# Web-based Supplementary Materials for Estimating differences in restricted mean lifetime using observational data subject to dependent censoring

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## 1 Notation

To begin, we review the essential notation:

$i$: subject

$j$: treatment $(j = 0, 1)$

$T_i$: death time

$C_{1i}$: independent censoring time

$C_{2i}$: dependent censoring time

$A_i$: treatment group, subject $i$

$A_{ij} = I(A_i = j)$: treatment group $j$ indicator

$U_i = T_i \wedge C_{1i} \wedge C_{2i}$ : observation time

$\Delta_{1i} = I(T_i \leq C_{1i} \wedge C_{2i})$: observed death indicator

$\Delta_{2i} = I(C_{2i} \leq T_i \wedge C_{1i})$: dependent censoring indicator

$\lambda_{ij}(t) = \lambda_{0j}(t) \exp(\beta_j^T Z_i)$: assumed model, death time

$\lambda_{ij}^C(t) = \lambda_{0j}^C(t) \exp\{\theta_j^T X_i(t)\}$: assumed model, dependent censoring time

$N_i(t) = I(U_i \leq t, \Delta_{1i} = 1)$: observed death counting process

$N_i^C(t) = I(U_i \leq t, \Delta_{2i} = 1)$: dependent censoring counting process

$Y_i(t) = I(U_i \geq t)$: at-risk process

$N_{ij}(t) = A_{ij}N_i(t)$: group $j$ death counting process

$N_{ij}^C(t) = A_{ij}N_i^C(t)$: group $j$ dependent censoring process

$Y_{ij}(t) = A_{ij}Y_i(t)$: group $j$ at-risk process

$dM_{ij}(t) = A_{ij}\{dN_i(t) - e^{\beta_j^T Z_i}Y_i(t)\lambda_{0j}(t)dt\}$: zero-mean process for death, group $j$

$dM_{ij}^C(t) = A_{ij}\{dN_i^C(t) - e^{\theta_j^T X_i(t)}Y_i(t)\lambda_{0j}^C(t)dt\}$: zero-mean process for dependent censoring, group $j$

$S_{ij}(t) = P(T_i \geq t|Z_i, A_i = j)$: survival function, conditional, group $j$

$\mu_{ij}(t) = \int_0^t S_{ij}(u)du$: restricted mean lifetime, conditional, group $j$

$\mu_{ij} = \mu_{ij}(L)$

$\mu_j = E_{Z_i}(\mu_{ij})$

$\delta = \mu_1 - \mu_0$: difference in restricted mean lifetime (parameter of interest)

# 2 Consistency of $\widehat{\beta}_j$ and $\widehat{\Lambda}_0$

## 2.1 Fundamental Identity

We first prove Fundamental Identity (10) of Robins (1993), which will be used below in proving consistency and is also used in proof of Lemma A.1 (Robins and Finkelstein, 2000):

$$Y_{ij}(t)e^{\Lambda_{ij}^C(t)} = A_{ij}I(T_i \geq t)\{1 - \int_0^t e^{\Lambda_{ij}^C(u)}dM_{ij}^C(u)\}. \tag{1}$$

Proof: we consider the four possible cases involving the only random variables in (1).

Case 1: $A_{ij} = 0$. LHS=0 and RHS=0, where LHS denotes for "left hand side" of equation (1) and RHS for right hand side.

Case 2: $A_{ij} = 1$ and $T_i < t$. LHS=0 and RHS=0.

Case 3: $A_{ij} = 1, T_i \geq t$, and $C_i \geq t$. $LHS = e^{\Lambda_{ij}^C(t)}$, and

$$\begin{aligned} \text{RHS} &= 1 - \int_0^t e^{\Lambda_{ij}^C(u)}\{dN_{ij}^C(u) - Y_{ij}(u)d\Lambda_{ij}^C(u)\} \\ &= 1 + \int_0^t e^{\Lambda_{ij}^C(u)} d\Lambda_{ij}^C(u) \\ &= 1 + \int_0^t d\{e^{\Lambda_{ij}^C(u)}\} = e^{\Lambda_{ij}^C(t)} = \text{LHS}. \end{aligned}$$

Case 4: $A_{ij} = 1, T_i \geq t$, and $C_i = s < t$. LHS=0, and

$$\begin{aligned} \text{RHS} &= 1 - \int_0^t e^{\Lambda_{ij}^C(u)}\{dN_{ij}^C(u) - Y_{ij}(u)d\Lambda_{ij}^C(u)\} \\ &= 1 - e^{\Lambda_{ij}^C(s)} + \int_0^s e^{\Lambda_{ij}^C(u)} d\Lambda_{ij}^C(u) \\ &= 1 - e^{\Lambda_{ij}^C(s)} + e^{\Lambda_{ij}^C(s)} - 1 = 0 = \text{ LHS}. \end{aligned}$$

Therefore, the fundamental identity is proved.

