78 a b r1^2 r2 + 39 b^2 r1^2 r2 -
   50 a m r1^2 r2 - 52 b m r1^2 r2 + 18 m^2 r1^2 r2 + 24 a r1^3 r2 + 26 b r1^3 r2 - 20 m r1^3 r2 +
   6 r1^4 r2 + 13 a^3 r2^2 + 39 a^2 b r2^2 + 36 a b^2 r2^2 + 10 b^3 r2^2 - 22 a^2 m r2^2 - 40 a b m r2^2 -
   18 b^2 m r2^2 + 6 a m^2 r2^2 + 6 b m^2 r2^2 + 39 a^2 r1 r2^2 + 78 a b r1 r2^2 + 36 b^2 r1 r2^2 - 52 a m r1 r2^2 -
   50 b m r1 r2^2 + 18 m^2 r1 r2^2 + 39 a r1^2 r2^2 + 39 b r1^2 r2^2 - 30 m r1^2 r2^2 + 13 r1^3 r2^2 +
   13 a^2 r2^3 + 24 a b r2^3 + 10 b^2 r2^3 - 18 a m r2^3 - 16 b m r2^3 + 6 m^2 r2^3 + 26 a r1 r2^3 +
   24 b r1 r2^3 - 20 m r1 r2^3 + 13 r1^2 r2^3 + 6 a r2^4 + 5 b r2^4 - 5 m r2^4 + 6 r1 r2^4 + r2^5) #1^2 +
(2 a^4 + 8 a^3 b + 12 a^2 b^2 + 8 a b^3 + 2 b^4 - a^3 m - 3 a^2 b m - 3 a b^2 m - b^3 m + 8 a^3 r1 + 24 a^2 b r1 +
   24 a b^2 r1 + 8 b^3 r1 - 7 a^2 m r1 - 14 a b m r1 - 7 b^2 m r1 + 12 a^2 r1^2 + 24 a b r1^2 + 12 b^2 r1^2 -
   11 a m r1^2 - 11 b m r1^2 + 8 a r1^3 + 8 b r1^3 - 5 m r1^3 + 2 r1^4 + 8 a^3 r2 + 24 a^2 b r2 +
   24 a b^2 r2 + 8 b^3 r2 - 7 a^2 m r2 - 14 a b m r2 - 7 b^2 m r2 + 24 a^2 r1 r2 + 48 a b r1 r2 +
   24 b^2 r1 r2 - 22 a m r1 r2 - 22 b m r1 r2 + 24 a r1^2 r2 + 24 b r1^2 r2 - 15 m r1^2 r2 +
   8 r1^3 r2 + 12 a^2 r2^2 + 24 a b r2^2 + 12 b^2 r2^2 - 11 a m r2^2 - 11 b m r2^2 + 24 a r1 r2^2 +
   24 b r1 r2^2 - 15 m r1 r2^2 + 12 r1^2 r2^2 + 8 a r2^3 + 8 b r2^3 - 5 m r2^3 + 8 r1 r2^3 + 2 r2^4) #1^3 +
(a^3 + 3 a^2 b + 3 a b^2 + b^3 + 3 a^2 r1 + 6 a b r1 + 3 b^2 r1 + 3 a r1^2 + 3 b r1^2 + r1^3 + 3 a^2 r2 + 6 a b r2 +
   3 b^2 r2 + 6 a r1 r2 + 6 b r1 r2 + 3 r1^2 r2 + 3 a r2^2 + 3 b r2^2 + 3 r1 r2^2 + r2^3) #1^4 &, 4]}
```

```
Simplify[
 Series[Det[JEqNBA2011 - x IdentityMatrix[4] /. x → -m r1 r2/((r1 + a) (r2 + b))], {m, 0, 1}]]
```

$$O[m]^2$$

This confirms that eq. (4.29) of BA2011 is consistent with matrix J_N given in their eq. (4.28).

- Assuming tight linkage (weak recombination)

The coordinates of the internal equilibrium under the assumption of tight linkage, *i.e.* $r \ll \min(a, m)$, up to and including first-order terms of r, can be obtained from eq. (4.2) in BA2011.

```
{pEqBA, qEqBA, DABEqBA} // MatrixForm
```

$$\left(\begin{array}{c} \frac{-a^2+b^2+6\,a\,(r1+r2)-4\,m\,(r1+r2)-(r1+r2)^2+(a-b+r1+r2)\sqrt{-8\,m\,(r1+r2)+(a+b+r1+r2)^2}}{8\,a\,(r1+r2)} \\ \frac{a^2-b^2+6\,b\,(r1+r2)-4\,m\,(r1+r2)-(r1+r2)^2+(-a+b+r1+r2)\sqrt{-8\,m\,(r1+r2)+(a+b+r1+r2)^2}}{8\,b\,(r1+r2)} \\ \frac{-8\,m^2\,(r1+r2)^2-4\,m\,(r1+r2)\left(a^2-2\,a\,b+b^2-2\,a\,(r1+r2)-2\,b\,(r1+r2)+(r1+r2)^2\right)+(a-b-r1-r2)\,(a+b-r1-r2)\,(a-b+r1+r2)\left(a+b+r1+r2-\sqrt{-8\,m\,(r\right.}{32\,a\,b\,(r1+r2)^2} \end{array}\right.$$

```
In[129]:= pEqWeakRec = FullSimplify[
     Series[pEqBA /. {r1 → ρ1 ϵ, r2 → ρ2 ϵ}, {ϵ, 0, 1}] /. {ρ1 → r1 / ϵ, ρ2 → r2 / ϵ} // Normal,
     Assumptions → {0 < a < b, 0 < m, 0 < r1, 0 < r2}] // FullSimplify;
   qEqWeakRec = FullSimplify[Series[qEqBA /. {r1 → ρ1 ϵ, r2 → ρ2 ϵ}, {ϵ, 0, 1}] /.
        {ρ1 → r1 / ϵ, ρ2 → r2 / ϵ} // Normal,
     Assumptions → {0 < a < b, 0 < m, 0 < r1, 0 < r2}] // FullSimplify;
   DABEqWeakRec = FullSimplify[Series[DABEqBA /. {r1 → ρ1 ϵ, r2 → ρ2 ϵ}, {ϵ, 0, 1}] /.
        {ρ1 → r1 / ϵ, ρ2 → r2 / ϵ} // Normal,
     Assumptions → {0 < a < b, 0 < m, 0 < r1, 0 < r2}] // FullSimplify;
```

```
{pEqWeakRec, qEqWeakRec, DABEqWeakRec} // TableForm
```

$$-\frac{-a^4+a^3\,(-3\,b+m)+a^2\,b\,(-3\,b+2\,m)+b\,(b-m)\,m\,(r1+r2)+a\left(-b^3+b^2\,m+b\,m\,(r1+r2)+m^2\,(r1+r2)\right)}{a\,(a+b)^3}$$

$$-\frac{-a^3\,b-b^4+b^3\,m+b\,m^2\,(r1+r2)+a^2\left(-3\,b^2+b\,m+m\,(r1+r2)\right)+a\left(-3\,b^3+2\,b^2\,m+b\,m\,(r1+r2)-m^2\,(r1+r2)\right)}{b\,(a+b)^3}$$

$$\frac{m\left(a^4\,b-b^2\,(b-m)\,m\,(r1+r2)+a^3\left(3\,b^2-m\,(r1+r2)-b\,(m+r1+r2)\right)+a^2\left(3\,b^3+b\,m\,(r1+r2)+m^2\,(r1+r2)-2\,b^2\,(m+r1+r2)\right)+a\,b\left(b^3+b\,m\,(r1+r2)-2\,m^2\,(r1+\right.\right.}{a\,b\,(a+b)^4}$$

Checking these against first-order terms of eq. (4.1) in BA2011:

$$1-\frac{m}{a+b}-\frac{r\,m}{(a+b)^2}\left(\frac{b}{a}-\frac{m}{a+b}\left(\frac{b}{a}-1\right)\right)-\text{pEqWeakRec /. \{r → r1 + r2\} // FullSimplify}$$

$$0$$

$$1-\frac{m}{a+b}-\frac{r\,m}{(a+b)^2}\left(\frac{a}{b}+\frac{m}{a+b}\left(1-\frac{a}{b}\right)\right)-\text{qEqWeakRec /. \{r → r1 + r2\} // FullSimplify}$$

$$0$$

$$\frac{m}{a+b}\left(1-\frac{m}{a+b}\right)-\frac{r\,m}{(a+b)^2}\left(1-\frac{m}{a+b}\left(1-\frac{m}{a+b}\right)\left(2-\frac{b}{a}-\frac{a}{b}\right)\right)-\text{DABEqWeakRec /. \{r → r1 + r2\} // FullSimplify}$$

$$0$$

```
In[132]:= ruleApplyEqWeakRec :=
    {p → pEqWeakRec, q → qEqWeakRec, n → nC, DAB → DABEqWeakRec, DAC → 0, DCB → 0, DACB → 0}
```

```
ruleApplyEqWeakRec
```

$$\left\{p \to -\frac{1}{a\,(a+b)^3}\left(-a^4 + a^3\,(-3\,b+m) + a^2\,b\,(-3\,b+2\,m) + b\,(b-m)\,m\,(r1+r2) + a\left(-b^3+b^2\,m+b\,m\,(r1+r2)+m^2\,(r1+r2)\right)\right),\right.$$

$$q \to -\frac{1}{b\,(a+b)^3}\left(-a^3\,b - b^4 + b^3\,m + b\,m^2\,(r1+r2) + a^2\left(-3\,b^2+b\,m+m\,(r1+r2)\right) + a\left(-3\,b^3+2\,b^2\,m+b\,m\,(r1+r2)-m^2\,(r1+r2)\right)\right),\ n \to nC,$$

$$DAB \to \frac{1}{a\,b\,(a+b)^4}\,m\left(a^4\,b - b^2\,(b-m)\,m\,(r1+r2) + a^3\left(3\,b^2 - m\,(r1+r2) - b\,(m+r1+r2)\right) + a^2\left(3\,b^3+b\,m\,(r1+r2)+m^2\,(r1+r2)-2\,b^2\,(m+r1+r2)\right) + a\,b\left(b^3+b\,m\,(r1+r2)-2\,m^2\,(r1+r2)-b^2\,(m+r1+r2)\right)\right),\ DAC \to 0,\ DCB \to 0,\ DACB \to 0\left.\right\}$$

```
JPrep // MatrixForm
```

$$\begin{pmatrix} a-m-2\,a\,p & 0 & b & 0 & 0 & 0 \\ 0 & b-m-2\,b\,q & a & 0 & 0 & 0 \\ -2\,a\,DAB+m\,q & -2\,b\,DAB+m\,p & a+b-m-2\,a\,p-2\,b\,q-r1-r2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -m & a & b \\ 0 & 0 & 0 & m\,p & a-m-2\,a\,p-r1 & 0 \\ 0 & 0 & 0 & m\,q & 0 & b-m-2\,\ldots \\ 0 & 0 & 0 & m\,(DAB-p\,q) & -2\,a\,DAB+m\,q & -2\,b\,DA\ldots \end{pmatrix}$$

Now we see the block structure claimed in eq. (4.27) of BA2011.

In[133]:=
```
JEqWeakRec = JPrep /. ruleApplyEqWeakRec // FullSimplify;
JEqWeakRec // MatrixForm
```

Out[134]//MatrixForm=

$$\begin{pmatrix} a-m+\frac{1}{4}\left(8\,a\left(-1+\frac{m}{a+b}\right)+\frac{8\,m\,(b\,(b-m)+a\,(b+m))\,(r1+r2)}{(a+b)^3}\right) & 0 \\ 0 & b-m+\frac{1}{4}\left(8\,b\left(-1+\frac{m}{a+b}\right)+\frac{8}{\ldots}\right. \\ \frac{m\left(b\left(8-\frac{8\,m}{a+b}\right)-\frac{8\,m\left(a^2+a\,(b-m)+b\,m\right)(r1+r2)}{(a+b)^3}-\frac{16\left(a\,b\,(a+b)^2\,(a+b-m)-\left(a^2+a\,(b-m)+b\,m\right)(b\,(b-m)+a\,(b+m))\,(r1+r2)\right)}{(a+b)^4}\right)}{8\,b} & \frac{m\left(a\left(8-\frac{8\,m}{a+b}\right)-\frac{8\,m\,(b\,(b-m)+a\,(b+m))\,(r1+r2)}{(a+b)^3}-\frac{16\left(a\,b\,(a+b)^2\,\ldots\right.}{\ldots}\right.}{8\,a} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

In[135]:=
```
JEqNWeakRec = JEqWeakRec[[{4, 5, 6, 7}, {4, 5, 6, 7}]];
JEqNWeakRec // MatrixForm
```

Out[136]//MatrixForm=

$$\begin{pmatrix} -m & a \\ \frac{m\left(a\left(8-\frac{8\,m}{a+b}\right)-\frac{8\,m\,(b\,(b-m)+a\,(b+m))\,(r1+r2)}{(a+b)^3}\right)}{8\,a} & \\ \frac{m\left(b\left(8-\frac{8\,m}{a+b}\right)-\frac{8\,m\left(a^2+a\,(b-m)+b\,m\right)(r1+r2)}{(a+b)^3}\right)}{8\,b} & \\ \frac{m\left(-\left(b\left(8-\frac{8\,m}{a+b}\right)-\frac{8\,m\left(a^2+a\,(b-m)+b\,m\right)(r1+r2)}{(a+b)^3}\right)\left(a\left(8-\frac{8\,m}{a+b}\right)-\frac{8\,m\,(b\,(b-m)+a\,(b+m))\,(r1+r2)}{(a+b)^3}\right)+\frac{64\,m\left(a\,b\,(a+b)^2\,(a+b-m)-\left(a^2+a\,(b-m)+b\,m\right)(b\,(b-m)+a\,(b+m))\,(r1+r2)\right)}{(a+b)^4}\right)}{64\,a\,b} & m\left(b\left(8-\ldots\right.\right. \end{pmatrix}$$

- Using the Ansatz $m_e = -\lambda_N = m\,z$

