Conceptual dynamical models for turbulence
Supplementary material

Andrew J. Majda and Yoonsang Lee

Department of Mathematics and Center for Atmosphere and Ocean Science, Courant Institute of Mathematical Sciences, New York University, New York, NY 10012

Table S1-S3 show equilibrium statistics of each mode from the negative large scale damping six dimensional conceptual models, $K = 5$, for various strengths of $\overline{F}$. In Table S1, the mean, variance, and standard deviation of $\overline{u}$ and effective damping of each small scale mode, $d_k + \gamma \langle \overline{u} \rangle$ are shown while Table S2 and S3 show variances and correlation times of the turbulent signal u, $\overline{u}$, and $u'_k, k = 1, 2, ..., 5$ respectively.

Table S1: Negative large scale damping with $K = 5$: mean, variance, and standard deviation of $\overline{u}$ and effective damping of small scale modes for various strengths of $\overline{F}$

$\overline{F}$	$\langle \overline{u} \rangle$	$\mathrm{Var}(\overline{u})$	$\mathrm{Std}(\overline{u})$	$d_1 + \gamma \langle \overline{u} \rangle$	$d_2 + \gamma \langle \overline{u} \rangle$	$d_3 + \gamma \langle \overline{u} \rangle$	$d_4 + \gamma \langle \overline{u} \rangle$	$d_5 + \gamma \langle \overline{u} \rangle$
0	1.4178	0.0000	0.0005	3.1467	3.2067	3.3067	3.4467	3.6267
-0.010	1.3654	0.0000	0.0005	3.0681	3.1281	3.2281	3.3681	3.5481
-0.054	0.8727	0.0000	0.0026	2.3290	2.3890	2.4890	2.6290	2.8090
-0.055	-0.6733	0.0397	0.1993	0.0025	0.0625	0.1625	0.3025	0.4825
-0.060	-0.6792	0.0416	0.2039	0.0013	0.0613	0.1613	0.3013	0.4813
-0.080	-0.6808	0.0511	0.2261	-0.0013	0.0587	0.1587	0.2987	0.4787
-0.100	-0.6814	0.0601	0.2452	-0.0021	0.0579	0.1579	0.2979	0.4779

Table S2: Negative large scale damping with $K = 5$: variances of the turbulent signal u, $\overline{u}$, and $u'_k, k = 1, 2, ..., 5$ for various strengths of $\overline{F}$

$\overline{F}$	$\mathrm{Var}(u)$	$\mathrm{Var}(\overline{u})$	$\mathrm{Var}(u'_1)$	$\mathrm{Var}(u'_2)$	$\mathrm{Var}(u'_3)$	$\mathrm{Var}(u'_4)$	$\mathrm{Var}(u'_5)$
0	0.0005	0.0000	0.0002	0.0001	0.0001	0.0001	0.0000
-0.010	0.0005	0.0000	0.0002	0.0001	0.0001	0.0001	0.0000
-0.054	0.0006	0.0000	0.0003	0.0001	0.0001	0.0001	0.0001
-0.055	0.1069	0.0397	0.0446	0.0174	0.0049	0.0014	0.0005
-0.060	0.1128	0.0416	0.0468	0.0181	0.0051	0.0014	0.0005
-0.080	0.1353	0.0511	0.0557	0.0209	0.0058	0.0016	0.0005
-0.100	0.1571	0.0601	0.0660	0.0227	0.0063	0.0018	0.0006

Table S3: Negative large scale damping with $K = 5$: correlation times of the turbulent signal u, $\overline{u}$, and $u'_k, k = 1, 2, ..., 5$ for various strengths of $\overline{F}$