## 2.2 Consistency

Since the estimation of $\beta_j$ and $\Lambda_{0j}$ through equations (12) and (13) is equivalent to the simultaneous solution of (10) and (11), consistency requires that equations (10) and (11) be unbiased at their true values, $\beta_j$ and $\Lambda_{0j}$; i.e., that

$$E\left\{\int_0^t W_{ij}(u)dM_{ij}(u)\right\} = 0 \tag{2}$$

$$E\left\{\int_0^\tau W_{ij}(t)Z_i\, dM_{ij}(t)\right\} = 0, \tag{3}$$

with $W_{ij}$ and $M_{ij}$ evaluated at the truth. If we can show that (2) holds, then (3) must also hold. First, note $Y_{ij}(t)dM_{ij}(t) = Y_{ij}(t)dM_{ij}^T(t)$, where $dM_{ij}^T(t) = A_{ij}\{dI(T_i \leq t) - I(T_i \geq t)d\Lambda_{ij}(t)$, and $I(T_i \geq t)dM_{ij}^T(t) = dM_{ij}^T(t)$. It follows that

$$
\begin{aligned}
& \int_0^t W_{ij}(u)dM_{ij}(u) \equiv \int_0^t e^{\Lambda_{ij}^C(u)}\kappa(u;Z_i,A_i)Y_{ij}(u)dM_{ij}^T(u) \\
= & \int_0^t I(T_i \geq u)\kappa(u;Z_i,A_i)\left\{1-\int_0^u e^{\Lambda_{ij}^C(s)}dM_{ij}^C(s)\right\}dM_{ij}^T(u) \quad \text{(Fundamental Identity)} \\
= & \int_0^t \kappa(u;Z_i,A_i)dM_{ij}^T(u) \qquad (4) \\
- & \int_0^t \int_s^t \kappa(u;Z_i,A_i)dM_{ij}^T(u)e^{\Lambda_{ij}^C(s)}dM_{ij}^C(s). \qquad (5)
\end{aligned}
$$

It is clear that (4) has mean zero due to the zero-mean property of Martingale integrals, while (5) also has mean zero due to the "no unmeasured confounders" assumption (specified by (9) in the main manuscript) along with the fact that (5) is a Martingale. Therefore, expression (2) is true. Expression (3) can be shown in a very similar fashion. Combining these two results, the estimating equation in (12) in the main manuscript is also an unbiased estimating equation.

# 3 Outline of Derivation

In this section, we outline the main steps of the derivation of the influence function of $\widehat{\delta}$. That is, we represent $n^{\frac{1}{2}}(\widehat{\delta}-\delta)$ as a summation of independent and identically distributed (iid) terms plus a term that converges in probability to zero. The derivation provided pertains to the case where one uses unstabilized inverse weighting; i.e., $\widehat{W}_{ij}(t) = e^{\widehat{\Lambda}_{ij}^C(t)}$; the proof for stabilized inverse weighting would be similar and is therefore omitted. The derivation is broken into several intermediate results, with each result involving the expression of a key quantity as a summation of iid terms:

1. $n^{\frac{1}{2}}(\widehat{\theta}_j-\theta_j)$

2. $n^{\frac{1}{2}}\{\widehat{\Lambda}_{0j}^C(t)-\Lambda_{0j}^C(t)\}$

3. $n^{\frac{1}{2}}\{\widehat{\Lambda}_{ij}^C(t) - \Lambda_{0j}^C(t)\}$

4. $n^{\frac{1}{2}}\{\widehat{W}_{ij}(t) - W_{ij}(t)\}$

5. $n^{\frac{1}{2}}(\widehat{\beta}_j - \beta_j)$

6. $n^{\frac{1}{2}}\{\widehat{\Lambda}_{0j}(t) - \Lambda_{0j}(t)\}$

7. $n^{\frac{1}{2}}\{\widehat{\Lambda}_{ij}(t) - \Lambda_{ij}(t)\}$

8. $n^{\frac{1}{2}}\{\widehat{S}_{ij}(t) - S_{ij}(t)\}$

9. $n^{\frac{1}{2}}(\widehat{\mu}_{ij} - \mu_{ij})$

10. $n^{\frac{1}{2}}(\widehat{\mu}_j - \mu_j)$, where $\mu_j = E_{Z_i}(\mu_{ij})$

11. $n^{\frac{1}{2}}(\widehat{\delta} - \delta)$, where $\delta = \mu_1 - \mu_0$.

