Supplementary material for the paper ”Platonic Scattering Cancellation for Bending Waves on a Thin Plate”

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Abstract

This is the supplementary materials section outlining the biharmonic equation derivation, its boundary conditions and the expression of scattering coefficients for the core-shell system.

I. SET-UP OF THE BIHARMONIC EQUATION IN ANISOTROPIC THIN-PLATES

A. Subdomain governing equation

To perform our simulations, we use the FEM commercial softaware COMSOL Multiphysics[1]. The biharmonic equation[2] is obtained by a set of two coupled PDE involving two independent functions V and U, where U is the displacement field of the plate. It is straightforward to show that V and U satisfy the set of equations[3,4]

$$\left\{\begin{array}{l} \nabla.(-\underline{\underline{\zeta}}^{-1}\,\nabla U) + \lambda^{-1}\,V = 0 \\ \nabla.(-\underline{\underline{\zeta}}^{-1}\,\nabla V) + \lambda^{-1}\,\beta_0^4\,U = 0\,, \end{array}\right. \tag{1}$$

where $\underline{\underline{\zeta}}$ is an inhomogeneous anisotropic $2D$ tensor and λ is an inhomogeneous coefficient of the material (to be specified). In cylindrical coordinates and assuming that ζ_r, ζ_θ (components of ζ) and λ depend only in radial coordinate, these equations become

$$\left\{\begin{array}{l} \frac{1}{r}\,\partial_r(\frac{r}{\zeta_r}\,\partial_r\,U) + \frac{1}{\zeta_\theta\,r^2}\,\partial_\theta^2\,U - \lambda^{-1}\,V = 0 \\ \frac{1}{r}\,\partial_r(\frac{r}{\zeta_r}\,\partial_r\,V) + \frac{1}{\zeta_\theta\,r^2}\,\partial_\theta^2\,V - \lambda^{-1}\,\beta_0^4\,U = 0\,. \end{array}\right. \tag{2}$$

The equation satisfied by the out-of-plane displacement U is then[3–6]

$$\nabla.(\underline{\underline{\zeta}}^{-1}\,\nabla\,(\lambda\,\nabla.(\underline{\underline{\zeta}}^{-1}\,\nabla U))) - \lambda^{-1}\,\beta_0^4\,U = 0\,. \tag{3}$$

In cylindrical coordinates and assuming that $U = \sum_{n=-\infty}^{\infty} U_n(r)e^{i\,n\,\theta}$, the equation satisfied by W_n writes

$$\begin{array}{l} \partial_r(\dfrac{r}{\zeta_r}\,\partial_r(\lambda(\dfrac{1}{r}\,\partial_r(\dfrac{r}{\zeta_r}\,\partial_r\,U_n) - \dfrac{n^2}{\zeta_\theta\,r^2}\,U_n))) \\ -\;\dfrac{n^2\,\lambda}{\zeta_\theta\,r}(\dfrac{1}{r}\,\partial_r(\dfrac{r}{\zeta_r}\,\partial_r\,U_n) - \dfrac{n^2}{\zeta_\theta\,r^2}\,U_n) \\ -\;r\,\lambda^{-1}\,\beta_0^4\,U_n = 0\,. \end{array} \tag{4}$$

The next step is to find the physical signification of the parameters used in Eq. (3) and to link them to the physical parameters of the elastic plate. To do so, we can suppose that the coefficients of Eq. (3) are constants and are expressed in term of the homogeneous parameters ρ_0, E_0, ν_0 and h_0 (density, Young modulus, Poisson ratio and hight of the plate respectively),

$$\{E_0^{-1}\frac{12\,(1-\nu_0^2)}{h_0^2}\rho_0\}\{\lambda^{-1}\,\zeta^2\} \Leftrightarrow \{E^{-1}\frac{12\,(1-\nu_0^2)}{h_0^2}\rho\}\,. \tag{5}$$

One possible physical choice is the following one[3,4]

$$\underline{\underline{\zeta}} = \underline{\underline{E}}^{-1/2}\,, \text{and} \quad \lambda = \rho^{-1}\,. \tag{6}$$