```
JEqWeakRec
```

$$\Big\{\Big\{-\frac{1}{(a+b)^3}\Big(a^4+a^3(3b-m)+a^2b(3b-m)+a(b+m)\big(b^2-2m(r1+r2)\big)+ b\,m\big(b^2-2b(r1+r2)+2m(r1+r2)\big)\Big),0,b,0,0,0,0\Big\},$$

$$\Big\{0,\frac{1}{(a+b)^3}\Big(-b^4+b^3m-a^3(b+m)+2bm^2(r1+r2)+a^2\big(-3b^2-bm+2m(r1+r2)\big)+ a\big(-3b^3+b^2m+2bm(r1+r2)-2m^2(r1+r2)\big)\Big),a,0,0,0,0\Big\},$$

$$\Big\{\frac{1}{b(a+b)^4}m\Big(-a^4b+b^2\big(b^3-b^2m+2bm(r1+r2)-3m^2(r1+r2)\big)+ ab\big(2b^3-3bm(r1+r2)+4m^2(r1+r2)+b^2(-m+2(r1+r2))\big)+ a^3\big(-2b^2+m(r1+r2)+b(m+2(r1+r2))\big)+ a^2\big(-4bm(r1+r2)-m^2(r1+r2)+b^2(m+4(r1+r2))\big)\Big),$$

$$\frac{1}{a(a+b)^4}m\Big(a^5+a^4(2b-m)+b^2(b-m)m(r1+r2)+a^3(2m(r1+r2)+b(-m+2(r1+r2)))+ ab\big(-b^3-4bm(r1+r2)+4m^2(r1+r2)+b^2(m+2(r1+r2))\big)+ a^2\big(-2b^3-3bm(r1+r2)-3m^2(r1+r2)+b^2(m+4(r1+r2))\big)\Big),$$

$$-\frac{a^2+b^2-2m(r1+r2)+b(-m+r1+r2)+a(2b-m+r1+r2)}{a+b},0,0,0,0\Big\},\{0,0,0,-m,a,b,0\},$$

$$\Big\{0,0,0,-\frac{1}{a(a+b)^3}m\Big(-a^4+a^3(-3b+m)+a^2b(-3b+2m)+ b(b-m)m(r1+r2)+a\big(-b^3+b^2m+bm(r1+r2)+m^2(r1+r2)\big)\Big),$$

$$a-m-r1+\frac{1}{(a+b)^3}2\Big(-a^4+a^3(-3b+m)+a^2b(-3b+2m)+b(b-m)m(r1+r2)+ a\big(-b^3+b^2m+bm(r1+r2)+m^2(r1+r2)\big)\Big),0,b\Big\},$$

$$\Big\{0,0,0,-\frac{1}{b(a+b)^3}m\Big(-a^3b-b^4+b^3m+bm^2(r1+r2)+a^2\big(-3b^2+bm+m(r1+r2)\big)+ a\big(-3b^3+2b^2m+bm(r1+r2)-m^2(r1+r2)\big)\Big),0,$$

$$b-m-r2+\frac{1}{(a+b)^3}2\Big(-a^3b-b^4+b^3m+bm^2(r1+r2)+a^2\big(-3b^2+bm+m(r1+r2)\big)+ a\big(-3b^3+2b^2m+bm(r1+r2)-m^2(r1+r2)\big)\Big),a\Big\},$$

$$\Big\{0,0,0,\frac{1}{ab(a+b)^6}m\Big(-\Big(-a^3b-b^4+b^3m+bm^2(r1+r2)+a^2\big(-3b^2+bm+m(r1+r2)\big)+ a\big(-3b^3+2b^2m+bm(r1+r2)-m^2(r1+r2)\big)\Big)\Big(-a^4+a^3(-3b+m)+ a^2b(-3b+2m)+b(b-m)m(r1+r2)+a\big(-b^3+b^2m+bm(r1+r2)+m^2(r1+r2)\big)\Big)+ (a+b)^2m\Big(a^4b-b^2(b-m)m(r1+r2)+a^3\big(3b^2-m(r1+r2)-b(m+r1+r2)\big)+ a^2\big(3b^3+bm(r1+r2)+m^2(r1+r2)-2b^2(m+r1+r2)\big)+ ab\big(b^3+bm(r1+r2)-2m^2(r1+r2)-b^2(m+r1+r2)\big)\Big)\Big),$$

$$\frac{1}{b(a+b)^4}m\Big(-a^4b+b^2\big(b^3-b^2m+2bm(r1+r2)-3m^2(r1+r2)\big)+ ab\big(2b^3-3bm(r1+r2)+4m^2(r1+r2)+b^2(-m+2(r1+r2))\big)+ a^3\big(-2b^2+m(r1+r2)+b(m+2(r1+r2))\big)+ a^2\big(-4bm(r1+r2)-m^2(r1+r2)+b^2(m+4(r1+r2))\big)\Big),$$

$$\frac{1}{a(a+b)^4}m\Big(a^5+a^4(2b-m)+b^2(b-m)m(r1+r2)+a^3(2m(r1+r2)+b(-m+2(r1+r2)))+ ab\big(-b^3-4bm(r1+r2)+4m^2(r1+r2)+b^2(m+2(r1+r2))\big)+$$

```
a^2 (-2 b^3 - 3 b m (r1 + r2) - 3 m^2 (r1 + r2) + b^2 (m + 4 (r1 + r2)))),
 - (a^2 + b^2 - 2 m (r1 + r2) + b (-m + r1 + r2) + a (2 b - m + r1 + r2)) / (a + b)}}
```

Applying the same Ansatz as above to the Jacobian obtained from the differential equations yields the following ratio between the effective and actual migration rate, $\frac{m_e}{m}$:

```
ruleApplyEqWeakRec
```

```
{p -> -1/(a (a + b)^3) (-a^4 + a^3 (-3 b + m) + a^2 b (-3 b + 2 m) +
     b (b - m) m (r1 + r2) + a (-b^3 + b^2 m + b m (r1 + r2) + m^2 (r1 + r2))),
 q -> -1/(b (a + b)^3) (-a^3 b - b^4 + b^3 m + b m^2 (r1 + r2) + a^2 (-3 b^2 + b m + m (r1 + r2)) +
     a (-3 b^3 + 2 b^2 m + b m (r1 + r2) - m^2 (r1 + r2))), n -> nC,
 DAB -> 1/(a b (a + b)^4) m (a^4 b - b^2 (b - m) m (r1 + r2) + a^3 (3 b^2 - m (r1 + r2) - b (m + r1 + r2)) +
     a^2 (3 b^3 + b m (r1 + r2) + m^2 (r1 + r2) - 2 b^2 (m + r1 + r2)) +
     a b (b^3 + b m (r1 + r2) - 2 m^2 (r1 + r2) - b^2 (m + r1 + r2))), DAC -> 0, DCB -> 0, DACB -> 0}
```

```
Simplify[
  Series[Det[(JEqWeakRec /. ruleApplyEqWeakRec) - x IdentityMatrix[7] /. x -> -m z],
   {m, 0, 1}], Assumptions -> {a ≥ 0, b ≥ 0, r1 ≥ 0, r2 ≥ 0}] // Normal
```

```
a b m (a + b + r1 + r2)^2 (a (b + r2) z + r1 (r2 (-1 + z) + b z))
```

```
Simplify[Solve[% == 0, z]]
```

```
{{z -> r1 r2 / ((a + r1) (b + r2))}}
```

Hence, we see that the effective migration rate does not only apply for weak migration, but also for weak recombination.

Graphical exploration of approximation

Generic

We compare the approximate effective migration rate to the exact (negative) eigenvalue of J_N computed numerically.

In[137]:=
```
approxEffMigRateACBFunc[a_, b_, m_, r1_, r2_] := m (r1/(a + r1)) (r2/(b + r2))
```

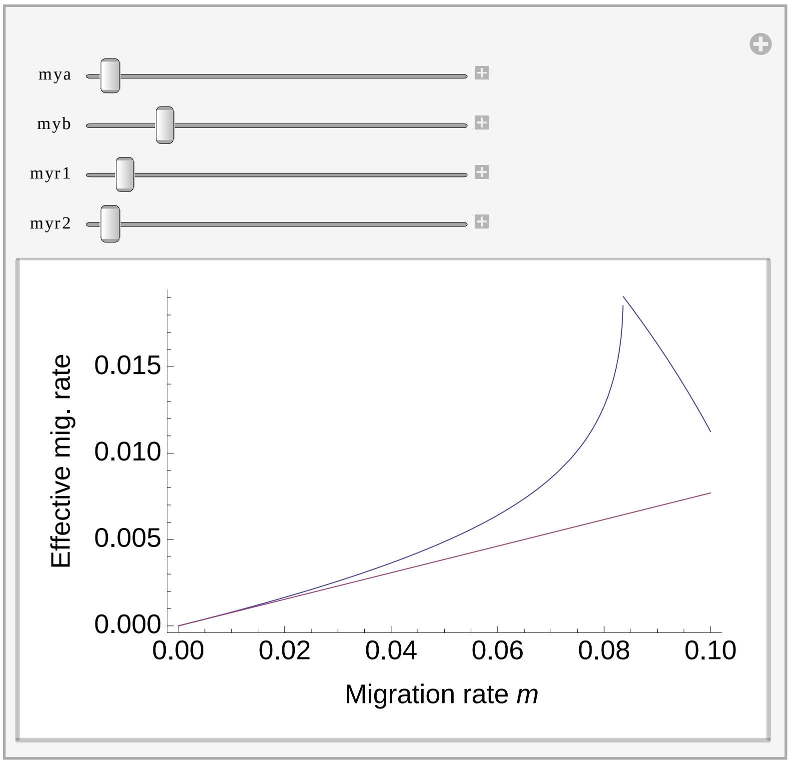
In[138]:=
```
exactEffMigRateACBFunc[a_,b_,m_,r1_,r2_]:=Module[{JN},JN={{-m,a,b,0},{1/(8 a (r1+r2)) m (-a
Return[-Max[Re[Eigenvalues[JN]]]]
]
```

```
mya = 0.002;
myb = 0.4;
mym = 0.0024;
myr1 = 0.01 * (10);
myr2 = 0.01 * (30);
{exactEffMigRateACBFunc[mya, myb, mym, myr1, myr2],
 approxEffMigRateACBFunc[mya, myb, mym, myr1, myr2],
 approxEffMigRateACBFunc[mya, myb, mym, myr1, myr2] /
   exactEffMigRateACBFunc[mya, myb, mym, myr1, myr2] - 1}
```

```
{0.00102183, 0.0010084, -0.0131391}
```

The exact (blue) and approximate (red) effective migration rate:

```
Manipulate[Plot[{exactEffMigRateACBFunc[mya, myb, m, myr1, myr2],
    approxEffMigRateACBFunc[mya, myb, m, myr1, myr2]}, {m, 0, .1},
  Frame → True, FrameStyle → Table[{Black, Opacity[0]}, {i, 1, 2}],
  FrameLabel → {"Migration rate m", "Effective mig. rate"},
  LabelStyle → {Directive[FontSize → 14], FontFamily → "Helvetica"}],
 {{mya, 0.02}, 0, 1}, {{myb, 0.4}, 0, 1}, {{myr1, 0.02}, 0, 1}, {{myr2, 0.02}, 0, 1}]
```



Order of loci: $\mathcal{A}$-$\mathcal{B}$-$\mathcal{C}$

We note that the definitions given in equations (1) to (37) above remain unchanged. The algorithm for the construction of difference equations under recombination also remains the same.

$$D_{ABC} = y_1 - p\,q\,n - p\,D_{CB} - q\,D_{AC} - n\,D_{AB} = y_1 - p\,q\,n - p\,[(y_1 + y_3) - q\,n] - q\,[(y_1 + y_2) - p\,n] - n\,[(y_1 + y_5) - p\,q] \quad (1)$$

- **Deriving the difference equations under recombination**

In[139]:=
```
gametes2 = {{A1, B1, C1}, {A1, B2, C1}, {A2, B1, C1},
   {A2, B2, C1}, {A1, B1, C2}, {A1, B2, C2}, {A2, B1, C2}, {A2, B2, C2}}
```

Out[139]=
```
{{A1, B1, C1}, {A1, B2, C1}, {A2, B1, C1}, {A2, B2, C1},
 {A1, B1, C2}, {A1, B2, C2}, {A2, B1, C2}, {A2, B2, C2}}
```

```
matings2 = Flatten[Table[{gametes2[[i]], gametes2[[j]]},
    {i, 1, Length[gametes2]}, {j, 1, Length[gametes2]}], 1]
Length[
 %]
```

```
{{{A1, B1, C1}, {A1, B1, C1}}, {{A1, B1, C1}, {A1, B2, C1}},
 {{A1, B1, C1}, {A2, B1, C1}}, {{A1, B1, C1}, {A2, B2, C1}},
 {{A1, B1, C1}, {A1, B1, C2}}, {{A1, B1, C1}, {A1, B2, C2}}, {{A1, B1, C1}, {A2, B1, C2}},
 {{A1, B1, C1}, {A2, B2, C2}}, {{A1, B2, C1}, {A1, B1, C1}}, {{A1, B2, C1}, {A1, B2, C1}},
 {{A1, B2, C1}, {A2, B1, C1}}, {{A1, B2, C1}, {A2, B2, C1}}, {{A1, B2, C1}, {A1, B1, C2}},
 {{A1, B2, C1}, {A1, B2, C2}}, {{A1, B2, C1}, {A2, B1, C2}}, {{A1, B2, C1}, {A2, B2, C2}},
 {{A2, B1, C1}, {A1, B1, C1}}, {{A2, B1, C1}, {A1, B2, C1}}, {{A2, B1, C1}, {A2, B1, C1}},
 {{A2, B1, C1}, {A2, B2, C1}}, {{A2, B1, C1}, {A1, B1, C2}}, {{A2, B1, C1}, {A1, B2, C2}},
 {{A2, B1, C1}, {A2, B1, C2}}, {{A2, B1, C1}, {A2, B2, C2}}, {{A2, B2, C1}, {A1, B1, C1}},
 {{A2, B2, C1}, {A1, B2, C1}}, {{A2, B2, C1}, {A2, B1, C1}}, {{A2, B2, C1}, {A2, B2, C1}},
 {{A2, B2, C1}, {A1, B1, C2}}, {{A2, B2, C1}, {A1, B2, C2}}, {{A2, B2, C1}, {A2, B1, C2}},
 {{A2, B2, C1}, {A2, B2, C2}}, {{A1, B1, C2}, {A1, B1, C1}}, {{A1, B1, C2}, {A1, B2, C1}},
 {{A1, B1, C2}, {A2, B1, C1}}, {{A1, B1, C2}, {A2, B2, C1}}, {{A1, B1, C2}, {A1, B1, C2}},
 {{A1, B1, C2}, {A1, B2, C2}}, {{A1, B1, C2}, {A2, B1, C2}}, {{A1, B1, C2}, {A2, B2, C2}},
 {{A1, B2, C2}, {A1, B1, C1}}, {{A1, B2, C2}, {A1, B2, C1}}, {{A1, B2, C2}, {A2, B1, C1}},
 {{A1, B2, C2}, {A2, B2, C1}}, {{A1, B2, C2}, {A1, B1, C2}}, {{A1, B2, C2}, {A1, B2, C2}},
 {{A1, B2, C2}, {A2, B1, C2}}, {{A1, B2, C2}, {A2, B2, C2}}, {{A2, B1, C2}, {A1, B1, C1}},
 {{A2, B1, C2}, {A1, B2, C1}}, {{A2, B1, C2}, {A2, B1, C1}}, {{A2, B1, C2}, {A2, B2, C1}},
 {{A2, B1, C2}, {A1, B1, C2}}, {{A2, B1, C2}, {A1, B2, C2}}, {{A2, B1, C2}, {A2, B1, C2}},
 {{A2, B1, C2}, {A2, B2, C2}}, {{A2, B2, C2}, {A1, B1, C1}}, {{A2, B2, C2}, {A1, B2, C1}},
 {{A2, B2, C2}, {A2, B1, C1}}, {{A2, B2, C2}, {A2, B2, C1}}, {{A2, B2, C2}, {A1, B1, C2}},
 {{A2, B2, C2}, {A1, B2, C2}}, {{A2, B2, C2}, {A2, B1, C2}}, {{A2, B2, C2}, {A2, B2, C2}}}
```

```
64
```

The gamete frequencies remain the same, as we have defined the list 'gametes2' appropriately:

```
{gametes2, gameteFreqs} // TableForm
```

A1 B1 C1	A1 B2 C1	A2 B1 C1	A2 B2 C1	A1 B1 C2	A1 B2 C2	A2 B1 C2	A2 B2 C2
y1	y2	y3	y4	y5	y6	y7	y8

Obviously, then, the pairs of frequencies applying to the matings remain the same, too:

```
{matings2, freqPairs} // TableForm
```

A1 B1 C1 A1 B1 C1	A1 B1 C1 A1 B2 C1	A1 B1 C1 A2 B1 C1	A1 B1 C1 A2 B2 C1	A1 B1 C1 A1 B1 C2	A1 B1 C1 A1 B2 C2	A1 B1 C1 A2 B1 C2
y1 y1	y1 y2	y1 y3	y1 y4	y1 y5	y1 y6	y1 y7