# 4 Derivation

In this section, we derive a a sequence of results which lead up to the derivation of the influence function for $\widehat{\delta}$.

## 4.1 $n^{\frac{1}{2}}(\widehat{\theta}_j - \theta_j)$

Through a linear Taylor series expansion, we have

$$n^{\frac{1}{2}}(\widehat{\theta}_j - \theta_j) = \Omega_j^C(\theta_j)^{-1} n^{-\frac{1}{2}} \sum_{i=1}^{n} U_{ij}^C(\theta_j) + o_p(1),$$

where the quantities

$$U_{ij}^C(\theta) = \int_0^\tau \{X_i(t) - \overline{x}_j(t;\theta)\} dM_{ij}^C(t;\theta)$$

are zero-mean and iid, with

$$
\begin{aligned}
r_{Cj}^{(k)}(t;\theta) &= E[Y_{ij}(t)X_i(t)^{\otimes k}e^{\theta^T X_i(t)}], k = 0,1,2 \\
\overline{x}_j(t;\theta) &= \frac{r_{Cj}^{(1)}(t;\theta)}{r_{Cj}^{(0)}(t;\theta)} \\
\Omega_j^C(\theta) &= E\Big[\int_0^\tau \Big\{\frac{r_{Cj}^{(2)}(t;\theta)}{r_{Cj}^{(0)}(t;\theta)} - \overline{x}_j(t;\theta)^{\otimes 2}\Big\} dN_{ij}^C(t)\Big].
\end{aligned}
$$

The quantity $M_{ij}^C(t;\theta_0)$ is a Martingale with respect to the filtration, $\mathcal{F}_{ij}^C(t) = \sigma\{N_{ij}^C(u), Y_{ij}(u), \widetilde{X}(s); u \in [0,t]\}$. Result 4.1 is based on standard Martingale theory, such as that described in Andersen and Gill (1982), Fleming and Harrington (1991) and Andersen et al (1993); see also closely related work by Tsiatis (1981).

## 4.2 $n^{\frac{1}{2}}\{\widehat{\Lambda}_{0j}^C(t) - \Lambda_{0j}^C(t)\}$

We can decompose the above-listed quantity as follows:

$$
\begin{aligned}
n^{\frac{1}{2}}\{\widehat{\Lambda}_{0j}^C(t) - \Lambda_{0j}^C(t)\} &= n^{\frac{1}{2}}\{\widehat{\Lambda}_{0j}^C(t;\widehat{\theta}_j) - \widehat{\Lambda}_{0j}^C(t;\theta_j)\} \quad (6) \\
&\quad + n^{\frac{1}{2}}\{\widehat{\Lambda}_{0j}^C(t;\theta_j) - \Lambda_{0j}^C(t)\}. \quad (7)
\end{aligned}
$$

We define the following quantities:

$$
\begin{aligned}
& R_{Cj}^{(k)}(t;\theta) = n^{-1}\sum_{i=1}^n Y_{ij}(t)X_i(t)^{\otimes k}e^{\theta^T X_i(t)} \\
& \overline{X}_j(t;\theta) = \frac{R_{Cj}^{(1)}(t;\theta)}{R_{Cj}^{(0)}(t;\theta)} \\
& \widehat{h}_{Cj}(t;\theta) = -n^{-1}\sum_{i=1}^n \int_0^t R_{Cj}^{(0)}(s;\theta)^{-1}\overline{X}_j(s;\theta)dN_{ij}^C(s) = -\int_0^t \overline{X}_j(s;\theta)d\widehat{\Lambda}_{0j}^C(s;\theta) \\
& h_{Cj}(t;\theta) = -\int_0^t \overline{x}_j(s;\theta)d\Lambda_{0j}^C(s).
\end{aligned}
$$