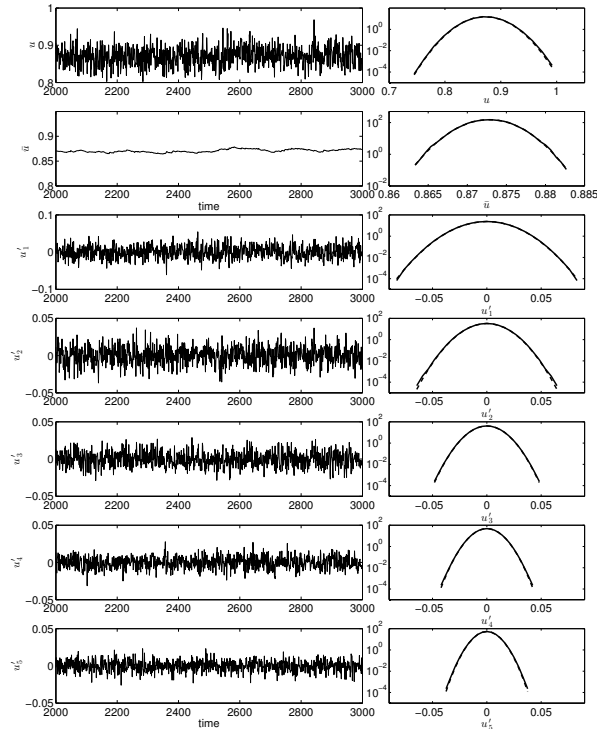| $\overline{F}$	$T_{cor}(u)$	$T_{cor}(\overline{u})$	$T_{cor}(u'_1)$	$T_{cor}(u'_2)$	$T_{cor}(u'_3)$	$T_{cor}(u'_4)$	$T_{cor}(u'_5)$
0	0.3030	5.0494	0.3174	0.3083	0.2937	0.2831	0.2734
-0.010	0.3102	5.6554	0.3318	0.3234	0.2917	0.2885	0.2799
-0.054	0.3895	36.07	0.4184	0.4073	0.3813	0.3717	0.3617
-0.055	24.95	34.18	29.41	15.53	6.373	4.122	2.833
-0.060	24.52	35.13	30.28	15.61	6.253	4.156	2.850
-0.080	20.32	36.13	33.18	17.88	4.817	4.037	2.987
-0.100	31.30	36.98	31.37	17.21	5.017	4.211	3.121

Fig. S1: Negative large scale damping with $K = 5$: time series (left column) and pdfs (right column) of the turbulent signal u, $\overline{u}$ and $u'_k, k = 1, 2, ..., 5$ with $\overline{F} = -0.054$. Note the logarithmic scale of pdfs in the y-axis. Dashed lines are Gaussian distributions with the same mean and variance.

Table S4-S6 show equilibrium statistics of each mode from the negative large scale damping six dimensional conceptual models, $K = 5$, for various strengths of $\overline{F}$. In Table S4, the mean, variance, and standard deviation of $\overline{u}$ and effective damping of each small scale mode, $d_k + \gamma\langle\overline{u}\rangle$ are shown while Table S5 and S6 show variances and correlation times of the turbulent signal u, $\overline{u}$, and $u'_k, k = 1, 2, ..., 5$ respectively.

Table S4: Positive large scale damping with $K = 5$: mean, variance, and standard deviation of $\overline{u}$ and effective damping of small scale modes for various strengths of $\overline{F}$

$\overline{F}$	$\langle\overline{u}\rangle$	$\mathrm{Var}(\overline{u})$	$\mathrm{Std}(\overline{u})$	$d_1 + \gamma\langle\overline{u}\rangle$	$d_2 + \gamma\langle\overline{u}\rangle$	$d_3 + \gamma\langle\overline{u}\rangle$	$d_4 + \gamma\langle\overline{u}\rangle$	$d_5 + \gamma\langle\overline{u}\rangle$
0	0.1700	0.0000	0.0070	1.2750	1.3350	1.4350	1.5750	1.7550
-0.010	-0.4519	0.0008	0.0289	0.3422	0.4022	0.5022	0.6422	0.8222
-0.030	-0.6504	0.0063	0.0793	0.0444	0.1044	0.2044	0.3444	0.5244
-0.054	-0.6779	0.0121	0.1102	0.0031	0.0631	0.1631	0.3031	0.4831
-0.055	-0.6784	0.0124	0.1113	0.0027	0.0624	0.1624	0.3024	0.4824
-0.060	-0.6808	0.0136	0.1167	-0.0012	0.0588	0.1588	0.2988	0.4788
-0.080	-0.6853	0.0186	0.1364	-0.0079	0.0521	0.1521	0.2921	0.4721
-0.100	-0.6870	0.0234	0.1530	-0.0105	0.0495	0.1495	0.2895	0.4695