```
MapThread[gametesProduced[#1[[1]], #1[[2]], #2[[1]], #2[[2]]] &,
  {matings2[[1 ;; 2]], freqPairs[[1 ;; 2]]}] // TableForm
```

A1 B1 C1	$(1-r1)\,(1-r2)\,y1^2 + r1\,(1-r2)\,y1^2 + (1-r1)\,r2\,y1^2 + r1\,r2\,y1^2$
A1 B1 C1 A1 B2 C1	$\frac{1}{2}\,(1-r1)\,(1-r2)\,y1\,y2 + \frac{1}{2}\,r1\,(1-r2)\,y1\,y2 + \frac{1}{2}\,(1-r1)\,r2\,y1\,y2 + \frac{1}{2}\,r1\,r2\,y1\,y2$ $\frac{1}{2}\,(1-r1)\,(1-r2)\,y1\,y2 + \frac{1}{2}\,r1\,(1-r2)\,y1\,y2 + \frac{1}{2}\,(1-r1)\,r2\,y1\,y2 + \frac{1}{2}\,r1\,r2\,y1\,y2$

In[140]:= **recSep2 =**
MapThread[gametesProduced[#1⟦1⟧, #1⟦2⟧, #2⟦1⟧, #2⟦2⟧] &, {matings2, freqPairs}];
TableForm[recSep2]

MapThread::mptd : Object matings2 at position {2, 1} in
MapThread[gametesProduced[#1⟦1⟧, #1⟦2⟧, #2⟦1⟧, #2⟦2⟧] &, {matings2, {{y1, y1}, {y1, y2}, {y1, y3}, {y1, y4}, {y1, y5}, {y1, y6}, {y1, y7}, {y1, y8}, {y2, y1}, {y2, y2}, {y2, y3}, {y2, y4}, {y2, y5}, {y2, y6}, {y2, y7}, {y2, y8}, {y3, y1}, {y3, y2}, {y3, y3}, {y3, y4}, ≪12≫, {y5, y1}, {y5, y2}, {y5, y3}, {y5, y4}, {y5, y5}, {y5, y6}, {y5, y7}, {y5, y8}, {y6, y1}, {y6, y2}, {y6, y3}, {y6, y4}, {y6, y5}, {y6, y6}, {y6, y7}, {y6, y8}, {y7, y1}, {y7, y2}, ≪14≫}}
] has only 0 of required 1 dimensions. ≫

MapThread::mptd : Object matings2 at position {2, 1} in
MapThread[gametesProduced[#1⟦1⟧, #1⟦2⟧, #2⟦1⟧, #2⟦2⟧] &, {matings2, {{y1, y1}, {y1, y2}, {y1, y3}, {y1, y4}, {y1, y5}, {y1, y6}, {y1, y7}, {y1, y8}, {y2, y1}, {y2, y2}, {y2, y3}, {y2, y4}, {y2, y5}, {y2, y6}, {y2, y7}, {y2, y8}, {y3, y1}, {y3, y2}, {y3, y3}, {y3, y4}, ≪12≫, {y5, y1}, {y5, y2}, {y5, y3}, {y5, y4}, {y5, y5}, {y5, y6}, {y5, y7}, {y5, y8}, {y6, y1}, {y6, y2}, {y6, y3}, {y6, y4}, {y6, y5}, {y6, y6}, {y6, y7}, {y6, y8}, {y7, y1}, {y7, y2}, ≪14≫}}
] has only 0 of required 1 dimensions. ≫

Out[141]//TableForm=
MapThread[gametesProduced[#1⟦1⟧, #1⟦2⟧, #2⟦1⟧, #2⟦2⟧] &,
{matings2, {{y1, y1}, {y1, y2}, {y1, y3}, {y1, y4}, {y1, y5}, {y1, y6}, {y1, y7}, {y1, y8},
{y2, y1}, {y2, y2}, {y2, y3}, {y2, y4}, {y2, y5}, {y2, y6}, {y2, y7}, {y2, y8},
{y3, y1}, {y3, y2}, {y3, y3}, {y3, y4}, {y3, y5}, {y3, y6}, {y3, y7}, {y3, y8},
{y4, y1}, {y4, y2}, {y4, y3}, {y4, y4}, {y4, y5}, {y4, y6}, {y4, y7}, {y4, y8},
{y5, y1}, {y5, y2}, {y5, y3}, {y5, y4}, {y5, y5}, {y5, y6}, {y5, y7}, {y5, y8},
{y6, y1}, {y6, y2}, {y6, y3}, {y6, y4}, {y6, y5}, {y6, y6}, {y6, y7}, {y6, y8},
{y7, y1}, {y7, y2}, {y7, y3}, {y7, y4}, {y7, y5}, {y7, y6}, {y7, y7}, {y7, y8},
{y8, y1}, {y8, y2}, {y8, y3}, {y8, y4}, {y8, y5}, {y8, y6}, {y8, y7}, {y8, y8}}}]

indices2 = Table[Position[recSep2⟦i, 1⟧, gametes2⟦2⟧], {i, Length[recSep2]}]

{{}, {{2}}, {}, {{2}}, {}, {{3}}, {}, {{3}}, {{2}}, {{1}}, {{2}}, {{1}}, {{3}}, {{1}},
{{3}}, {{1}}, {}, {{2}}, {}, {}, {}, {{3}}, {}, {}, {{2}}, {{1}}, {}, {}, {{3}},
{{1}}, {}, {}, {}, {{3}}, {}, {{3}}, {}, {}, {}, {}, {{3}}, {{1}}, {{3}}, {{1}}, {},
{}, {}, {}, {}, {{3}}, {}, {}, {}, {}, {}, {}, {{3}}, {{1}}, {}, {}, {}, {}, {}, {}}

Flatten[Table[Part[recSep2, i, 2]⟦Flatten[indices2⟦i⟧]⟧, {i, Length[indices2]}]] //
Total // FullSimplify

$y2^2 + r1\,y4\,y5 - r1\,r2\,y4\,y5 + r2\,y3\,y6 - r1\,r2\,y3\,y6 + r1\,y4\,y6 + r2\,y4\,y6 - 2\,r1\,r2\,y4\,y6 + y2\,(y3 - r1\,y3 + y4 + y5 - r2\,y5 + y6 + y7 + r1\,(-1 + r2)\,y7 + y8 + r1\,(-1 + 2\,r2)\,y8 - r2\,(y7 + y8)) + y1\,(y2 + r2\,y6 + r1\,(y4 + r2\,y8))$

- **Differential equations under recombination**
- In terms of gamete frequencies

recTilde1 = sumPerOffspringGamete[gametes2, 1, recSep2] // FullSimplify

$y1^2 + r2\,(y2 + y3 + y4)\,y5 + r1\,(-r2\,y4\,y5 + y3\,(y5 - 2\,r2\,y5 + y6 - r2\,y6) + y2\,(y3 + r2\,y7)) + y1\,(y2 + y3 + y4 - r1\,y4 + y5 + y6 - r2\,y6 + y7 + r1\,(-1 + 2\,r2)\,y7 + y8 + r1\,(-1 + r2)\,y8 - r2\,(y7 + y8))$

The corresponding continuous-time differential equation:

yTilde1D =
FullSimplify[Series[recTilde1 - y1 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]

$y1^2 + r2\,(y2 + y3 + y4)\,y5 + r1\,y3\,(y2 + y5 + y6) + y1\,(-1 + y2 + y3 + y4 - r1\,y4 + y5 + y6 - r2\,y6 + y7 + y8 - (r1 + r2)\,(y7 + y8))$

Collect[yTilde1D, {r1, r2}]

$-y1 + y1^2 + y1\,y2 + y1\,y3 + y1\,y4 + y1\,y5 + y1\,y6 + y1\,y7 + y1\,y8 + r2\,((y2 + y3 + y4)\,y5 - y1\,y6 - y1\,(y7 + y8)) + r1\,(-y1\,y4 + y3\,(y2 + y5 + y6) - y1\,(y7 + y8))$

FullSimplify[(-y1 + y1^2 + y1 y2 + y1 y3 + y1 y4 + y1 y5 + y1 y6 + y1 y7 + y1 y8),
Assumptions → {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]

0

We note that 'yTilde1D' can be simplified to

In[142]:=
```
yTilde1DotRec := -r1 (y1 (1 - p) - y3 p) - r2 (y1 (1 - n) - y5 n)

yTilde1DotRec - yTilde1D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7, n → y1 + y2 + y3 + y4} //.
   {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify
```
0
```
recTilde2 = sumPerOffspringGamete[gametes2, 2, recSep2] // FullSimplify
```
y2^2 + r1 y4 y5 - r1 r2 y4 y5 + r2 y3 y6 - r1 r2 y3 y6 + r1 y4 y6 + r2 y4 y6 - 2 r1 r2 y4 y6 +
 y2 (y3 - r1 y3 + y4 + y5 - r2 y5 + y6 + y7 + r1 (-1 + r2) y7 + y8 + r1 (-1 + 2 r2) y8 - r2 (y7 + y8)) +
 y1 (y2 + r2 y6 + r1 (y4 + r2 y8))
```
yTilde2D =
 FullSimplify[Series[recTilde2 - y2 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]
```
y2^2 + r2 (y1 + y3 + y4) y6 + r1 y4 (y1 + y5 + y6) +
 y2 (-1 + y1 + y3 - r1 y3 + y4 + y5 - r2 y5 + y6 + y7 + y8 - (r1 + r2) (y7 + y8))
```
Collect[yTilde2D, {r1, r2}]
```
-y2 + y1 y2 + y2^2 + y2 y3 + y2 y4 + y2 y5 + y2 y6 + y2 y7 + y2 y8 +
 r2 (-y2 y5 + (y1 + y3 + y4) y6 - y2 (y7 + y8)) + r1 (-y2 y3 + y4 (y1 + y5 + y6) - y2 (y7 + y8))
```
FullSimplify[(-y2 + y1 y2 + y2^2 + y2 y3 + y2 y4 + y2 y5 + y2 y6 + y2 y7 + y2 y8),
 Assumptions → {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]
```
0

We note that 'yTilde2D' can be simplified to

In[143]:=
```
yTilde2DotRec := -r1 (y2 (1 - p) - y4 p) - r2 (y2 (1 - n) - y6 n)

yTilde2DotRec - yTilde2D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7, n → y1 + y2 + y3 + y4} //.
   {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify
```
0
```
recTilde3 = sumPerOffspringGamete[gametes2, 3, recSep2] // FullSimplify
```
y3^2 + y3 y4 + y3 y5 - r1 y3 y5 - r2 y3 y5 + 2 r1 r2 y3 y5 + r1 r2 y4 y5 + y3 y6 -
 r1 y3 y6 - r2 y3 y6 + r1 r2 y3 y6 + y3 y7 + r2 y4 y7 - (-1 + r1) y2 (y3 + r2 y7) +
 y3 y8 - r2 y3 y8 + y1 (y3 + r2 y7 + r1 (y4 + y7 - 2 r2 y7 + y8 - r2 y8))
```
yTilde3D =
 FullSimplify[Series[recTilde3 - y3 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]
```
y3^2 + r2 (y1 + y2 + y4) y7 + r1 y1 (y4 + y7 + y8) +
 y3 (-1 + y1 + y2 - r1 y2 + y4 + y5 + y6 - (r1 + r2) (y5 + y6) + y7 + y8 - r2 y8)
```
Collect[yTilde3D, {r1, r2}]
```
-y3 + y1 y3 + y2 y3 + y3^2 + y3 y4 + y3 y5 + y3 y6 + y3 y7 + y3 y8 +
 r2 (-y3 (y5 + y6) + (y1 + y2 + y4) y7 - y3 y8) + r1 (-y2 y3 - y3 (y5 + y6) + y1 (y4 + y7 + y8))
```
FullSimplify[(-y3 + y1 y3 + y2 y3 + y3^2 + y3 y4 + y3 y5 + y3 y6 + y3 y7 + y3 y8),
 Assumptions → {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]
```
0

We note that 'yTilde3D' can be simplified to

In[144]:=
```
yTilde3DotRec := -r1 (y3 p - y1 (1 - p)) - r2 (y3 (1 - n) - y7 n)

yTilde3DotRec - yTilde3D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7, n → y1 + y2 + y3 + y4} //.
   {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify
```
0

```
recTilde4 = sumPerOffspringGamete[gametes2, 4, recSep2] // FullSimplify

y4 (y1 + y2 + y3 + y4 - (-1 + r2) (y5 + y6 + y7)) + (r2 (y1 + y2 + y3) + y4) y8 +
 r1 ((-1 + r2) y4 y5 + (r2 y3 - y4 + 2 r2 y4) y6 + y2 (y3 + y7 - r2 y7 + y8 - 2 r2 y8) - y1 (y4 + r2 y8))

yTilde4D =
 FullSimplify[Series[recTilde4 - y4 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]

(-1 + y1) y4 + y4 (y2 + y3 + y4 - (-1 + r2) (y5 + y6 + y7)) +
 (r2 (y1 + y2 + y3) + y4) y8 + r1 (-y4 (y1 + y5 + y6) + y2 (y3 + y7 + y8))

Collect[yTilde4D, {r1, r2}]

(-1 + y1) y4 + y2 y4 + y3 y4 + y4^2 + y4 y5 + y4 y6 + y4 y7 + y4 y8 +
 r2 (-y4 (y5 + y6 + y7) + (y1 + y2 + y3) y8) + r1 (-y4 (y1 + y5 + y6) + y2 (y3 + y7 + y8))

FullSimplify[((-1 + y1) y4 + y2 y4 + y3 y4 + y4^2 + y4 y5 + y4 y6 + y4 y7 + y4 y8),
 Assumptions → {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]

0
```

We note that 'yTilde4D' can be simplified to

```
In[145]:= yTilde4DotRec := -r1 (y4 p - y2 (1 - p)) - r2 (y4 (1 - n) - y8 n)

yTilde4DotRec - yTilde4D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7, n → y1 + y2 + y3 + y4} //.
   {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify

0

recTilde5 = sumPerOffspringGamete[gametes2, 5, recSep2] // FullSimplify

y3 y5 - r1 y3 y5 - r2 y3 y5 + 2 r1 r2 y3 y5 + y4 y5 - r1 y4 y5 - r2 y4 y5 +
 r1 r2 y4 y5 + y5^2 + r1 r2 y3 y6 + y5 y6 + y5 y7 + r1 y6 y7 - (-1 + r2) y2 (y5 + r1 y7) +
 y5 y8 - r1 y5 y8 + y1 (y5 + r1 y7 + r2 (y6 + y7 - 2 r1 y7 + y8 - r1 y8))

yTilde5D =
 FullSimplify[Series[recTilde5 - y5 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]

y5^2 + r1 (y1 + y2 + y6) y7 + r2 y1 (y6 + y7 + y8) +
 y5 (-1 + y1 + y2 - r2 y2 + y3 - r1 y3 + y4 - r2 (y3 + y4) + y6 + y7 + y8 - r1 (y4 + y8))

Collect[yTilde5D, {r1, r2}]

-y5 + y1 y5 + y2 y5 + y3 y5 + y4 y5 + y5^2 + y5 y6 + y5 y7 + y5 y8 +
 r1 (-y3 y5 + (y1 + y2 + y6) y7 - y5 (y4 + y8)) + r2 (-y2 y5 + (-y3 - y4) y5 + y1 (y6 + y7 + y8))

FullSimplify[(-y5 + y1 y5 + y2 y5 + y3 y5 + y4 y5 + y5^2 + y5 y6 + y5 y7 + y5 y8),
 Assumptions → {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]

0
```

We note that 'yTilde5D' can be simplified to

```
In[146]:= yTilde5DotRec := -r1 (y5 (1 - p) - y7 p) - r2 (y5 n - y1 (1 - n))

yTilde5DotRec - yTilde5D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7, n → y1 + y2 + y3 + y4} //.
   {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify

0

recTilde6 = sumPerOffspringGamete[gametes2, 6, recSep2] // FullSimplify

y6 (y1 + y2 + y3 - r1 y3 + y4 + y5 + y6 + y7 - r1 (y4 + y7)) + (r1 (y1 + y2 + y5) + y6) y8 +
 r2 (-(y1 + y3 + y4) y6 + y2 (y5 + y7 - r1 y7 + y8 - 2 r1 y8) + r1 (y4 y5 + y3 y6 + 2 y4 y6 - y1 y8))

yTilde6D =
 FullSimplify[Series[recTilde6 - y6 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]

y6 (-1 + y1 + y2 + y3 - r1 y3 + y4 + y5 + y6 + y7 - r1 (y4 + y7)) +
 (r1 (y1 + y2 + y5) + y6) y8 + r2 (-(y1 + y3 + y4) y6 + y2 (y5 + y7 + y8))
```