By the Weak Law of Large Numbers (WLLN), $R_{Cj}^{(k)}(t;\theta)$ converges in probability to $r_{Cj}^{(k)}(t;\theta)$ and, through the Continuous Mapping Theorem, $\overline{X}_j(t;\theta) \xrightarrow{p} \overline{x}_j(t;\theta)$. In addition, using the uniform convergence in probability of $\widehat{\Lambda}_{0j}^C(t)$ to $\Lambda_{0j}^C(t)$ (e.g., Andersen and Gill, 1982) and continuity, it follows that $\widehat{h}_{Cj}(t;\theta) \xrightarrow{p} h_{Cj}(t;\theta)$. Returning to the quantity of interest, we can write (6) as

$$(6) \;=\; n^{-\frac{1}{2}} \sum_{i=1}^{n} \int_0^t \{R_{Cj}^{(0)}(s;\widehat{\theta}_j)^{-1} - R_{Cj}^{(0)}(s;\theta_j)^{-1}\} dN_{ij}^C(s).$$

Through a Taylor expansion,

$$\begin{aligned}(6) \;&=\; \widehat{h}_{Cj}^T(t;\theta_j)\Omega_j^C(\theta_j)^{-1} n^{-\frac{1}{2}} \sum_{i=1}^{n} U_{ij}^C(\theta_j) + o_p(1) \\ &=\; h_{Cj}^T(t;\theta_j)\Omega_j^C(\theta_j)^{-1} n^{-\frac{1}{2}} \sum_{i=1}^{n} U_{ij}^C(\theta_j) + o_p(1),\end{aligned}$$

with the last equality holding asymptotically using Slutsky's Theorem. We can write (7) as

$$\begin{aligned}(7) \;&=\; n^{-\frac{1}{2}} \sum_{i=1}^{n} \int_0^t R_{Cj}^{(0)}(s;\theta_j)^{-1} dM_{ij}^C(s) \\ &=\; n^{-\frac{1}{2}} \sum_{i=1}^{n} \int_0^t r_{Cj}^{(0)}(s;\theta_j)^{-1} dM_{ij}^C(s) + o_p(1).\end{aligned}$$

with the last equality following from standard Martingale results. Combining the re-expressions of (6) and (7), we have

$$n^{\frac{1}{2}}\{\widehat{\Lambda}_{0j}^C(t) - \Lambda_{0j}^C(t)\} = n^{-\frac{1}{2}} \sum_{i=1}^{n} \Phi_{ij}^C(t;\theta_j) + o_p(1),$$

where

$$\Phi_{ij}^C(t;\theta) = h_{Cj}^T(t;\theta)\Omega_j^C(\theta)^{-1}U_{ij}^C(\theta) + \int_0^t r_{Cj}^{(0)}(s;\theta)^{-1} dM_{ij}^C(s) = \int_0^t d\Phi_{ij}^C(s;\theta),$$

and

$$d\Phi_{ij}^C(s;\theta) = -\overline{x}_j^T(s;\theta) d\Lambda_{0j}^C(s)\Omega_j^C(\theta)^{-1}U_{ij}^C(\theta) + r_{Cj}^{(0)}(s;\theta)^{-1} dM_{ij}^C(s).$$

Result 4.2 is available through well-established Martingale results. We provided the details above since several key quantities needed to be introduced for later use anyway.

### 4.3 $n^{\frac{1}{2}}\{\widehat{\Lambda}_{ij}^C(t) - \Lambda_{ij}^C(t)\}$

We decompose the above quantity as

$$\begin{aligned} n^{\frac{1}{2}}\{\widehat{\Lambda}_{ij}^C(t) - \Lambda_{ij}^C(t)\} &= n^{\frac{1}{2}}\Big\{\int_0^t e^{\widehat{\theta}_j^T X_i(s)} Y_{ij}(s) d\widehat{\Lambda}_{0j}^C(s) - \int_0^t e^{\theta_j^T X_i(s)} Y_{ij}(s) d\widehat{\Lambda}_{0j}^C(s)\Big\} \quad (8) \\ &+ n^{\frac{1}{2}}\Big\{\int_0^t e^{\theta_j^T X_i(s)} Y_{ij}(s) d\widehat{\Lambda}_{0j}^C(s) - \int_0^t e^{\theta_j^T X_i(s)} Y_{ij}(s) d\Lambda_{0j}^C(s)\Big\}. \quad (9) \end{aligned}$$