Table S5: Positive large scale damping with $K = 5$: variances of the turbulent signal u, $\overline{u}$, and $u'_k, k = 1, 2, ..., 5$ for various strengths of $\overline{F}$

$\overline{F}$	$\mathrm{Var}(u)$	$\mathrm{Var}(\overline{u})$	$\mathrm{Var}(u'_1)$	$\mathrm{Var}(u'_2)$	$\mathrm{Var}(u'_3)$	$\mathrm{Var}(u'_4)$	$\mathrm{Var}(u'_5)$
0	0.0012	0.0000	0.0005	0.0003	0.0002	0.0001	0.0001
-0.010	0.0045	0.0008	0.0018	0.0009	0.0005	0.0003	0.0002
-0.030	0.0218	0.0063	0.0092	0.0041	0.0015	0.0006	0.0003
-0.054	0.0436	0.0121	0.0190	0.0087	0.0025	0.0008	0.0004
-0.055	0.0445	0.0124	0.0195	0.0089	0.0026	0.0008	0.0004
-0.060	0.0488	0.0136	0.0216	0.0099	0.0028	0.0009	0.0004
-0.080	0.0667	0.0186	0.0311	0.0129	0.0033	0.0010	0.0004
-0.100	0.0849	0.0234	0.0421	0.0147	0.0037	0.0011	0.0004

Table S6: Positive large scale damping with $K = 5$: correlation times of the turbulent signal u, $\overline{u}$, and $u'_k, k = 1, 2, ..., 5$ for various strengths of $\overline{F}$