B. Boundary conditions in weak formulation

When considering propagation in finite media, the biharmonic equation [Eq. (3)] is generally supplied with appropriate boundary conditions that are of three types: simply supported, fixed (clamped or Dirichlet-type) and freely vibrating (Neumann-type). In terms of cylindrical coordinates, they can be written respectively as follows (for a plate of radius a)

$$U|_{r=a} = 0\,, \qquad M_r|_{r=a} = 0\,, \tag{7}$$

$$U|_{r=a} = 0\,, \qquad \frac{\partial U}{\partial r}|_{r=a} = 0\,, \tag{8}$$

$$M_r|_{r=a} = 0\,, \qquad (V_r - \frac{1}{r}\frac{\partial M_{rt}}{\partial \theta})|_{r=a} = 0\,, \tag{9}$$

with $M_r = -D[\partial_r^2 U + \nu(1/r\partial_r U + 1/r^2\partial_\theta^2 U)]$, $M_{rt} = D(1-\nu)(1/r\partial_{r,\theta}^2 U - 1/r^2\partial_\theta U]$ and $V_r = -D\partial_r(\partial_r^2 U + 1/r\partial_r U + 1/r^2\partial_\theta^2 U) - 1/r\partial_\theta M_{rt}$. The first condition given in Eq. (7) means the plate does not experience any deflection and that bending moments are zero. The second condition in Eq. (8) says that the boundary of the plate does not experience any deflection and that it must be horizontal (the derivative is zero). The last one given in Eq. (9) expresses that the plate is freely vibrating.

The first two conditions can be easily implemented in the commercial software Comsol[1]. Howerver, the third one poses a serious convergence problems and is hard to formulate in a correct manner.

On General form (PDE module of Comsol), the order of equations is important, since in weak formulation the first equation is multiplied by the test function U and the second one by the test function test(V). The natural boundary conditions appear when integrating by part the following system

$$\begin{cases} -\nabla\cdot\nabla V + \beta^4 U = 0 \\ -\nabla\cdot\nabla U + V = 0 \end{cases} \tag{10}$$

They are given by

$$\begin{cases} \mathbf{n}\cdot\nabla V = g_U \\ \mathbf{n}\cdot\nabla U = g_V \end{cases} \tag{11}$$

where the U and V indices on the boundary flux terms are ther to remind with which test function they should be multiplied.

As most of the cases we have studied were linked with cylindrical geometries, we will consider the special case of constant radius circles (with normal vector $\mathbf{n}$ pointing towards the origin). The system (11) becomes

$$\begin{cases} \partial V/\partial r = -g_U \\ \partial U/\partial r = -g_V \end{cases} \tag{12}$$

The first condition in (9) which means that there is no bending at the boundary of the plate can be written for a constant r in cylindrical coordinates as

$$\frac{\partial^2 U}{\partial r^2} + \nu\Big(\frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\Big) = 0 \tag{13}$$

Using the development of the Laplacien in cylindrical coordinates permit us to re-writes Equation (13) in this form

$$V - \Big(\frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\Big) + \nu\Big(\frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\Big) = 0 \tag{14}$$

This can also be written as follows in order to compare with (12)

$$\frac{\partial U}{\partial r} = r\Big(\frac{V}{1-\nu} - \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\Big) \tag{15}$$

which clearly means that

$$g_V = -r\Big(\frac{V}{1-\nu} - \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\Big) \tag{16}$$

The last step is to express this equation in cartesian system because as we know the package Comsol isn't adapted to other types of coordinates (cylindrical or spherical). To do so, we will use the correspondance

$$\begin{cases} \partial/\partial r \Rightarrow (-\mathbf{n}\cdot\nabla) = -\partial/\partial n \\ \partial/r\partial r \Rightarrow (-\mathbf{t}\cdot\nabla) = -\partial/\partial t \end{cases} \tag{17}$$

where **t** is the tangential unitary vector (perpendicular with the normal vector **n**. We remark that this vector is always twicely applied, so that its direction doesn't matter. This double application is also important for the weak formulation which consists in integrating by part and by using a function test.

Under all these assumption, Equation (16) turns to be implemented in the following weak form

$$\text{test}(V)g_V = -r\Big(\frac{\text{test}(V)V}{1-\nu} + \nabla_T\text{test}(V)\cdot\nabla_T U\Big) \tag{18}$$

We now turn to the the second condition in (9) expressing that the generalized Kirchhoff stress is zero can be written

$$\frac{\partial}{\partial r}\Big(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\Big) + \frac{(1-\nu)}{r}\frac{\partial}{\partial \theta}\Big(\frac{1}{r}\frac{\partial^2 U}{\partial r\partial \theta} - \frac{1}{r^2}\frac{\partial U}{\partial \theta}\Big) \tag{19}$$

This condition can be transformed, and we finally get the expression of g_U

$$g_U = \frac{(1-\nu)}{r}\frac{\partial}{\partial \theta}\Big(\frac{1}{r}\frac{\partial}{\partial \theta}\Big(\frac{\partial U}{\partial r}\Big) - \frac{1}{r^2}\frac{\partial U}{\partial \theta}\Big) \tag{20}$$