Considering the first term,

$$(8) = n^{\frac{1}{2}} \int_0^t \{e^{\widehat{\theta}_j^T X_i(s)} - e^{\theta_j^T X_i(s)}\} Y_{ij}(s) d\widehat{\Lambda}_{0j}^C(s).$$

By a Taylor expansion, we obtain

$$\begin{aligned} n^{\frac{1}{2}}\{e^{\widehat{\theta}_j^T X_i(s)} - e^{\theta_j^T X_i(s)}\} &= X_i^T(s) e^{\theta_j^T X_i(s)} n^{\frac{1}{2}}(\widehat{\theta}_j - \theta_j) + o_p(1) \\ &= X_i^T(s) e^{\theta_j^T X_i(s)} \Omega_j^C(\theta_j)^{-1} n^{-\frac{1}{2}} \sum_{\ell=1}^n U_{\ell j}^C(\theta_j) + o_p(1), \end{aligned}$$

by Result 4.1. Since $\widehat{\Lambda}_{0j}^C(t) \xrightarrow{p} \Lambda_{0j}^C(t)$ for $t \in [0, \tau]$ (Andersen and Gill, 1982), we obtain

$$(8) = \int_0^t X_i^T(s) d\Lambda_{ij}^C(s) \Omega_j^C(\theta_j)^{-1} n^{-\frac{1}{2}} \sum_{\ell=1}^n U_{\ell j}^C(\theta_j) + o_p(1).$$

The second term in the above decomposition can be written as

$$\begin{aligned} (9) &= n^{\frac{1}{2}} \int_0^t e^{\theta_j^T X_i(s)} Y_{ij}(s) d\{\widehat{\Lambda}_{0j}^C(s) - \Lambda_{0j}^C(s)\} \\ &= \int_0^t e^{\theta_j^T X_i(s)} Y_{ij}(s) n^{-\frac{1}{2}} \sum_{\ell=1}^n d\Phi_{\ell j}^C(s; \theta_j), \end{aligned}$$

where the second equality holds by Result 4.2. Combining the above re-expressions for (8) and (9) leads to

$$\begin{aligned} n^{\frac{1}{2}}\{\widehat{\Lambda}_{ij}^C(t) - \Lambda_{ij}^C(t)\} &= \int_0^t \{X_i(s) - \overline{x}_j(s; \theta_j)\}^T d\Lambda_{ij}^C(s) \Omega_j^C(\theta_j)^{-1} n^{-\frac{1}{2}} \sum_{\ell=1}^n U_{\ell j}^C(\theta_j) \\ &+ n^{-\frac{1}{2}} \sum_{\ell=1}^n \int_0^t e^{\theta_j^T X_i(s)} Y_{ij}(s) r_{Cj}^{(0)}(s; \theta_j)^{-1} dM_{\ell j}^C(s) \\ &= D_{ij}^T(t) \Omega_j^C(\theta_j)^{-1} n^{-\frac{1}{2}} \sum_{\ell=1}^n U_{\ell j}^C(\theta_j) + n^{-\frac{1}{2}} \sum_{\ell=1}^n J_{i\ell j}^C(t), \end{aligned}$$

where we define

$$D_{ij}(t) = \int_0^t \{X_i(s) - \overline{x}_j(s;\theta_j)\} d\Lambda_{ij}^C(s) = \int_0^t dD_{ij}(s),$$
$$J_{i\ell j}^C(t) = \int_0^t e^{\theta_j^T X_i(s)} Y_{ij}(s) r_{Cj}^{(0)}(s;\theta_j)^{-1} dM_{\ell j}^C(s).$$

## 4.4 $n^{\frac{1}{2}}\{\widehat{W}_{ij}(t) - W_{ij}(t)\}$

As $W_{ij}(t) = e^{\Lambda_{ij}^C(t)}$ and $\widehat{W}_{ij}(t) = e^{\widehat{\Lambda}_{ij}^C(t)}$, then

$$n^{\frac{1}{2}}\{\widehat{W}_{ij}(t) - W_{ij}(t)\} \;=\; n^{\frac{1}{2}}\{e^{\widehat{\Lambda}_{ij}^C(t)} - e^{\Lambda_{ij}^C(t)}\}.$$

Therefore, applying a Taylor expansion,

$$\begin{aligned} n^{\frac{1}{2}}\{\widehat{W}_{ij}(t) - W_{ij}(t)\} &= W_{ij}(t) n^{\frac{1}{2}}\{\widehat{\Lambda}_{ij}^C(t) - \Lambda_{ij}^C(t)\} + o_p(1) \\ &= W_{ij}(t) n^{-\frac{1}{2}} \sum_{\ell=1}^n \{D_{ij}^T(t)\Omega_j^C(\theta_j)^{-1} U_{\ell j}^C(\theta_j) + J_{i\ell j}^C(t)\} + o_p(1), \end{aligned}$$

where the final equality follows from Result 4.3.