| $\overline{F}$	$T_{cor}(u)$	$T_{cor}(\overline{u})$	$T_{cor}(u'_1)$	$T_{cor}(u'_2)$	$T_{cor}(u'_3)$	$T_{cor}(u'_4)$	$T_{cor}(u'_5)$
0	1.7502	79.34	0.7502	0.7213	0.6923	0.6280	0.5771
-0.010	3.1739	33.18	2.885	2.452	2.019	1.587	1.251
-0.030	7.502	16.98	9.013	8.800	6.059	3.404	2.019
-0.054	14.42	17.31	12.98	11.54	7.933	4.328	2.308
-0.055	14.00	18.00	12.83	11.25	7.817	4.353	2.400
-0.060	16.78	18.75	15.87	11.71	7.517	4.478	2.511
-0.080	23.01	22.15	28.35	13.31	6.780	4.517	2.671
-0.100	28.83	28.18	42.13	13.71	6.275	4.328	2.783
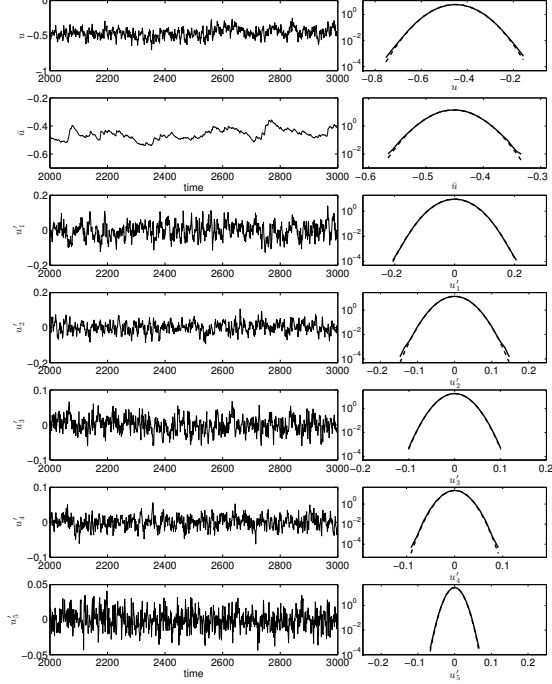
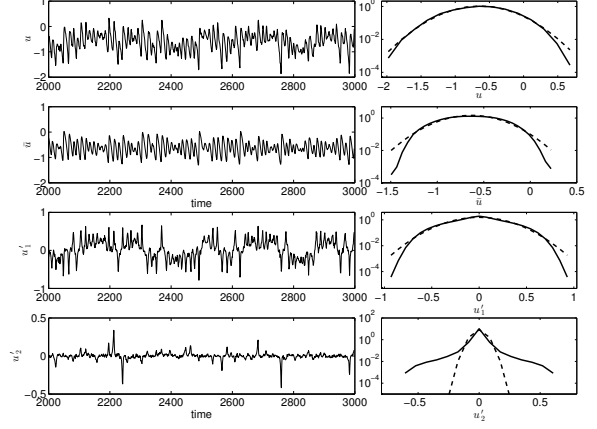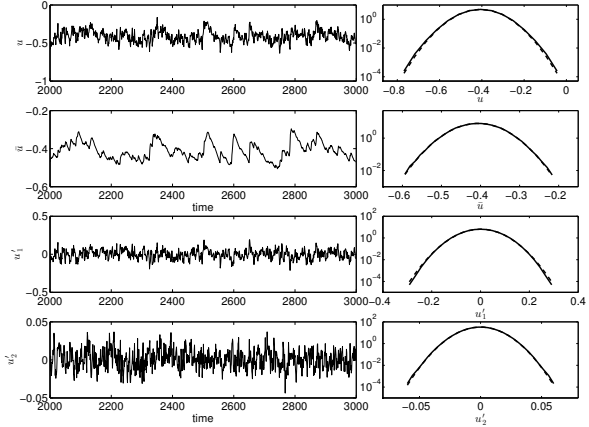(a) $\overline{F} = -0.010$
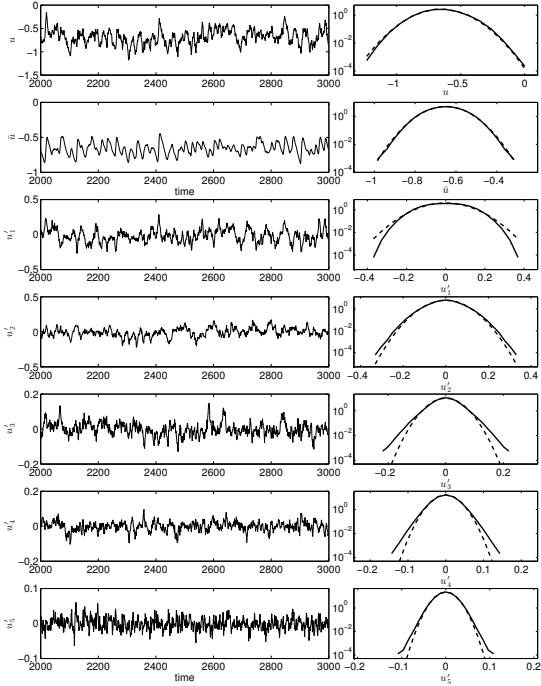
(b) $\overline{F} = -0.030$
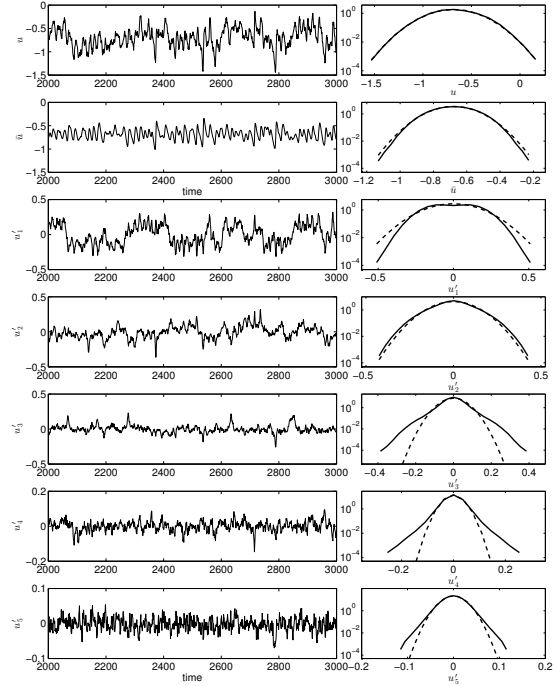
(c) $\overline{F} = -0.055$