```
Collect[yTilde6D, {r1, r2}]
```

```
-y6 + y1 y6 + y2 y6 + y3 y6 + y4 y6 + y5 y6 + y6^2 + y6 y7 + y6 y8 +
 r1 (-y3 y6 - y6 (y4 + y7) + (y1 + y2 + y5) y8) + r2 (- (y1 + y3 + y4) y6 + y2 (y5 + y7 + y8))
```

```
FullSimplify[(-y6 + y1 y6 + y2 y6 + y3 y6 + y4 y6 + y5 y6 + y6^2 + y6 y7 + y6 y8),
 Assumptions → {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]
```

```
0
```

We note that ‘yTilde6D’ can be simplified to

```
In[147]:= yTilde6DotRec := -r1 (y6 (1 - p) - y8 p) - r2 (y6 n - y2 (1 - n))
```

```
yTilde6DotRec - yTilde6D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7, n → y1 + y2 + y3 + y4} //.
   {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify
```

```
0
```

```
recTilde7 = sumPerOffspringGamete[gametes2, 7, recSep2] // FullSimplify
```

```
y7 (y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8) +
 r1 (y4 (y5 - r2 y5) + y3 (y5 - 2 r2 y5 - r2 y6) + ((-1 + 2 r2) y1 + (-1 + r2) y2 - y6) y7 +
    (r2 y1 + y5) y8) + r2 (- (y1 + y2 + y4) y7 + y3 (y5 + y6 + y8))
```

```
yTilde7D =
 FullSimplify[Series[recTilde7 - y7 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]
```

```
y7 (-1 + y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8) +
 r1 (- (y1 + y2 + y6) y7 + y5 (y3 + y4 + y8)) + r2 (- (y1 + y2 + y4) y7 + y3 (y5 + y6 + y8))
```

```
Collect[yTilde7D, {r1, r2}]
```

```
y7 (-1 + y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8) +
 r1 (- (y1 + y2 + y6) y7 + y5 (y3 + y4 + y8)) + r2 (- (y1 + y2 + y4) y7 + y3 (y5 + y6 + y8))
```

```
FullSimplify[(y7 (-1 + y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8)),
 Assumptions → {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]
```

```
0
```

We note that ‘yTilde7D’ can be simplified to

```
In[148]:= yTilde7DotRec := -r1 (y7 p - y5 (1 - p)) - r2 (y7 n - y3 (1 - n))
```

```
yTilde7DotRec - yTilde7D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7, n → y1 + y2 + y3 + y4} //.
   {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify
```

```
0
```

```
recTilde8 = sumPerOffspringGamete[gametes2, 8, recSep2] // FullSimplify
```

```
r1 y6 (y3 + y4 + y7) - r1 (y1 + y2 + y5) y8 + y8 (y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8) +
 r2 (y4 (y5 - r1 y5 + y6 - 2 r1 y6 + y7) - (y1 + y2 + y3) y8 + r1 (-y3 y6 + y2 y7 + y1 y8 + 2 y2 y8))
```

```
yTilde8D =
 FullSimplify[Series[recTilde8 - y8 /. recScale, {ϵ, 0, 1}] /. recBackScale // Normal]
```

```
r1 y6 (y3 + y4 + y7) + r2 y4 (y5 + y6 + y7) - r2 (y1 + y2 + y3) y8 -
 r1 (y1 + y2 + y5) y8 + y8 (-1 + y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8)
```

```
Collect[yTilde8D, {r1, r2}]
```

```
y8 (-1 + y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8) +
 r2 (y4 (y5 + y6 + y7) + (-y1 - y2 - y3) y8) + r1 (y6 (y3 + y4 + y7) + (-y1 - y2 - y5) y8)
```

```
FullSimplify[(y8 (-1 + y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8)),
 Assumptions → {y8 == 1 - y1 - y2 - y3 - y4 - y5 - y6 - y7, y1 + y2 + y3 + y4 + y5 + y6 + y7 + y8 == 1}]
```

```
0
```

We note that ‘yTilde8D’ can be simplified to

In[149]:=
```
yTilde8DotRec := -r1 (y8 p - y6 (1 - p)) - r2 (y8 n - y4 (1 - n))
```

```
yTilde8DotRec - yTilde8D /. {p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7, n → y1 + y2 + y3 + y4} //.
  {y1 → 1 - (y2 + y3 + y4 + y5 + y6 + y7 + y8)} // Simplify
```

```
0
```

- In terms of allele frequencies and LD

```
allToGam
```

```
{p → y1 + y2 + y5 + y6, q → y1 + y3 + y5 + y7, n → y1 + y2 + y3 + y4}
```

In[150]:=
```
pTildeDotRec := D[pDef /. {y1 → y1[t], y2 → y2[t], y5 → y5[t], y6 → y6[t]}, t] /.
    {y1'[t] → yTilde1DotRec, y2'[t] → yTilde2DotRec, y5'[t] → yTilde5DotRec,
     y6'[t] → yTilde6DotRec} /. gamToAllLD // FullSimplify
qTildeDotRec := D[qDef /. {y1 → y1[t], y3 → y3[t], y5 → y5[t], y7 → y7[t]}, t] /.
    {y1'[t] → yTilde1DotRec, y3'[t] → yTilde3DotRec, y5'[t] → yTilde5DotRec,
     y7'[t] → yTilde7DotRec} /. gamToAllLD // FullSimplify
nTildeDotRec := D[nDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t], y4 → y4[t]}, t] /.
    {y1'[t] → yTilde1DotRec, y2'[t] → yTilde2DotRec, y3'[t] → yTilde3DotRec,
     y4'[t] → yTilde4DotRec} /. gamToAllLD // FullSimplify
DACTildeDotRec := D[DACDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t],
        y4 → y4[t], y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
     {y1'[t] → yTilde1DotRec, y2'[t] → yTilde2DotRec, y3'[t] → yTilde3DotRec,
      y4'[t] → yTilde4DotRec, y5'[t] → yTilde5DotRec, y6'[t] → yTilde6DotRec,
      y7'[t] → yTilde7DotRec, y8'[t] → yTilde8DotRec} /.
    {x_[t] → x} /. gamToAllLD // FullSimplify
DCBTildeDotRec := D[DCBDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t],
        y4 → y4[t], y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
     {y1'[t] → yTilde1DotRec, y2'[t] → yTilde2DotRec, y3'[t] → yTilde3DotRec,
      y4'[t] → yTilde4DotRec, y5'[t] → yTilde5DotRec, y6'[t] → yTilde6DotRec,
      y7'[t] → yTilde7DotRec, y8'[t] → yTilde8DotRec} /.
    {x_[t] → x} /. gamToAllLD // FullSimplify
DABTildeDotRec := D[DABDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t],
        y4 → y4[t], y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
     {y1'[t] → yTilde1DotRec, y2'[t] → yTilde2DotRec, y3'[t] → yTilde3DotRec,
      y4'[t] → yTilde4DotRec, y5'[t] → yTilde5DotRec, y6'[t] → yTilde6DotRec,
      y7'[t] → yTilde7DotRec, y8'[t] → yTilde8DotRec} /.
    {x_[t] → x} /. gamToAllLD // FullSimplify
DACBTildeDotRec := D[DACBDef /. {y1 → y1[t], y2 → y2[t], y3 → y3[t],
        y4 → y4[t], y5 → y5[t], y6 → y6[t], y7 → y7[t], y8 → y8[t]}, t] /.
     {y1'[t] → yTilde1DotRec, y2'[t] → yTilde2DotRec, y3'[t] → yTilde3DotRec,
      y4'[t] → yTilde4DotRec, y5'[t] → yTilde5DotRec, y6'[t] → yTilde6DotRec,
      y7'[t] → yTilde7DotRec, y8'[t] → yTilde8DotRec} /.
    {x_[t] → x} /. gamToAllLD // FullSimplify
```

```
{pTildeDotRec, qTildeDotRec, nTildeDotRec, DACTildeDotRec,
  DCBTildeDotRec, DABTildeDotRec, DACBTildeDotRec} // TableForm
```

```
0
0
0
-DAC (r1 + r2)
-DCB r2
-DAB r1
-DACB (r1 + r2)
```

- **A comment to the differential equations under migration and selection**

The differential equations under migration and selection are independent of the ordering of the loci, as long as gametes and their frequencies are defined consistently (such as given above).

■ Differential equations under selection, migration and recombination

■ In terms of gamete frequencies

In[157]:=

```
yTilde1Dot := y1DotSel + y1DotMig + yTilde1DotRec
yTilde2Dot := y2DotSel + y2DotMig + yTilde2DotRec
yTilde3Dot := y3DotSel + y3DotMig + yTilde3DotRec
yTilde4Dot := y4DotSel + y4DotMig + yTilde4DotRec

yTilde5Dot := y5DotSel + y5DotMig + yTilde5DotRec
yTilde6Dot := y6DotSel + y6DotMig + yTilde6DotRec
yTilde7Dot := y7DotSel + y7DotMig + yTilde7DotRec
yTilde8Dot := y8DotSel + y8DotMig + yTilde8DotRec
```

```
{yTilde1Dot, yTilde2Dot, yTilde3Dot, yTilde4Dot, yTilde5Dot,
   yTilde6Dot, yTilde7Dot, yTilde8Dot} // Simplify // TableForm
```

```
-m y1 + r1 ((-1 + p) y1 + p y3) + r2 ((-1 + n) y1 + n y5) + y1 (b (y2 + y4 + y6 + y8) + a (y3 + y4 + y7 + y8)
-m y2 + r1 ((-1 + p) y2 + p y4) + r2 ((-1 + n) y2 + n y6) + y2 (-b (y1 + y3 + y5 + y7) + a (y3 + y4 + y7 + y8
-m y3 - r1 ((-1 + p) y1 + p y3) + r2 ((-1 + n) y3 + n y7) + y3 (-a (y1 + y2 + y5 + y6) + b (y2 + y4 + y6 + y8
m (nC - y4) - r1 ((-1 + p) y2 + p y4) + y4 (-a (y1 + y2 + y5 + y6) - b (y1 + y3 + y5 + y7)) + r2 ((-1 + n) y4
-m y5 - r2 ((-1 + n) y1 + n y5) + r1 ((-1 + p) y5 + p y7) + y5 (b (y2 + y4 + y6 + y8) + a (y3 + y4 + y7 + y8)
-m y6 - r2 ((-1 + n) y2 + n y6) + r1 ((-1 + p) y6 + p y8) + y6 (-b (y1 + y3 + y5 + y7) + a (y3 + y4 + y7 + y8
-m y7 - r2 ((-1 + n) y3 + n y7) - r1 ((-1 + p) y5 + p y7) + y7 (-a (y1 + y2 + y5 + y6) + b (y2 + y4 + y6 + y8
(-a (y1 + y2 + y5 + y6) - b (y1 + y3 + y5 + y7)) y8 - m (-1 + nC + y8) - r2 ((-1 + n) y4 + n y8) - r1 ((-1 +
```

```
Map[Collect[#, {m, r1, r2}] &, {yTilde1Dot, yTilde2Dot, yTilde3Dot,
   yTilde4Dot, yTilde5Dot, yTilde6Dot, yTilde7Dot, yTilde8Dot}] // TableForm
```

```
-m y1 + r1 (-(1 - p) y1 + p y3) + r2 (-(1 - n) y1 + n y5) + y1 (b (y2 + y4 + y6 + y8) + a (y3 + y4 + y7 + y8)
-m y2 + r1 (-(1 - p) y2 + p y4) + r2 (-(1 - n) y2 + n y6) + y2 (-b (y1 + y3 + y5 + y7) + a (y3 + y4 + y7 + y8
-m y3 + r1 ((1 - p) y1 - p y3) + r2 (-(1 - n) y3 + n y7) + y3 (-a (y1 + y2 + y5 + y6) + b (y2 + y4 + y6 + y8)
m (nC - y4) + r1 ((1 - p) y2 - p y4) + y4 (-a (y1 + y2 + y5 + y6) - b (y1 + y3 + y5 + y7)) + r2 (-(1 - n) y4
-m y5 + r2 ((1 - n) y1 - n y5) + r1 (-(1 - p) y5 + p y7) + y5 (b (y2 + y4 + y6 + y8) + a (y3 + y4 + y7 + y8))
-m y6 + r2 ((1 - n) y2 - n y6) + r1 (-(1 - p) y6 + p y8) + y6 (-b (y1 + y3 + y5 + y7) + a (y3 + y4 + y7 + y8)
-m y7 + r2 ((1 - n) y3 - n y7) + r1 ((1 - p) y5 - p y7) + y7 (-a (y1 + y2 + y5 + y6) + b (y2 + y4 + y6 + y8))
m (1 - nC - y8) + (-a (y1 + y2 + y5 + y6) - b (y1 + y3 + y5 + y7)) y8 + r2 ((1 - n) y4 - n y8) + r1 ((1 - p) y
```

■ In terms of allele frequencies and LD

```
nDotMig
```

```
m (-n + nC)
```

```
nDotSel
```

```
a DAC + b DCB
```

In[165]:=

```
pTildeDot := pDotSel + pDotMig + pTildeDotRec
qTildeDot := qDotSel + qDotMig + qTildeDotRec
nTildeDot := nDotSel + nDotMig + nTildeDotRec
DACTildeDot := DACDotSel + DACDotMig + DACTildeDotRec
DCBTildeDot := DCBDotSel + DCBDotMig + DCBTildeDotRec
DABTildeDot := DABDotSel + DABDotMig + DABTildeDotRec
DACBTildeDot := DACBDotSel + DACBDotMig + DACBTildeDotRec
```

```
{pTildeDot, qTildeDot, nTildeDot, DACTildeDot,
   DCBTildeDot, DABTildeDot, DACBTildeDot} // FullSimplify // TableForm
```

```
b DAB - (m + a (-1 + p)) p
a DAB - (m + b (-1 + q)) q
a DAC + b DCB + m (-n + nC)
b DACB + m (n - nC) p + a (DAC - 2 DAC p) - DAC (m + r1 + r2)
a DACB + m (n - nC) q + b (DCB - 2 DCB q) - DCB (m + r2)
-DAB m + m p q + DAB (a + b - 2 a p - 2 b q) - DAB r1
a (-2 DAB DAC + DACB - 2 DACB p) + b (DACB - 2 DAB DCB - 2 DACB q) + m (-DACB + DCB p + DAC q + (n - nC) (D
```

```
Map[Collect[#, {m, r1, r2}] &, {pTildeDot, qTildeDot, nTildeDot,
   DACTildeDot, DCBTildeDot, DABTildeDot, DACBTildeDot}] // TableForm
```

```
b DAB − m p − a (−1 + p) p
a DAB − m q − b (−1 + q) q
a DAC + b DCB + m (−n + nC)
b DACB + a (DAC − 2 DAC p) + m (−DAC − (−n + nC) p) − DAC r1 − DAC r2
a DACB + b DCB (1 − 2 q) + m (−DCB − (−n + nC) q) − DCB r2
DAB (a + b − 2 a p − 2 b q) + m (−DAB + p q) − DAB r1
a (−2 DAB DAC + DACB − 2 DACB p) + b (DACB − 2 DAB DCB − 2 DACB q) + m (−DACB + DCB p + DAC q + (n − nC) (I
```

```
Map[Collect[#, {DAC, DAB, DCB, DACB}] &, {pTildeDot, qTildeDot, nTildeDot,
   DACTildeDot, DCBTildeDot, DABTildeDot, DACBTildeDot}] // TableForm
```

```
b DAB − m p − a (−1 + p) p
a DAB − m q − b (−1 + q) q
a DAC + b DCB + m (−n + nC)
b DACB − m (−n + nC) p + DAC (a − m − 2 a p − r1 − r2)
a DACB − m (−n + nC) q + DCB (−m + b (1 − 2 q) − r2)
m p q + DAB (a + b − m − 2 a p − 2 b q − r1)
DAB (−2 b DCB + m (n − nC)) + DCB m p − m (n − nC) p q + DAC (−2 a DAB + m q) + DACB (a + b − m − 2 a p − 2 b q
```

```
TM1 =
 Map[Collect[#, {DAC, DAB, DCB, DACB}] &, {pTildeDot, qTildeDot, nTildeDot, DACTildeDot,
    DCBTildeDot, DABTildeDot, DACBTildeDot}] /. {r1 → rAB, r2 → rCB}; TM1 // TableForm
```