## 4.5 $n^{\frac{1}{2}}(\widehat{\beta}_j - \beta_j)$

By a Taylor expansion, we obtain

$$n^{\frac{1}{2}}(\widehat{\beta}_j - \beta_j) \;=\; \Omega_j(\beta_j)^{-1} n^{-\frac{1}{2}} \sum_{i=1}^n V_{ij}(\beta_j, \widehat{W}) + o_p(1),$$

where we define

$$V_{ij}(\beta, W) = \int_0^\tau \{Z_i - \overline{Z}_j(t;\beta,W)\} W_{ij}(t) dN_{ij}(t)$$
$$R_j^{(k)}(t;\beta,W) = n^{-1} \sum_{i=1}^n W_{ij}(t) Y_{ij}(t) Z_i^{\otimes k} e^{\beta^T Z_i}$$
$$\overline{Z}_j(t;\beta,W) = \frac{R_j^{(1)}(t;\beta,W)}{R_j^{(0)}(t;\beta,W)}.$$

In addition, we define the following notation:

$$r_j^{(k)}(t;\beta,W) = E\{W_{ij}(t)Y_{ij}(t)Z_i^{\otimes k}e^{\beta^T Z_i}\}$$
$$\overline{z}_j(t;\beta,W) = \frac{r_j^{(1)}(t;\beta,W)}{r_j^{(0)}(t;\beta,W)}.$$

The term $n^{-\frac{1}{2}}\sum_{i=1}^n V_{ij}(\beta_j,\widehat{W})$ can be decomposed as follows,

$$\begin{aligned} n^{-\frac{1}{2}}\sum_{i=1}^n V_{ij}(\beta_j,\widehat{W}) &= n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^\tau \{Z_i - \overline{Z}_j(t;\beta_j,\widehat{W})\}\widehat{W}_{ij}(t)dM_{ij}(t) \\ &= n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^\tau \{Z_i - \overline{Z}_j(t;\beta_j,W)\}W_{ij}(t)dM_{ij}(t) \qquad (10) \\ &\quad - n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^\tau \{\overline{Z}_j(t;\beta_j,\widehat{W}) - \overline{Z}_j(t;\beta_j,W)\}\widehat{W}_{ij}(t)dM_{ij}(t) \qquad (11) \\ &\quad + n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^\tau \{Z_i - \overline{Z}_j(t;\beta_j,W)\}\{\widehat{W}_{ij}(t) - W_{ij}(t)\}dM_{ij}(t). \qquad (12) \end{aligned}$$

Through techniques from empirical processes (e.g., Bilias, Gu and Ying, 1998; Lin, Wei, Yang and Ying, 2000), it can be shown that (10) $= n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^\tau \{Z_i - \overline{z}_j(t;\beta_j,W)\}W_{ij}(t)dM_{ij}(t)$, asymptotically. Through the Functional Delta Method, combined with a lot of tedious algebra, (11) can be shown to converge in probability to 0. Additionally, using Result 4.4,

$$\begin{aligned} (12) &= n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^\tau \{Z_i - \overline{Z}_j(t;\beta_j,W)\}W_{ij}(t)n^{-1}\sum_{\ell=1}^n D_{ij}^T(t)\Omega_j^C(\theta_j)^{-1}U_{\ell j}^C(\theta_j)dM_{ij}(t) \qquad (13) \\ &\quad + n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^\tau \{Z_i - \overline{Z}_j(t;\beta_j,W)\}W_{ij}(t)n^{-1}\sum_{\ell=1}^n J_{i\ell j}^C(t)dM_{ij}(t) + o_p(1). \qquad (14) \end{aligned}$$

