Appendix A. Proofs

Lemma 1. In the Dynamic Programming Matrix $C_{m\times n}$ for tackling the *k-difference Problem*, the values of elements along each diagonal are monotonically non-decreasing.

Proof: For any two adjacent elements along a diagonal, if the data flow to the right bottom element is from the left top one, then as the penalty (or distance) is non-negative and accumulative, the right bottom element cannot be smaller than the left top one; otherwise, without loss of generality, suppose the data flow to the right bottom element[1] $C_{i,j}$ is from its top neighbor $C_{i-1,j}$, and suppose $C_{i,j}<C_{i-1,j-1}$, then $C_{i-1,j}=C_{i,j}-1<C_{i-1,j-1}-1$, so the data flow to $C_{i-1,j}$ are from neither $C_{i-1,j-1}$ nor $C_{i-2,j-1}$, otherwise result in $C_{i-1,j-1}<C_{i-1,j}<C_{i-1,j-1}-1$ or $C_{i-2,j-1}\leq C_{i-1,j}<C_{i-1,j-1}-1$, but from the recurrence logic, $C_{i-2,j-1}\geq C_{i-1,j-1}-1$, contradictory. The same reasoning is applied for all the elements above $C_{i,j}$, and finally we have $C_{0,j}=C_{i,j}-i<C_{i-1,j-1}-i$. But from the recurrence logic, we have $C_{0,j-1}\geq C_{1,j}-1\geq\ldots\geq C_{i-1,j-1}-(i-1)$, so $C_{0,j-1}-C_{0,j}>1$. Which is not true for the recurrence start condition. In conclusion, in all cases $C_{i,j}\geq C_{i-1,j-1}$. ■

Lemma 2. If the penalty for an insertion/deletion is D, the maximum penalty for a mismatch is P, then $\Delta h_{i,j} = C_{i,j} - C_{i,j-1} \in [-D, D]$; $\Delta v_{i,j} = C_{i,j} - C_{i-1,j} \in [-D, D]$; $\Delta d_{i,j} = C_{i,j} - C_{i-1,j-1} \in [0, \min(P, 2D)]$.

Proof: From the dynamic programming recurrence, $C_{i,j} \leq C_{i,j-1} + D$, thus $C_{i,j} - C_{i,j-1} \leq D$. From Lemma 1, $C_{i,j} \geq C_{i-1,j-1} \geq C_{i,j-1} - D$, thus $C_{i,j} - C_{i,j-1} \geq -D$, hence, $C_{i,j} - C_{i,j-1} \in [-D, D]$; similarly, $C_{i,j} - C_{i-1,j} \in [-D, D]$; From Lemma 1, $C_{i,j} - C_{i-1,j-1} \geq 0$, From the dynamic programming recurrence $C_{i,j} - C_{i-1,j-1} \leq P$; also $C_{i,j} \leq C_{i-1,j} + D \leq C_{i-1,j-1} + 2D$, thus $C_{i,j} - C_{i-1,j-1} \in [0, \min(P, 2D)]$. ■

[1] $C_{i,j}$ and C[i,j] denote the same element in our nomenclature