```
b DAB − m p − a (−1 + p) p
a DAB − m q − b (−1 + q) q
a DAC + b DCB + m (−n + nC)
b DACB − m (−n + nC) p + DAC (a − m − 2 a p − rAB − rCB)
a DACB − m (−n + nC) q + DCB (−m + b (1 − 2 q) − rCB)
m p q + DAB (a + b − m − 2 a p − 2 b q − rAB)
DAB (−2 b DCB + m (n − nC)) + DCB m p − m (n − nC) p q + DAC (−2 a DAB + m q) + DACB (a + b − m − 2 a p − 2 b q
```

For comparison, we print the differential equations for the case of the ordering $\mathcal{A} - C - \mathcal{B}$ from above:

```
Map[Collect[#, {m, r1, r2}] &,
  {pDot, qDot, nDot, DACDot, DCBDot, DABDot, DACBDot}] // TableForm
```

```
b DAB − m p − a (−1 + p) p
a DAB − m q − b (−1 + q) q
a DAC + b DCB + m (−n + nC)
b DACB + a (DAC − 2 DAC p) + m (−DAC − (−n + nC) p) − DAC r1
a DACB + b DCB (1 − 2 q) + m (−DCB − (−n + nC) q) − DCB r2
DAB (a + b − 2 a p − 2 b q) + m (−DAB + p q) − DAB r1 − DAB r2
a (−2 DAB DAC + DACB − 2 DACB p) + b (DACB − 2 DAB DCB − 2 DACB q) + m (−DACB + DCB p + DAC q + (n − nC) (I
```

```
Map[Collect[#, {DAC, DAB, DCB, DACB}] &,
  {pDot, qDot, nDot, DACDot, DCBDot, DABDot, DACBDot}] // TableForm
```

```
b DAB − m p − a (−1 + p) p
a DAB − m q − b (−1 + q) q
a DAC + b DCB + m (−n + nC)
b DACB − m (−n + nC) p + DAC (a − m − 2 a p − r1)
a DACB − m (−n + nC) q + DCB (−m + b (1 − 2 q) − r2)
m p q + DAB (a + b − m − 2 a p − 2 b q − r1 − r2)
DAB (−2 b DCB + m (n − nC)) + DCB m p − m (n − nC) p q + DAC (−2 a DAB + m q) + DACB (a + b − m − 2 a p − 2 b q
```

```
TM2 = Map[Collect[#, {DAC, DAB, DCB, DACB}] &,
    {pDot, qDot, nDot, DACDot, DCBDot, DABDot, DACBDot}] /. {r1 → rAC, r2 → rCB};
TM2 //
 TableForm
```

```
b DAB - m p - a (-1 + p) p
a DAB - m q - b (-1 + q) q
a DAC + b DCB + m (-n + nC)
b DACB - m (-n + nC) p + DAC (a - m - 2 a p - rAC)
a DACB - m (-n + nC) q + DCB (-m + b (1 - 2 q) - rCB)
m p q + DAB (a + b - m - 2 a p - 2 b q - rAC - rCB)
DAB (-2 b DCB + m (n - nC)) + DCB m p - m (n - nC) p q + DAC (-2 a DAB + m q) + DACB (a + b - m - 2 a p - 2 b q
```

A comparison shows that only the differentials of $\dot{D}_{\text{AC}}$ and $\dot{D}_{\text{AB}}$ change.

```
MapThread[Simplify[#1 == #2] &, {TM1, TM2}] // TableForm
```

```
True
True
True
DAC (rAB - rAC + rCB) == 0
True
DAB rAB == DAB (rAC + rCB)
DACB rAB == DACB rAC
```

```
MapThread[Simplify[#1 == #2] &,
  {Map[Collect[#, {m, r1, r2}] &, {pTildeDot, qTildeDot, nTildeDot, DACTildeDot,
     DCBTildeDot, DABTildeDot, DACBTildeDot}], Map[Collect[#, {m, r1, r2}] &,
    {pDot, qDot, nDot, DACDot, DCBDot, DABDot, DACBDot}]}] // TableForm
```

```
True
True
True
DAC r2 == 0
True
DAB r2 == 0
True
```

Internal equilibrium

■ Coordinates

We recall the differential equations:

```
In[172]:= diffEqsTilde = Map[Collect[#, {m, r1, r2}] &, {pTildeDot, qTildeDot,
      DABTildeDot, nTildeDot, DACTildeDot, DCBTildeDot, DACBTildeDot}];
   diffEqsTilde // TableForm
```

```
Out[173]//TableForm=
b DAB - m p - a (-1 + p) p
a DAB - m q - b (-1 + q) q
DAB (a + b - 2 a p - 2 b q) + m (-DAB + p q) - DAB r1
a DAC + b DCB + m (-n + nC)
b DACB + a (DAC - 2 DAC p) + m (-DAC - (-n + nC) p) - DAC r1 - DAC r2
a DACB + b DCB (1 - 2 q) + m (-DCB - (-n + nC) q) - DCB r2
a (-2 DAB DAC + DACB - 2 DACB p) + b (DACB - 2 DAB DCB - 2 DACB q) + m (-DACB + DCB p + DAC q + (n - nC) (I
```

BA2011 (eq. 3.15) showed the coordinates of the internal stable equilibrium to be

```
In[174]:= R2 := √((a + b + r)^2 - 8 m r1)
```

In[175]:=
```
pEqTilde := 1/(8 a r) (b^2 - a^2 + 6 a r - r^2 - 4 m r + (a - b + r) R2) //. {r → r1}
qEqTilde := 1/(8 b r) (a^2 - b^2 + 6 b r - r^2 - 4 m r + (b - a + r) R2) //. {r → r1}
DABEqTilde := 1/(32 a b r^2) ((a - b - r) (a + b - r) (a - b + r) ((a + b + r) - R2) -
    4 m r (a^2 + b^2 + r^2 - 2 a b - 2 a r - 2 b r) - 8 m^2 r^2) //. {r → r1}
nEqTilde := nC
DACEqTilde := 0
DCBEqTilde := 0
DACBEqTilde := 0
```

We have used the assumption that higher-order recombination terms can be ignored, and therefore $r = r_1 + r_2$.

In[182]:=
```
ruleApplyEqTilde := {p → pEqTilde, q → qEqTilde, n → nEqTilde,
  DAB → DABEqTilde, DAC → DACEqTilde, DCB → DCBEqTilde, DACB → DACBEqTilde}
```

We have

```
diffEqsTilde /. ruleApplyEqTilde // FullSimplify
```

```
{0, 0, 0, 0, 0, 0, 0}
```

which confirms that this is indeed an equilibrium. We omit the proof that this equilibrium is asymptotically stable (*cf.* Bürger and Akerman 2011). Instead, we directly proceed to the computation of the Jacobian matrix.

- **Jacobian matrix and effective migration rate**
- Generic

In[183]:=
```
JTilde := Map[Table[D[#, {i}], {i, {p, q, DAB, n, DAC, DCB, DACB}}] &, diffEqsTilde]
JTilde // MatrixForm
```

Out[184]//MatrixForm=

-m - a (-1 + p) - a p	0	b	
0	-m - b (-1 + q) - b q	a	
-2 a DAB + m q	-2 b DAB + m p	a + b - m - 2 a p - 2 b q - r1	
0	0	0	
-2 a DAC + m (n - nC)	0	0	
0	-2 b DCB + m (n - nC)	0	
-2 a DACB + m (DCB - (n - nC) q)	-2 b DACB + m (DAC - (n - nC) p)	-2 a DAC - 2 b DCB + m (n - nC)	m (DA

```
D[diffEqsTilde⟦7⟧, DACB]
```

```
-m + a (1 - 2 p) + b (1 - 2 q) - r1 - r2
```

```
FullSimplify[Normal[Series[% /. ruleApplyEqTilde, {m, 0, 1}]],
 Assumptions → {0 < a < b, 0 < r1, 0 < r2}]
```

$$-a - b - r1 + \frac{m (a + b + 3 r1)}{a + b + r1} - r2$$

```
D[diffEqs⟦7⟧, DACB]
```

```
-m + a (1 - 2 p) + b (1 - 2 q) - r1 - r2
```

```
FullSimplify[Normal[Series[% /. ruleApplyEq, {m, 0, 1}]],
 Assumptions → {0 < a < b, 0 < r1, 0 < r2}]
```

$$-a - b + 3 m - r1 - r2 - \frac{2 (a + b) m}{a + b + r1 + r2}$$

$$\% - \left(-a - b - r1 - r2 + \frac{m (a + b + 3 (r1 + r2))}{a + b + r1 + r2}\right) \text{ // FullSimplify}$$

```
0
```

For comparison

```
J // MatrixForm
```

−m − a (−1 + p) − a p	0	b	
0	−m − b (−1 + q) − b q	a	
−2 a DAB + m q	−2 b DAB + m p	a + b − m − 2 a p − 2 b q − r1 − r2	
0	0	0	
−2 a DAC + m (n − nC)	0	0	
0	−2 b DCB + m (n − nC)	0	
−2 a DACB + m (DCB − (n − nC) q)	−2 b DACB + m (DAC − (n − nC) p)	−2 a DAC − 2 b DCB + m (n − nC)	m (D

As an intermediate step, we set n, D_{AC}, D_{CB}, and D_{ACB} to their equilibrium values n_c, 0, 0, and 0, respectively.

In[185]:=
```
JTildePrep = JTilde /. {n → nC, DAC → 0, DCB → 0, DACB → 0} // FullSimplify;
JTildePrep // MatrixForm
```

Out[186]//MatrixForm=

a − m − 2 a p	0	b	0	0	0
0	b − m − 2 b q	a	0	0	0
−2 a DAB + m q	−2 b DAB + m p	a + b − m − 2 a p − 2 b q − r1	0	0	0
0	0	0	−m	a	b
0	0	0	m p	a − m − 2 a p − r1 − r2	0
0	0	0	m q	0	b − m − 2
0	0	0	m (DAB − p q)	−2 a DAB + m q	−2 b DA

Now we plug in the equilibrium coordinates into the generic matrix of first-order partial derivatives:

In[187]:=
```
JEqGenericTilde = JTildePrep /. ruleApplyEqTilde // FullSimplify;
JEqGenericTilde // MatrixForm
```

Out[188]//MatrixForm=

$$a - m - \frac{-a^2+b^2+6\,a\,r1-4\,m\,r1-r1^2+(a-b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}}{4\,r1}$$

$$0$$

$$\frac{8\,m^2\,r1^2+4\,m\,r1\left(a^2+(b-r1)^2-2\,a\,(b+r1)\right)-(a-b-r1)\,(a+b-r1)\,(a-b+r1)\left(a+b+r1-\sqrt{-8\,m\,r1+(a+b+r1)^2}\right)+2\,m\,r1\left(a^2-b^2+6\,b\,r1-4\,m\,r1-r1^2+(-a\cdots\right.}{16\,b\,r1^2}$$

$$0$$

$$0$$

$$0$$

$$0$$

This is a complicated matrix. Below, we instead plug in the weak-migration approximation to the internal equilibrium.

In[189]:=
```
JEqGenericTildemSmall := Simplify[Normal[Series[JEqGenericTilde, {m, 0, 1}]],
  Assumptions → {0 < a < b, 0 < m < b, 0 < r1, 0 < r2}]
JEqGenericTildemSmall // MatrixForm
```

Out[190]//MatrixForm=

$-a + \frac{m\,(a-b+r1)}{a+b+r1}$	0	b	0	0	0
0	$-b + \frac{m\,(-a+b+r1)}{a+b+r1}$	a	0	0	0
$\frac{m\,(-a+b+r1)}{a+b+r1}$	$\frac{m\,(a-b+r1)}{a+b+r1}$	$-a - b - r1 + \frac{m\,(a+b+3\,r1)}{a+b+r1}$	0	0	0
0	0	0	−m	a	b
0	0	0	m	$-a - r1 + \frac{m\,(a-b+r1)}{a+b+r1} - r2$	0
0	0	0	m	0	$-b + \frac{m\,(-a+b+r1)}{a+b+r1} - r2$
0	0	0	−m	$\frac{m\,(-a+b+r1)}{a+b+r1}$	$\frac{m\,(a-b+r1)}{a+b+r1}$

For comparison

```
JEqmSmall // MatrixForm
```

$$\begin{pmatrix} -a+m-\frac{2\,b\,m}{a+b+r1+r2} & 0 & b & 0 & 0 & \\ 0 & -b+m-\frac{2\,a\,m}{a+b+r1+r2} & a & 0 & 0 & \\ \frac{m\,(-a+b+r1+r2)}{a+b+r1+r2} & m-\frac{2\,b\,m}{a+b+r1+r2} & -a-b-r1-r2+\frac{m\,(a+b+3\,(r1+r2))}{a+b+r1+r2} & 0 & 0 & \\ 0 & 0 & 0 & -m & a & \\ 0 & 0 & 0 & m & -a+m-r1-\frac{2\,b\,m}{a+b+r1+r2} & \\ 0 & 0 & 0 & m & 0 & -b+m\cdots \\ 0 & 0 & 0 & -m & \frac{m\,(-a+b+r1+r2)}{a+b+r1+r2} & m\cdots \end{pmatrix}$$

In[191]:=
```
JEqNGenericTilde := JEqGenericTilde[[4 ;; 7, 4 ;; 7]]
JEqNGenericTilde // MatrixForm
```

Out[192]//MatrixForm=

$$\begin{pmatrix} \cdots -m \\ \cdots \frac{m\left(-a^2+b^2+6\,a\,r1-4\,m\,r1-r1^2+(a-b+r1)\sqrt{-8\,m\,r1+(a+b\cdots}\right.}{8\,a\,r1} \\ \cdots \frac{m\left(a^2-b^2+6\,b\,r1-4\,m\,r1-r1^2+(-a+b+r1)\sqrt{-8\,m\,r1+(a+b\cdots}\right.}{8\,b\,r1} \\ \frac{m\left(-\left(-a^2+b^2+6\,a\,r1-4\,m\,r1-r1^2+(a-b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}\right)\left(a^2-b^2+6\,b\,r1-4\,m\,r1-r1^2+(-a+b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}\right)+2\left(-8\,m^2\,r1^2-4\,m\cdots\right.\right.}{64\,a\,b\,r1^2} \end{pmatrix}$$

In[193]:=
```
JEqNGenericTildemSmall := Simplify[Normal[Series[JEqNGenericTilde, {m, 0, 1}]],
  Assumptions → {0 < a < b, 0 < m < b, 0 < r1, 0 < r2}]
JEqNGenericTildemSmall // MatrixForm
```

Out[194]//MatrixForm=

$$\begin{pmatrix} -m & a & b & 0 \\ m & -a-r1+\frac{m\,(a-b+r1)}{a+b+r1}-r2 & 0 & b \\ m & 0 & -b+\frac{m\,(-a+b+r1)}{a+b+r1}-r2 & a \\ -m & \frac{m\,(-a+b+r1)}{a+b+r1} & \frac{m\,(a-b+r1)}{a+b+r1} & -a-b-r1+\frac{m\,(a+b+3\,r1)}{a+b+r1}-r2 \end{pmatrix}$$

```
TM3 = JEqNGenericTildemSmall /. {r1 → rAB, r2 → rBC};
TM3 // MatrixForm
```

$$\begin{pmatrix} -m & a & b & 0 \\ m & -a-rAB+\frac{m\,(a-b+rAB)}{a+b+rAB}-rBC & 0 & b \\ m & 0 & -b+\frac{m\,(-a+b+rAB)}{a+b+rAB}-rBC & a \\ -m & \frac{m\,(-a+b+rAB)}{a+b+rAB} & \frac{m\,(a-b+rAB)}{a+b+rAB} & -a-b-rAB+\frac{m\,(a+b+3\,rAB)}{a+b+rAB}-rBC \end{pmatrix}$$

For comparison:

```
JEqNmSmall // MatrixForm
```

$$\begin{pmatrix} -m & a & b & 0 \\ m & -a+m-r1-\frac{2\,b\,m}{a+b+r1+r2} & 0 & b \\ m & 0 & -b+m-r2-\frac{2\,a\,m}{a+b+r1+r2} & a \\ -m & \frac{m\,(-a+b+r1+r2)}{a+b+r1+r2} & m-\frac{2\,b\,m}{a+b+r1+r2} & -a-b-r1-r2+\frac{m\,(a+b+3\,(r1+r2))}{a+b+r1+r2} \end{pmatrix}$$

```
TM4 = JEqNmSmall /. {r1 → rAC, r2 → rCB};
TM4 // MatrixForm
```

$$\begin{pmatrix} -m & a & b & 0 \\ m & -a+m-rAC-\frac{2\,b\,m}{a+b+rAC+rCB} & 0 & b \\ m & 0 & -b+m-rCB-\frac{2\,a\,m}{a+b+rAC+rCB} & a \\ -m & \frac{m\,(-a+b+rAC+rCB)}{a+b+rAC+rCB} & m-\frac{2\,b\,m}{a+b+rAC+rCB} & -a-b-rAC-rCB+\frac{m\,(a+b+3\,(rAC+rCB))}{a+b+rAC+rCB} \end{pmatrix}$$

```
FullSimplify[JEqNGenericTildemSmall - JEqNmSmall] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & r2\left(-1-\frac{2\,b\,m}{(a+b+r1)\,(a+b+r1+r2)}\right) & 0 & 0 \\ 0 & 0 & -\frac{2\,a\,m\,r2}{(a+b+r1)\,(a+b+r1+r2)} & 0 \\ 0 & -\frac{2\,a\,m\,r2}{(a+b+r1)\,(a+b+r1+r2)} & -\frac{2\,b\,m\,r2}{(a+b+r1)\,(a+b+r1+r2)} & -\frac{2\,(a+b)\,m\,r2}{(a+b+r1)\,(a+b+r1+r2)} \end{pmatrix}$$

- Using the Ansatz $m_e = -\lambda_N = m\,z$

```
JEqGenericTilde // MatrixForm
```

$$a-m-\frac{-a^2+b^2+6\,a\,r1-4\,m\,r1-r1^2+(a-b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}}{4\,r1}$$

$$0$$

$$\frac{8\,m^2\,r1^2+4\,m\,r1\left(a^2+(b-r1)^2-2\,a\,(b+r1)\right)-(a-b-r1)\,(a+b-r1)\,(a-b+r1)\left(a+b+r1-\sqrt{-8\,m\,r1+(a+b+r1)^2}\right)+2\,m\,r1\left(a^2-b^2+6\,b\,r1-4\,m\,r1-r1^2+(-a\right.}{16\,b\,r1^2}$$

$$0$$

$$0$$

$$0$$

$$0$$

```
Simplify[Series[Det[JEqGenericTilde - x IdentityMatrix[7] /. x → -m z], {m, 0, 1}],
  Assumptions → {a ≥ 0, b ≥ 0, r1 ≥ 0, r2 ≥ 0}] // Normal
```

$$a\,b\,m\,(a+b+r1)\,(a+r1+r2)\left(b^2\,z+r2\,(r1\,(-1+z)+r2\,(-1+z)+a\,z)+b\,((a+r1)\,z+r2\,(-1+2\,z))\right)$$

```
Simplify[Solve[% == 0, z]]
```

$$\left\{\left\{z\to\frac{r2\,(b+r1+r2)}{(b+r2)\,(a+b+r1+r2)}\right\}\right\}$$

which is identical to eq. (4.30) in BA2011. Or, alternatively and more directly:

```
JEqNGenericTildemSmall // MatrixForm
```

$$\begin{pmatrix} -m & a & b & 0 \\ m & -a-r1+\frac{m\,(a-b+r1)}{a+b+r1}-r2 & 0 & b \\ m & 0 & -b+\frac{m\,(-a+b+r1)}{a+b+r1}-r2 & a \\ -m & \frac{m\,(-a+b+r1)}{a+b+r1} & \frac{m\,(a-b+r1)}{a+b+r1} & -a-b-r1+\frac{m\,(a+b+3\,r1)}{a+b+r1}-r2 \end{pmatrix}$$

```
Simplify[Series[Det[JEqNGenericTildemSmall - x IdentityMatrix[4] /. x → -m z], {m, 0, 1}],
  Assumptions → {a ≥ 0, b ≥ 0, r1 ≥ 0, r2 ≥ 0}] // Normal
```

$$-m\,(a+r1+r2)\left(b^2\,z+r2\,(r1\,(-1+z)+r2\,(-1+z)+a\,z)+b\,((a+r1)\,z+r2\,(-1+2\,z))\right)$$

```
Simplify[Solve[% == 0, z]]
```

$$\left\{\left\{z\to\frac{r2\,(b+r1+r2)}{(b+r2)\,(a+b+r1+r2)}\right\}\right\}$$

- Assuming weak migration

The coordinates of the internal equilibrium under the assumption of weak migration, up to and including first-order terms of *m*, can be obtained from eq. (4.1) in BA2011.

```
In[195]:= pEqTildeWeakMig = FullSimplify[Series[pEqTilde, {m, 0, 1}] // Normal,
              Assumptions → {0 < a < b, 0 < m, 0 < r1, 0 < r2}];
           qEqTildeWeakMig = FullSimplify[Series[qEqTilde, {m, 0, 1}] // Normal,
              Assumptions → {0 < a < b, 0 < m, 0 < r1, 0 < r2}];
           DABEqTildeWeakMig = FullSimplify[Series[DABEqTilde, {m, 0, 1}] // Normal,
              Assumptions → {0 < a < b, 0 < m, 0 < r1, 0 < r2}];
           {pEqTildeWeakMig, qEqTildeWeakMig, DABEqTildeWeakMig} // TableForm
```

$$1-\frac{m\,(a+r1)}{a\,(a+b+r1)}$$
$$\frac{a\,b+(b-m)\,(b+r1)}{b\,(a+b+r1)}$$
$$\frac{m}{a+b+r1}$$

Checking these against first-order terms of eq. (4.1) in BA2011:

$$1-\frac{m}{a}\left(1-\frac{b}{a+b+r}\right)-\text{pEqTildeWeakMig /. \{r → r1\} // FullSimplify}$$

0

$$1-\frac{m}{b}\left(1-\frac{a}{a+b+r}\right)-\text{qEqTildeWeakMig /. \{r → r1\} // FullSimplify}$$

0

$$\frac{m}{a+b+r}-\text{DABEqTildeWeakMig /. \{r → r1\} // FullSimplify}$$

0

```
In[198]:= ruleApplyEqWeakMigTilde := {p → pEqTildeWeakMig, q → qEqTildeWeakMig,
             n → nC, DAB → DABEqTildeWeakMig, DAC → 0, DCB → 0, DACB → 0}
           ruleApplyEqWeakMigTilde
```

$$\left\{p\to 1-\frac{m\,(a+r1)}{a\,(a+b+r1)},\ q\to\frac{a\,b+(b-m)\,(b+r1)}{b\,(a+b+r1)},\ n\to nC,\ DAB\to\frac{m}{a+b+r1},\ DAC\to 0,\ DCB\to 0,\ DACB\to 0\right\}$$

```
JTildePrep // MatrixForm
```

a − m − 2 a p	0	b	0	0	0
0	b − m − 2 b q	a	0	0	0
−2 a DAB + m q	−2 b DAB + m p	a + b − m − 2 a p − 2 b q − r1	0	0	0
0	0	0	−m	a	b
0	0	0	m p	a − m − 2 a p − r1 − r2	0
0	0	0	m q	0	b − m − 2
0	0	0	m (DAB − p q)	−2 a DAB + m q	−2 b DA

Now we see the block structure claimed in eq. (4.27) of BA2011.

In[199]:=
```
JEqTilde = JTildePrep /. ruleApplyEqWeakMigTilde // FullSimplify;
JEqTilde // MatrixForm
```

Out[200]//MatrixForm=

$$\begin{pmatrix} -a+m-\frac{2\,b\,m}{a+b+r1} & 0 & b & 0 & \\ 0 & -b+m-\frac{2\,a\,m}{a+b+r1} & a & 0 & \\ \frac{m\,(-a\,b+(b-m)\,(b+r1))}{b\,(a+b+r1)} & \frac{m\,\left(a^2-a\,(b+m-r1)-m\,r1\right)}{a\,(a+b+r1)} & -a-b+3\,m-r1-\frac{2\,(a+b)\,m}{a+b+r1} & 0 & \\ 0 & 0 & 0 & -m & \\ 0 & 0 & 0 & m\left(1-\frac{m\,(a+r1)}{a\,(a+b+r1)}\right) & -a-m-r1+ \\ 0 & 0 & 0 & \frac{m\,(a\,b+(b-m)\,(b+r1))}{b\,(a+b+r1)} & \\ 0 & 0 & 0 & \frac{m\left(m-\frac{(a\,b+(b-m)\,(b+r1))\left(1-\frac{m\,(a+r1)}{a\,(a+b+r1)}\right)}{b}\right)}{a+b+r1} & \frac{m\,(-a\,b+(b}{b\,(a\text{-}} \end{pmatrix}$$

In[201]:=
```
JEqNTilde = JEqTilde[[{4, 5, 6, 7}, {4, 5, 6, 7}]];
JEqNTilde // MatrixForm
```

Out[202]//MatrixForm=

$$\begin{pmatrix} -m & a & b & 0 \\ m\left(1-\frac{m\,(a+r1)}{a\,(a+b+r1)}\right) & -a-m-r1+\frac{2\,m\,(a+r1)}{a+b+r1}-r2 & 0 & b \\ \frac{m\,(a\,b+(b-m)\,(b+r1))}{b\,(a+b+r1)} & 0 & -\frac{(b+r1)\,(b-m+r2)+a\,(b+m+r2)}{a+b+r1} & a \\ \frac{m\left(m-\frac{(a\,b+(b-m)\,(b+r1))\left(1-\frac{m\,(a+r1)}{a\,(a+b+r1)}\right)}{b}\right)}{a+b+r1} & \frac{m\,(-a\,b+(b-m)\,(b+r1))}{b\,(a+b+r1)} & \frac{m\,\left(a^2-a\,(b+m-r1)-m\,r1\right)}{a\,(a+b+r1)} & -\frac{a^2+b^2+r1\,(-3\,m+r1+r2)+b\,(-m+2}{a+b+r} \end{pmatrix}$$

```
JEqTildemSmall := Simplify[Normal[Series[JEqTilde, {m, 0, 1}]]]
JEqTildemSmall // MatrixForm
```

$$\begin{pmatrix} -a+m-\frac{2\,b\,m}{a+b+r1} & 0 & b & 0 & 0 & 0 \\ 0 & -b+m-\frac{2\,a\,m}{a+b+r1} & a & 0 & 0 & 0 \\ \frac{m\,(-a+b+r1)}{a+b+r1} & \frac{m\,(a-b+r1)}{a+b+r1} & -a-b-r1+\frac{m\,(a+b+3\,r1)}{a+b+r1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -m & a & b \\ 0 & 0 & 0 & m & -a-r1+\frac{m\,(a-b+r1)}{a+b+r1}-r2 & 0 \\ 0 & 0 & 0 & m & 0 & -b+\frac{m\,(-a+b+r1)}{a+b+r1}-r2 \\ 0 & 0 & 0 & -m & \frac{m\,(-a+b+r1)}{a+b+r1} & \frac{m\,(a-b+r1)}{a+b+r1} \end{pmatrix}$$

```
JEqNTildemSmall := JEqTildemSmall[[4 ;; 7, 4 ;; 7]]
JEqNTildemSmall // MatrixForm
```

$$\begin{pmatrix} -m & a & b & 0 \\ m & -a-r1+\frac{m\,(a-b+r1)}{a+b+r1}-r2 & 0 & b \\ m & 0 & -b+\frac{m\,(-a+b+r1)}{a+b+r1}-r2 & a \\ -m & \frac{m\,(-a+b+r1)}{a+b+r1} & \frac{m\,(a-b+r1)}{a+b+r1} & -a-b-r1+\frac{m\,(a+b+3\,r1)}{a+b+r1}-r2 \end{pmatrix}$$

```
JEqNTildemSmall - JEqNGenericTildemSmall
```

```
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

- Using the Ansatz $m_e = -\lambda_N = m\,z$

```
JEqTilde
```

$$\left\{\left\{-a+m-\frac{2\,b\,m}{a+b+r1},\,0,\,b,\,0,\,0,\,0,\,0\right\},\,\left\{0,\,-b+m-\frac{2\,a\,m}{a+b+r1},\,a,\,0,\,0,\,0,\,0\right\},\right.$$
$$\left\{\frac{m\,(-a\,b+(b-m)\,(b+r1))}{b\,(a+b+r1)},\,\frac{m\left(a^2-a\,(b+m-r1)-m\,r1\right)}{a\,(a+b+r1)},\right.$$
$$\left.-a-b+3\,m-r1-\frac{2\,(a+b)\,m}{a+b+r1},\,0,\,0,\,0,\,0\right\},\,\{0,\,0,\,0,\,-m,\,a,\,b,\,0\},$$
$$\left\{0,\,0,\,0,\,m\left(1-\frac{m\,(a+r1)}{a\,(a+b+r1)}\right),\,-a-m-r1+\frac{2\,m\,(a+r1)}{a+b+r1}-r2,\,0,\,b\right\},$$
$$\left\{0,\,0,\,0,\,\frac{m\,(a\,b+(b-m)\,(b+r1))}{b\,(a+b+r1)},\,0,\,-\frac{(b+r1)\,(b-m+r2)+a\,(b+m+r2)}{a+b+r1},\,a\right\},\,\left\{0,\,0,\,0,\right.$$
$$\frac{m\left(m-\frac{(a\,b+(b-m)\,(b+r1))\left(1-\frac{m\,(a+r1)}{a\,(a+b+r1)}\right)}{b}\right)}{a+b+r1},\,\frac{m\,(-a\,b+(b-m)\,(b+r1))}{b\,(a+b+r1)},\,\frac{m\left(a^2-a\,(b+m-r1)-m\,r1\right)}{a\,(a+b+r1)},$$
$$\left.\left.-\frac{1}{a+b+r1}\left(a^2+b^2+r1\,(-3\,m+r1+r2)+b\,(-m+2\,r1+r2)+a\,(2\,b-m+2\,r1+r2)\right)\right\}\right\}$$

Applying the same Ansatz as above to the Jacobian obtained from the differential equations yields the following ratio between the effective and actual migration rate, $\frac{m_e}{m}$:

```
ruleApplyEqWeakMigTilde
```

$$\left\{p\to 1-\frac{m\,(a+r1)}{a\,(a+b+r1)},\,q\to\frac{a\,b+(b-m)\,(b+r1)}{b\,(a+b+r1)},\right.$$
$$\left.n\to nC,\,DAB\to\frac{m}{a+b+r1},\,DAC\to 0,\,DCB\to 0,\,DACB\to 0\right\}$$

```
Simplify[
  Series[Det[(JEqTilde /. ruleApplyEqWeakMigTilde) - x IdentityMatrix[7] /. x → -m z],
   {m, 0, 1}], Assumptions → {a ≥ 0, b ≥ 0, r1 ≥ 0, r2 ≥ 0}] // Normal
```

$$a\,b\,m\,(a+b+r1)\,(a+r1+r2)$$
$$\left(b^2\,z+r2\,(r1\,(-1+z)+r2\,(-1+z)+a\,z)+b\,((a+r1)\,z+r2\,(-1+2\,z))\right)$$

```
Simplify[Solve[% == 0, z]]
```

$$\left\{\left\{z\to\frac{r2\,(b+r1+r2)}{(b+r2)\,(a+b+r1+r2)}\right\}\right\}$$

Or, alternatively and more directly:

```
JEqNTildemSmall // MatrixForm
```

$$\begin{pmatrix} -m & a & b & 0 \\ m & -a-r1+\frac{m\,(a-b+r1)}{a+b+r1}-r2 & 0 & b \\ m & 0 & -b+\frac{m\,(-a+b+r1)}{a+b+r1}-r2 & a \\ -m & \frac{m\,(-a+b+r1)}{a+b+r1} & \frac{m\,(a-b+r1)}{a+b+r1} & -a-b-r1+\frac{m\,(a+b+3\,r1)}{a+b+r1}-r2 \end{pmatrix}$$

- Assuming tight linkage (weak recombination)

The coordinates of the internal equilibrium under the assumption of tight linkage, *i.e.* $r \ll \min(a,\,m)$, up to and including first-order terms of r, can be obtained from eq. (4.2) in BA2011.