Fig. S2: Positive large scale damping with $K = 5$: time series (left column) and pdfs (right column) of the turbulent signal u, $\overline{u}$ and u'_k, $k = 1, 2, ..., 5$ with $\overline{F} = -0.010, -0.030$, and -0.055. The tails of u'_3, u'_4, and u'_5 become fatter for larger $\overline{F}$ in magnitude. Note the logarithmic scale of pdfs in the y-axis. Dashed lines are Gaussian distributions with the same mean and variance.
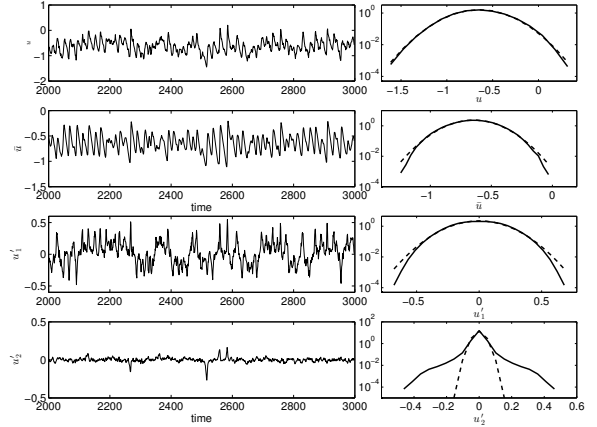
4
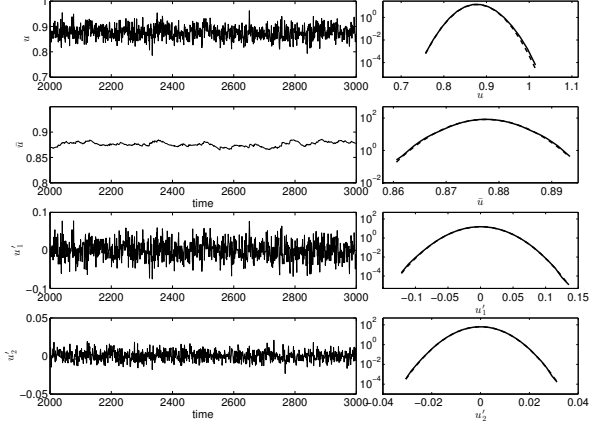
(a) $\overline{F} = -0.054$

(b) $\overline{F} = -0.055$

Fig. S3: Negative large scale damping with $K = 2$: time series (left column) and pdfs (right column) of the turbulent signal u, $\overline{u}$ and $u'_k, k = 1, 2$ with $\overline{F} = -0.054$ (a) and -0.055 (b). Note the logarithmic scale of pdfs in the y-axis. Dashed lines are Gaussian distributions with the same mean and variance.
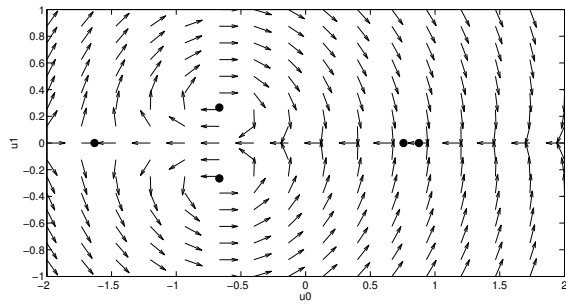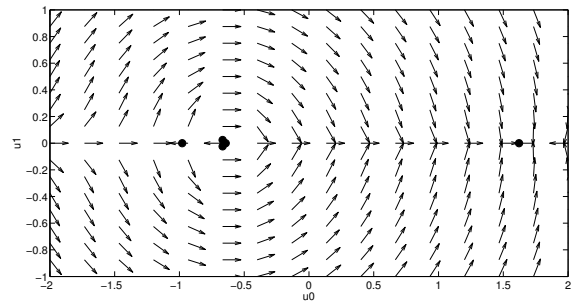

(a) $\overline{F} = -0.010$

(b) $\overline{F} = -0.055$

Fig. S4: Positive large scale damping with $K = 2$: time series (left column) and pdfs (right column) of the turbulent signal u, $\overline{u}$ and $u'_k, k = 1, 2$ with $\overline{F} = -0.010$ (a) and -0.055 (b). Note the logarithmic scale of pdfs in the y-axis. Dashed lines are Gaussian distributions with the same mean and variance.

(a) $\overline{F} = -0.054$ (b) $\overline{F} = 0.051$

Fig. S5: Phase portrait of the two mode, $K = 1$, without noise [7] for $\overline{F} = -0.054$ (a) and $\overline{F} = 0.051$ (b). Critical points are marked with black circles. The rightmost critical points are stable