Finally, we can integrate by parts and multiply by a function test test(U) to obtain the appropriate form

$$\text{test}(U)g_U = -(1-\nu)\nabla_T\text{test}(V)\cdot\Big(\nabla_T\Big(\frac{\partial U}{\partial r}\Big) - \frac{1}{r}\nabla_T U\Big) \tag{21}$$

II. SCATTERING COEFFICIENTS

The incident field is a plane wave $e^{ik_0 r cos\,\theta}$ and can be developed in this way

$$U^{inc} = \sum_n \varepsilon_n i^n\, J_n(k_0\, r)\, \cos n\theta \tag{22}$$

The scattered field must satisfy the radiation condition and can be developed in term of the cylindrical Hankel functions and the modified Bessel ones

$$U^{scatt} = \sum_n \varepsilon_n i^n [A_n\, H_n^{(1)}(k_0\, r) + B_n\, K_n(k_0\, r)] \cos n\theta \tag{23}$$

Inside the cloaking shell ($a_s < r < a_c$), the field must remain finite at $r = a_s$, then

$$U^{cloak} = \sum_n \varepsilon_n i^n [C_n\, Y_n(k_c\, r) + D_n\, K_n(k_c\, r) + E_n\, J_n(k_c\, r) + F_n\, I_n(k_c\, r)] \cos n\theta \tag{24}$$

The field inside the obstacle is given by

$$U^{int} = \sum_n \varepsilon_n i^n [G_n\, J_n(k_s\, r) + H_n\, I_n(k_s\, r)] \cos n\theta \tag{25}$$

The scatterd field can be made identically zero if the scattering coefficients $A_n = B_n = 0$ for every n. As pointed out, in the far-field we have $B_n = 0$. Thus, we have to calculate the inverse matrix and to find the conditions to impose to have a zero-scattered field.

The scattering coefficients A_n of the core-shell systems of Fig. 1 are given in Eq. (2) of the manuscript. These are expressed as ratios $A_n = \tilde{A}_n/d_n$. $\tilde{A}_n$ are the determinants given in Eq. (4) of the manuscript. The remaining terms d_n are also determinants and they could be expressed as:

$$\begin{vmatrix} H_n^{(1)}(k_0a_c) & K_n(k_0a_c) & -Y_n(k_ca_c) & -K_n(k_ca_c) & -J_n(k_ca_c) & -I_n(k_ca_c) & 0 & 0 \\ 0 & 0 & Y_n(k_ca_s) & K_n(k_ca_c) & J_n(k_ca_c) & I_n(k_ca_c) & -J_n(k_sa_s) & -I_n(k_sa_s) \\ k_0H_n^{(1)'}(k_0a_c) & k_0K_n'(k_0a_c) & -k_cY_n'(k_ca_c) & -k_cK_n'(k_ca_c) & -k_cJ_n'(k_ca_c) & -k_cI_n'(k_ca_c) & 0 & 0 \\ 0 & 0 & k_cY_n'(k_ca_s) & k_cK_n'(k_ca_s) & k_cJ_n'(k_ca_s) & k_cI_n'(k_ca_s) & -kJ_n'(k_sa_s) & -kI_n'(k_sa_s) \\ S_H(k_0a_c) & S_K(k_0a_c) & -S_Y(k_ca_c) & -S_K(k_ca_c) & S_J(k_ca_c) & S_I(k_ca_c) & 0 & 0 \\ 0 & 0 & S_Y(k_ca_s) & S_K(k_ca_s) & S_J(k_ca_s) & S_I(k_ca_s) & -S_J(k_sa_s) & S_I(k_sa_s) \\ T_H(k_0a_c) & T_K(k_0a_c) & -T_Y(k_ca_c) & -T_K(k_ca_c) & T_J(k_ca_c) & T_I(k_ca_c) & 0 & 0 \\ 0 & 0 & T_Y(k_ca_s) & T_K(k_ca_s) & T_J(k_ca_s) & T_I(k_ca_s) & -T_J(k_sa_s) & T_I(k_sa_s) \end{vmatrix} \tag{26}$$

with same notations of the parameters as in the manuscript.

The scattering coefficients from clamped obstacles and stress-free holes of the same radius a_s could also be obtained from the general case above by removing the sixth and eighth lines and last two columns, and second and fourth lines and last two columns from the 8×8 determinants of density-dependent objects, respectively. One obtains thus 6×6 determinants that can be used to describe scattering from these obstacles.

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