```
{pEqTilde, qEqTilde, DABEqTilde} // MatrixForm
```

$$\begin{pmatrix} \frac{-a^2+b^2+6\,a\,r1-4\,m\,r1-r1^2+(a-b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}}{8\,a\,r1} \\ \frac{a^2-b^2+6\,b\,r1-4\,m\,r1-r1^2+(-a+b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}}{8\,b\,r1} \\ \frac{-8\,m^2\,r1^2-4\,m\,r1\left(a^2-2\,a\,b+b^2-2\,a\,r1-2\,b\,r1+r1^2\right)+(a-b-r1)\,(a+b-r1)\,(a-b+r1)\left(a+b+r1-\sqrt{-8\,m\,r1+(a+b+r1)^2}\right)}{32\,a\,b\,r1^2} \end{pmatrix}$$

```
In[203]:= pEqTildeWeakRec = FullSimplify[Series[pEqTilde, {r1, 0, 1}] // Normal,
              Assumptions → {0 < a < b, 0 < m, 0 < r1, 0 < r2}] // FullSimplify;
          qEqTildeWeakRec = FullSimplify[Series[qEqTilde, {r1, 0, 1}] // Normal,
              Assumptions → {0 < a < b, 0 < m, 0 < r1, 0 < r2}] // FullSimplify;
          DABEqTildeWeakRec = FullSimplify[Series[DABEqTilde, {r1, 0, 1}] // Normal,
              Assumptions → {0 < a < b, 0 < m, 0 < r1, 0 < r2}] // FullSimplify;
```

```
{pEqTildeWeakRec, qEqTildeWeakRec, DABEqTildeWeakRec} // TableForm
```

$$\frac{a\,(a+b)^2\,(a+b-m)-m\,(b\,(b-m)+a\,(b+m))\,r1}{a\,(a+b)^3}$$

$$1-\frac{m}{a+b}-\frac{m\left(a^2+a\,(b-m)+b\,m\right)r1}{b\,(a+b)^3}$$

$$\frac{m\left((a+b)^2\,(a+b-m)-\frac{\left(a^2+a\,(b-m)+b\,m\right)\,(b\,(b-m)+a\,(b+m))\,r1}{a\,b}\right)}{(a+b)^4}$$

Checking these against first-order terms of eq. (4.1) in BA2011:

$$1-\frac{m}{a+b}-\frac{r\,m}{(a+b)^2}\left(\frac{b}{a}-\frac{m}{a+b}\left(\frac{b}{a}-1\right)\right)-\text{pEqTildeWeakRec /. \{r → r1\} // FullSimplify}$$

0

$$1-\frac{m}{a+b}-\frac{r\,m}{(a+b)^2}\left(\frac{a}{b}+\frac{m}{a+b}\left(1-\frac{a}{b}\right)\right)-\text{qEqTildeWeakRec /. \{r → r1\} // FullSimplify}$$

0

$$\frac{m}{a+b}\left(1-\frac{m}{a+b}\right)-\frac{r\,m}{(a+b)^2}\left(1-\frac{m}{a+b}\left(1-\frac{m}{a+b}\right)\left(2-\frac{b}{a}-\frac{a}{b}\right)\right)-\text{DABEqTildeWeakRec /. \{r → r1\} // FullSimplify}$$

0

```
In[206]:= ruleApplyEqWeakRecTilde := {p → pEqTildeWeakRec, q → qEqTildeWeakRec,
            n → nC, DAB → DABEqTildeWeakRec, DAC → 0, DCB → 0, DACB → 0}
          ruleApplyEqWeakRecTilde
```

$$\left\{p\to\frac{a\,(a+b)^2\,(a+b-m)-m\,(b\,(b-m)+a\,(b+m))\,r1}{a\,(a+b)^3},\ q\to 1-\frac{m}{a+b}-\frac{m\left(a^2+a\,(b-m)+b\,m\right)r1}{b\,(a+b)^3},\right.$$
$$\left.n\to nC,\ DAB\to\frac{m\left((a+b)^2\,(a+b-m)-\frac{\left(a^2+a\,(b-m)+b\,m\right)\,(b\,(b-m)+a\,(b+m))\,r1}{a\,b}\right)}{(a+b)^4},\ DAC\to 0,\ DCB\to 0,\ DACB\to 0\right\}$$

```
JTildePrep // MatrixForm
```

$$\begin{pmatrix} a-m-2\,a\,p & 0 & b & 0 & 0 & 0 \\ 0 & b-m-2\,b\,q & a & 0 & 0 & 0 \\ -2\,a\,DAB+m\,q & -2\,b\,DAB+m\,p & a+b-m-2\,a\,p-2\,b\,q-r1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -m & a & b \\ 0 & 0 & 0 & m\,p & a-m-2\,a\,p-r1-r2 & 0 \\ 0 & 0 & 0 & m\,q & 0 & b-m-2 \\ 0 & 0 & 0 & m\,(DAB-p\,q) & -2\,a\,DAB+m\,q & -2\,b\,DA \end{pmatrix}$$

Now we see the block structure claimed in eq. (4.27) of BA2011.

In[207]:=
```
JEqWeakRecTilde = JTildePrep /. ruleApplyEqWeakRecTilde // FullSimplify;
JEqWeakRecTilde // MatrixForm
```

Out[208]//MatrixForm=

$$\left(\begin{array}{cc}
-m+\frac{-a (a+b)^2 (a+b-2 m)+2 m (b (b-m)+a (b+m)) r1}{(a+b)^3} & 0 \\
0 & \frac{-(a+b)^2 (b (b-m)+a (b+m))+2 m \left(a^2+a (b-m)+b m\right) r1}{(a+b)^3} \\
\frac{-(a-b) b (a+b)^2 (a+b-m) m+m (2 b (a+b)+(a-3 b) m) \left(a^2+a (b-m)+b m\right) r1}{b (a+b)^4} & \frac{a (a-b) (a+b)^2 (a+b-m) m+m \left(2 a^2+2 a b-3 a m+b m\right) (b (b-m)+a (b+ \cdots}{a (a+b)^4} \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{array}\right.$$

In[209]:=
```
JEqNWeakRecTilde = JEqWeakRecTilde[[{4, 5, 6, 7}, {4, 5, 6, 7}]];
JEqNWeakRecTilde // MatrixForm
```

Out[210]//MatrixForm=

$$\left(\begin{array}{cc}
-m & \\
\frac{m \left(a (a+b)^2 (a+b-m)-m (b (b-m)+a (b+m)) r1\right)}{a (a+b)^3} & -m-r1+\frac{-\cdots}{\cdots} \\
m \left(1-\frac{m}{a+b}-\frac{m \left(a^2+a (b-m)+b m\right) r1}{b (a+b)^3}\right) & \\
\frac{m \left(-a b (a+b)^4 (a+b-2 m) (a+b-m)+(a+b)^2 (a+b-2 m) m \left(a^3+b^3-(a-b)^2 m\right) r1-m^2 \left(a^2+a (b-m)+b m\right) (b (b-m)+a (b+m)) r1^2\right)}{a b (a+b)^6} & \frac{-(a-b) b (a+b)^2 (\cdots}{\cdots}
\end{array}\right.$$

- Using the Ansatz $m_e = -\lambda_N = m\,z$

```
JEqWeakRecTilde
```

$$\Big\{\Big\{-m+\frac{-a\,(a+b)^2\,(a+b-2\,m)+2\,m\,(b\,(b-m)+a\,(b+m))\,r1}{(a+b)^3},\,0,\,b,\,0,\,0,\,0,\,0\Big\},$$

$$\Big\{0,\,\frac{-(a+b)^2\,(b\,(b-m)+a\,(b+m))+2\,m\,\left(a^2+a\,(b-m)+b\,m\right)\,r1}{(a+b)^3},\,a,\,0,\,0,\,0,\,0\Big\},$$

$$\Big\{\frac{1}{b\,(a+b)^4}\left(-(a-b)\,b\,(a+b)^2\,(a+b-m)\,m+m\,(2\,b\,(a+b)+(a-3\,b)\,m)\,\left(a^2+a\,(b-m)+b\,m\right)\,r1\right),$$

$$\frac{1}{a\,(a+b)^4}\left(a\,(a-b)\,(a+b)^2\,(a+b-m)\,m+m\left(2\,a^2+2\,a\,b-3\,a\,m+b\,m\right)\,(b\,(b-m)+a\,(b+m))\,r1\right),$$

$$-a-b+m-\frac{(a+b-2\,m)\,r1}{a+b},\,0,\,0,\,0,\,0\Big\},\,\{0,\,0,\,0,\,-m,\,a,\,b,\,0\},$$

$$\Big\{0,\,0,\,0,\,\frac{m\left(a\,(a+b)^2\,(a+b-m)-m\,(b\,(b-m)+a\,(b+m))\,r1\right)}{a\,(a+b)^3},$$

$$-m-r1+\frac{-a\,(a+b)^2\,(a+b-2\,m)+2\,m\,(b\,(b-m)+a\,(b+m))\,r1}{(a+b)^3}-r2,\,0,\,b\Big\},$$

$$\Big\{0,\,0,\,0,\,m\left(1-\frac{m}{a+b}-\frac{m\left(a^2+a\,(b-m)+b\,m\right)\,r1}{b\,(a+b)^3}\right),\,0,$$

$$-m+\frac{-b\,(a+b)^2\,(a+b-2\,m)+2\,m\left(a^2+a\,(b-m)+b\,m\right)\,r1}{(a+b)^3}-r2,\,a\Big\},$$

$$\Big\{0,\,0,\,0,\,\frac{1}{a\,b\,(a+b)^6}\,m\,\Big(-a\,b\,(a+b)^4\,(a+b-2\,m)\,(a+b-m)+(a+b)^2\,(a+b-2\,m)\,m$$

$$\left(a^3+b^3-(a-b)^2\,m\right)\,r1-m^2\left(a^2+a\,(b-m)+b\,m\right)\,(b\,(b-m)+a\,(b+m))\,r1^2\Big),\,\frac{1}{b\,(a+b)^4}$$

$$\left(-(a-b)\,b\,(a+b)^2\,(a+b-m)\,m+m\,(2\,b\,(a+b)+(a-3\,b)\,m)\,\left(a^2+a\,(b-m)+b\,m\right)\,r1\right),$$

$$\frac{1}{a\,(a+b)^4}\left(a\,(a-b)\,(a+b)^2\,(a+b-m)\,m+m\left(2\,a^2+2\,a\,b-3\,a\,m+b\,m\right)\,(b\,(b-m)+a\,(b+m))\,r1\right),$$

$$-\frac{a^2+b^2-2\,m\,r1+b\,(-m+r1+r2)+a\,(2\,b-m+r1+r2)}{a+b}\Big\}\Big\}$$

Applying the same Ansatz as above to the Jacobian obtained from the differential equations yields the following ratio between the effective and actual migration rate, $\frac{m_e}{m}$:

```
ruleApplyEqWeakRecTilde
```

$$\Big\{p\to\frac{a\,(a+b)^2\,(a+b-m)-m\,(b\,(b-m)+a\,(b+m))\,r1}{a\,(a+b)^3},\,q\to 1-\frac{m}{a+b}-\frac{m\left(a^2+a\,(b-m)+b\,m\right)\,r1}{b\,(a+b)^3},$$

$$n\to nC,\,DAB\to\frac{m\left((a+b)^2\,(a+b-m)-\frac{\left(a^2+a\,(b-m)+b\,m\right)\,(b\,(b-m)+a\,(b+m))\,r1}{a\,b}\right)}{(a+b)^4},\,DAC\to 0,\,DCB\to 0,\,DACB\to 0\Big\}$$

```
Simplify[Series[Det[
    (JEqWeakRecTilde /. ruleApplyEqWeakRecTilde) - x IdentityMatrix[7] /. x -> -m z],
   {m, 0, 1}], Assumptions → {a ≥ 0, b ≥ 0, r1 ≥ 0, r2 ≥ 0}] // Normal
```

$$a\,b\,m\,(a+b+r1)\,(a+r1+r2)$$
$$\left(b^2\,z+r2\,(r1\,(-1+z)+r2\,(-1+z)+a\,z)+b\,((a+r1)\,z+r2\,(-1+2\,z))\right)$$

```
Simplify[Solve[% == 0, z]]
```

$$\Big\{\Big\{z\to\frac{r2\,(b+r1+r2)}{(b+r2)\,(a+b+r1+r2)}\Big\}\Big\}$$

Hence, we see that the effective migration rate does not only apply for weak migration, but also for weak recombination.

- **Graphical exploration of approximation**
- Generic

We compare the approximate effective migration rate to the exact (negative) eigenvalue of J_N computed numerically.

In[211]:=
```
approxEffMigRateABCFunc[a_, b_, m_, r1_, r2_] := m (r2 (b + r1 + r2))/((b + r2) (a + b + r1 + r2))
```

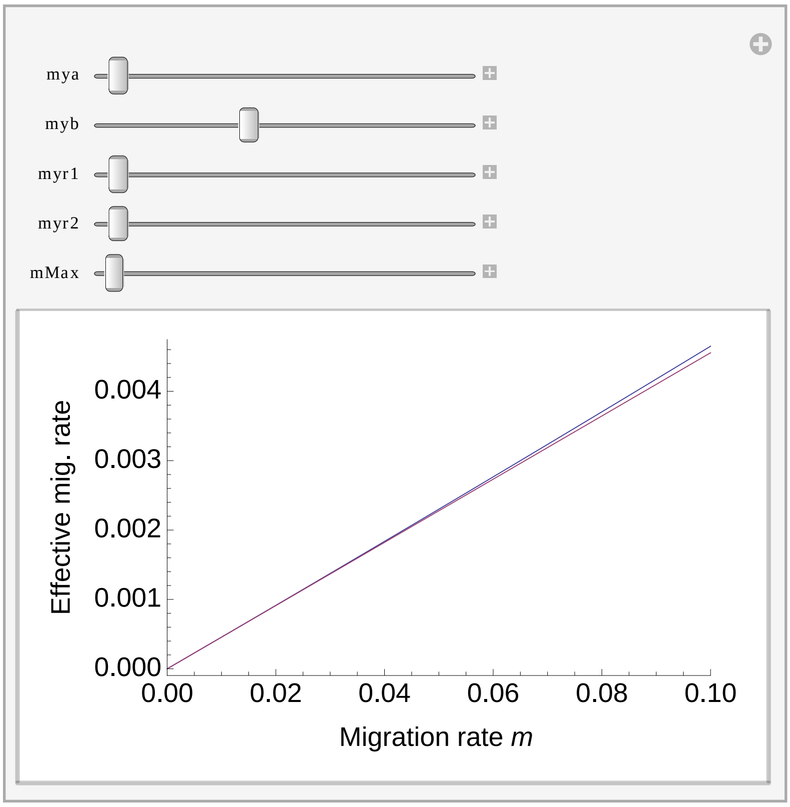
In[212]:=
```
exactEffMigRateABCFunc[a_,b_,m_,r1_,r2_]:=Module[{JN},JN={{-m,a,b,0},{(m (-a^2+b^2+6 a r1
Return[-Max[Re[Eigenvalues[JN]]]]
]
```

```
mya = 0.002;
myb = 0.4;
mym = 0.0024;
myr1 = 0.01 * (10);
myr2 = 0.01 * (30);
{exactEffMigRateABCFunc[mya, myb, mym, myr1, myr2],
 approxEffMigRateABCFunc[mya, myb, mym, myr1, myr2],
 approxEffMigRateABCFunc[mya, myb, mym, myr1, myr2] /
   exactEffMigRateABCFunc[mya, myb, mym, myr1, myr2] - 1}
```

```
{0.00102813, 0.00102601, -0.00206156}
```

The exact (blue) and approximate (red) effective migration rate:

```
Manipulate[Plot[{exactEffMigRateABCFunc[mya, myb, m, myr1, myr2],
    approxEffMigRateABCFunc[mya, myb, m, myr1, myr2]},
   {m, 0, mMax}, PlotRange → {{0, mMax}, Automatic}, Frame → True,
   FrameStyle → Table[{Black, Opacity[0]}, {i, 1, 2}],
   FrameLabel → {"Migration rate m", "Effective mig. rate"},
   LabelStyle → {Directive[FontSize → 14], FontFamily → "Helvetica"}],
  {{mya, 0.02}, 0, 1}, {{myb, 0.4}, 0, 1}, {{myr1, 0.02}, 0, 1},
  {{myr2, 0.02}, 0, 1}, {{mMax, 0.1}, 0, 10}]
```



Remark

The only difference between configurations $\mathcal{A}-\mathcal{C}-\mathcal{B}$ and $\mathcal{A}-\mathcal{B}-\mathcal{C}$ seems to be in the fourth element of the fourth row of the respective Jacobian matrices J_N. We double check this in the following:

```
JMTestACB :=
 {{-m, a, b, 0}, {m, -a - rAC + (m (a - b + rAB))/(a + b + rAB), 0, b}, {m, 0, -b - rCB + (m (b - a + rAB))/(a + b + rAB), a},
  {-m, (m (b - a + rAB))/(a + b + rAB), (m (a - b + rAB))/(a + b + rAB), -a - b - rAB + (m (a + b + 3 rAB))/(a + b + rAB)}}
JMTestABC := {{-m, a, b, 0}, {m, -a - rAC + (m (a - b + rAB))/(a + b + rAB), 0, b},
  {m, 0, -b - rCB + (m (b - a + rAB))/(a + b + rAB), a},
  {-m, (m (b - a + rAB))/(a + b + rAB), (m (a - b + rAB))/(a + b + rAB), -a - b - rAC + (m (a + b + 3 rAB))/(a + b + rAB)}}
FullSimplify[
  Normal[Series[Det[JMTestACB - x IdentityMatrix[4]] /. x → -m (rAC rCB)/((a + rAC) (b + rCB)),
    {m, 0, 1}]]] /. {rAB → rAC + rCB}
```

```
0
```

```
FullSimplify[
  Normal[Series[Det[JMTestABC - x IdentityMatrix[4]] /. x → -m rCB (b + rAC) / ((b + rCB) (a + b + rAC)),
    {m, 0, 1}]]] /. {rAC → rAB + rCB}
0
```

This confirms that the only difference is in the fourth element of the fourth row of the two matrices.

Order of loci: $\mathcal{C}$-$\mathcal{A}$-$\mathcal{B}$

For reasons of symmetry, we directly obtain

$$m_e = m \frac{r_1 (a + r_1 + r_2)}{(a + r_1) (a + b + r_1 + r_2)}. \tag{1}$$

- **Graphical exploration of approximation**
- Generic

We compare the approximate effective migration rate to the exact (negative) eigenvalue of J_N computed numerically.

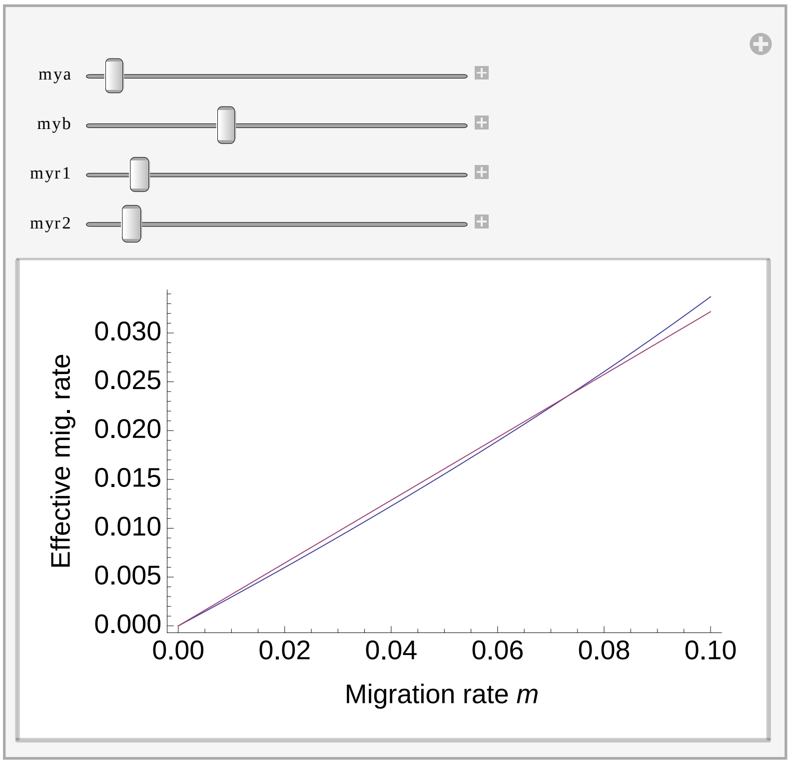
```
In[213]:= approxEffMigRateCABFunc[a_, b_, m_, r1_, r2_] := m r1 (a + r1 + r1) / ((a + r1) (a + b + r1 + r2))
```

```
In[214]:= exactEffMigRateCABFunc[a_,b_,m_,r1_,r2_]:=Module[{JN},JN={{-m,b,a,0},{m (a^2-b^2+6 b r2-
  Return[-Max[Re[Eigenvalues[JN]]]]
  ]
```

```
mya = 0.002;
myb = 0.4;
mym = 0.0024;
myr1 = 0.01 * (10);
myr2 = 0.01 * (30);
{exactEffMigRateCABFunc[mya, myb, mym, myr1, myr2],
 approxEffMigRateCABFunc[mya, myb, mym, myr1, myr2],
 approxEffMigRateCABFunc[mya, myb, mym, myr1, myr2] /
   exactEffMigRateCABFunc[mya, myb, mym, myr1, myr2] - 1}
{0.000969913, 0.000592636, -0.38898}
```

The exact (blue) and approximate (red) effective migration rate:

```
Manipulate[Plot[{exactEffMigRateCABFunc[mya, myb, m, myr1, myr2],
   approxEffMigRateCABFunc[mya, myb, m, myr1, myr2]}, {m, 0, .1},
  Frame → True, FrameStyle → Table[{Black, Opacity[0]}, {i, 1, 2}],
  FrameLabel → {"Migration rate m", "Effective mig. rate"},
  LabelStyle → {Directive[FontSize → 14], FontFamily → "Helvetica"}],
 {{mya, 0.02}, 0, 1}, {{myb, 0.4}, 0, 1}, {{myr1, 0.02}, 0, 1}, {{myr2, 0.02}, 0, 1}]
```



$$\Big\{\Big\{a-m-\frac{1}{4\,r1}\Big(-a^2+b^2+6\,a\,r1-4\,m\,r1-r1^2+(a-b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big),0,b,0,0,0,0\Big\},\Big\{0,b-m-\frac{1}{4\,r1}\Big(a^2-b^2+6\,b\,r1-4\,m\,r1-r1^2+(-a+b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big),a,0,0,0,0\Big\},$$

$$\Big\{\frac{1}{16\,b\,r1^2}\Big(8\,m^2\,r1^2+4\,m\,r1\,\big(a^2+(b-r1)^2-2\,a\,(b+r1)\big)-(a-b-r1)\,(a+b-r1)\,(a-b+r1)\Big(a+b+r1-\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big)+2\,m\,r1\Big(a^2-b^2+6\,b\,r1-4\,m\,r1-r1^2+(-a+b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big)\Big),$$

$$\frac{1}{16\,a\,r1^2}\Big(8\,m^2\,r1^2+4\,m\,r1\,\big(a^2+(b-r1)^2-2\,a\,(b+r1)\big)-(a-b-r1)\,(a+b-r1)\,(a-b+r1)\Big(a+b+r1-\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big)+2\,m\,r1\Big(-a^2+b^2+6\,a\,r1-4\,m\,r1-r1^2+(a-b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big)\Big),$$

$$\frac{1}{2}\Big(-a-b+2\,m-r1-\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big),0,0,0,0\Big\},\{0,0,0,-m,a,b,0\},$$

$$\Big\{0,0,0,\frac{1}{8\,a\,r1}m\Big(-a^2+b^2+6\,a\,r1-4\,m\,r1-r1^2+(a-b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big),a-m-r1-\frac{1}{4\,r1}\Big(-a^2+b^2+6\,a\,r1-4\,m\,r1-r1^2+(a-b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big)-r2,0,b\Big\},$$

$$\Big\{0,0,0,\frac{1}{8\,b\,r1}m\Big(a^2-b^2+6\,b\,r1-4\,m\,r1-r1^2+(-a+b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big),0,b-m-\frac{1}{4\,r1}\Big(a^2-b^2+6\,b\,r1-4\,m\,r1-r1^2+(-a+b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big)-r2,a\Big\},$$

$$\Big\{0,0,0,\frac{1}{64\,a\,b\,r1^2}\,m\Big(-\Big(-a^2+b^2+6\,a\,r1-4\,m\,r1-r1^2+(a-b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big)\Big(a^2-b^2+6\,b\,r1-4\,m\,r1-r1^2+(-a+b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big)+2\Big(-8\,m^2\,r1^2-4\,m\,r1\,\big(a^2+(b-r1)^2-2\,a\,(b+r1)\big)+(a-b-r1)\,(a+b-r1)\,(a-b+r1)\Big(a+b+r1-\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big)\Big)\Big),$$

$$\frac{1}{16\,b\,r1^2}\Big(8\,m^2\,r1^2+4\,m\,r1\,\big(a^2+(b-r1)^2-2\,a\,(b+r1)\big)-(a-b-r1)\,(a+b-r1)\,(a-b+r1)\Big(a+b+r1-\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big)+2\,m\,r1\Big(a^2-b^2+6\,b\,r1-4\,m\,r1-r1^2+(-a+b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big)\Big),$$

$$\frac{1}{16\,a\,r1^2}\Big(8\,m^2\,r1^2+4\,m\,r1\,\big(a^2+(b-r1)^2-2\,a\,(b+r1)\big)-(a-b-r1)\,(a+b-r1)\,(a-b+r1)\Big(a+b+r1-\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big)+2\,m\,r1\Big(-a^2+b^2+6\,a\,r1-4\,m\,r1-r1^2+(a-b+r1)\sqrt{-8\,m\,r1+(a+b+r1)^2}\Big)\Big),$$

$$\frac{1}{2}\Big(-a-b+2\,m-r1-\sqrt{-8\,m\,r1+(a+b+r1)^2}-2\,r2\Big)\Big\}\Big\}\ /.\ \{r1\to r2,\ r2\to r1,\ a\to b,\ b\to a\}$$

$$\Big\{\Big\{b-m-\frac{1}{4\,r2}\left(a^2-b^2+6\,b\,r2-4\,m\,r2-r2^2+(-a+b+r2)\sqrt{-8\,m\,r2+(a+b+r2)^2}\right),0,a,0,0,0,0\Big\},\Big\{0,a-m-\frac{1}{4\,r2}\left(-a^2+b^2+6\,a\,r2-4\,m\,r2-r2^2+(a-b+r2)\sqrt{-8\,m\,r2+(a+b+r2)^2}\right),b,0,0,0,0\Big\},$$

$$\Big\{\frac{1}{16\,a\,r2^2}\Big(8\,m^2\,r2^2+4\,m\,r2\left(b^2+(a-r2)^2-2\,b\,(a+r2)\right)-(-a+b-r2)\,(a+b-r2)\,(-a+b+r2)\left(a+b+r2-\sqrt{-8\,m\,r2+(a+b+r2)^2}\right)+2\,m\,r2\left(-a^2+b^2+6\,a\,r2-4\,m\,r2-r2^2+(a-b+r2)\sqrt{-8\,m\,r2+(a+b+r2)^2}\right)\Big),$$

$$\frac{1}{16\,b\,r2^2}\Big(8\,m^2\,r2^2+4\,m\,r2\left(b^2+(a-r2)^2-2\,b\,(a+r2)\right)-(-a+b-r2)\,(a+b-r2)\,(-a+b+r2)\left(a+b+r2-\sqrt{-8\,m\,r2+(a+b+r2)^2}\right)+2\,m\,r2\left(a^2-b^2+6\,b\,r2-4\,m\,r2-r2^2+(-a+b+r2)\sqrt{-8\,m\,r2+(a+b+r2)^2}\right)\Big),$$

$$\frac{1}{2}\left(-a-b+2\,m-r2-\sqrt{-8\,m\,r2+(a+b+r2)^2}\right),0,0,0,0\Big\},\{0,0,0,-m,b,a,0\},$$

$$\Big\{0,0,0,\frac{1}{8\,b\,r2}m\left(a^2-b^2+6\,b\,r2-4\,m\,r2-r2^2+(-a+b+r2)\sqrt{-8\,m\,r2+(a+b+r2)^2}\right),b-m-r1-r2-\frac{1}{4\,r2}\left(a^2-b^2+6\,b\,r2-4\,m\,r2-r2^2+(-a+b+r2)\sqrt{-8\,m\,r2+(a+b+r2)^2}\right),0,a\Big\},$$

$$\Big\{0,0,0,\frac{1}{8\,a\,r2}m\left(-a^2+b^2+6\,a\,r2-4\,m\,r2-r2^2+(a-b+r2)\sqrt{-8\,m\,r2+(a+b+r2)^2}\right),0,a-m-r1-\frac{1}{4\,r2}\left(-a^2+b^2+6\,a\,r2-4\,m\,r2-r2^2+(a-b+r2)\sqrt{-8\,m\,r2+(a+b+r2)^2}\right),b\Big\},$$

$$\Big\{0,0,0,\frac{1}{64\,a\,b\,r2^2}m\Big(\left(-a^2+b^2+6\,a\,r2-4\,m\,r2-r2^2+(a-b+r2)\sqrt{-8\,m\,r2+(a+b+r2)^2}\right)\left(-a^2+b^2-6\,b\,r2+4\,m\,r2+r2^2-(-a+b+r2)\sqrt{-8\,m\,r2+(a+b+r2)^2}\right)+2\Big(-8\,m^2\,r2^2-4\,m\,r2\left(b^2+(a-r2)^2-2\,b\,(a+r2)\right)+(-a+b-r2)\,(a+b-r2)\,(-a+b+r2)\left(a+b+r2-\sqrt{-8\,m\,r2+(a+b+r2)^2}\right)\Big)\Big),$$

$$\frac{1}{16\,a\,r2^2}\Big(8\,m^2\,r2^2+4\,m\,r2\left(b^2+(a-r2)^2-2\,b\,(a+r2)\right)-(-a+b-r2)\,(a+b-r2)\,(-a+b+r2)\left(a+b+r2-\sqrt{-8\,m\,r2+(a+b+r2)^2}\right)+2\,m\,r2\left(-a^2+b^2+6\,a\,r2-4\,m\,r2-r2^2+(a-b+r2)\sqrt{-8\,m\,r2+(a+b+r2)^2}\right)\Big),$$

$$\frac{1}{16\,b\,r2^2}\Big(8\,m^2\,r2^2+4\,m\,r2\left(b^2+(a-r2)^2-2\,b\,(a+r2)\right)-(-a+b-r2)\,(a+b-r2)\,(-a+b+r2)\left(a+b+r2-\sqrt{-8\,m\,r2+(a+b+r2)^2}\right)+2\,m\,r2\left(a^2-b^2+6\,b\,r2-4\,m\,r2-r2^2+(-a+b+r2)\sqrt{-8\,m\,r2+(a+b+r2)^2}\right)\Big),$$

$$\frac{1}{2}\left(-a-b+2\,m-2\,r1-r2-\sqrt{-8\,m\,r2+(a+b+r2)^2}\right)\Big\}\Big\}$$