Switching the order of summation, we have

$$\begin{aligned} (13) &= n^{-1}\sum_{i=1}^n \int_0^\tau \{Z_i - \overline{Z}_j(t;\beta_j,W)\}W_{ij}(t)D_{ij}^T(t)dM_{ij}(t)\Omega_j^C(\theta_j)^{-1}n^{-\frac{1}{2}}\sum_{\ell=1}^n U_{\ell j}^C(\theta_j) \\ &= \widehat{H}_j(t)\Omega_j^C(\theta_j)^{-1}n^{-\frac{1}{2}}\sum_{\ell=1}^n U_{\ell j}^C(\theta_j) \\ &= H_j(t)\Omega_j^C(\theta_j)^{-1}n^{-\frac{1}{2}}\sum_{\ell=1}^n U_{\ell j}^C(\theta_j) + o_p(1), \end{aligned}$$

where we define

$$\widehat{H}_j(t) = n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\{Z_i - \overline{Z}_j(t;\beta_j,W)\}W_{ij}(t)D_{ij}^T(t)dM_{ij}(t)$$
$$H_j(t) = E\Big[\int_0^{\tau}\{Z_i - \overline{z}_j(t;\beta_j,W)\}W_{ij}(t)D_{ij}^T(t)dM_{ij}(t)\Big].$$

Switching the order of summation and integration,

$$\begin{aligned}
(14) \quad &= n^{-\frac{1}{2}}\sum_{\ell=1}^{n}\int_0^{\tau}\Big[n^{-1}\sum_{i=1}^{n}e^{\theta_j^T X_i(s)}Y_{ij}(s)\int_s^{\tau}\{Z_i - \overline{Z}_j(t;\beta_j,W)\}W_{ij}(t)dM_{ij}(t)\Big]r_{Cj}^{(0)}(s,\theta_j)^{-1}dM_{\ell j}^C(s)\\
&= n^{-\frac{1}{2}}\sum_{\ell=1}^{n}\int_0^{\tau}\widehat{G}_j(s,\tau)r_{Cj}^{(0)}(s,\theta_j)^{-1}dM_{\ell j}^C(s)\\
&= n^{-\frac{1}{2}}\sum_{\ell=1}^{n}\int_0^{\tau}G_j(s,\tau)r_{Cj}^{(0)}(s,\theta_j)^{-1}dM_{\ell j}^C(s),
\end{aligned}$$

where we set

$$\widehat{G}_j(t_1,t_2) = n^{-1}\sum_{i=1}^{n}e^{\theta_j^T X_i(t_1)}Y_{ij}(t_1)\int_{t_1}^{t_2}\{Z_i - \overline{Z}_j(t;\beta_j,W)\}W_{ij}(t)dM_{ij}(t)$$
$$G_j(t_1,t_2) = E\Big[e^{\theta_j^T X_i(t_1)}Y_{ij}(t_1)\int_{t_1}^{t_2}\{Z_i - \overline{z}_j(t;\beta_j,W)\}W_{ij}(t)dM_{ij}(t)\Big].$$

Combining the re-expressions of (10), (11) and (12), we have

$$n^{\frac{1}{2}}(\widehat{\beta}_j - \beta_j) = \Omega_j(\beta_j)^{-1}n^{-\frac{1}{2}}\sum_{i=1}^{n}U_{ij}(\beta_j) + o_p(1),$$

where we set

$$\begin{aligned}
U_{ij}(\beta) = &\int_0^{\tau}\{Z_i - \overline{z}_j(t;\beta,W)\}W_{ij}(t)dM_{ij}(t)\\
&+ H_j(t)\Omega_j^C(\theta_j)^{-1}U_{ij}^C(\theta_j)\\
&+ \int_0^{\tau}G_j(t,\tau)r_{Cj}^{(0)}(t;\theta_j)^{-1}dM_{ij}^C(t).
\end{aligned}$$

### 4.6 $n^{\frac{1}{2}}\{\widehat{\Lambda}_{0j}(t) - \Lambda_{0j}(t)\}$

We work with the following decomposition,

$$n^{\frac{1}{2}}\{\widehat{\Lambda}_{0j}(t) - \Lambda_{0j}(t)\} = n^{\frac{1}{2}}[\widehat{\Lambda}_{0j}\{t;\widehat{W},R(\widehat{\beta}_j,\widehat{W})\} - \widehat{\Lambda}_{0j}\{t;\widehat{W},R(\beta_j,\widehat{W})\}] \quad (15)$$

$$+ n^{\frac{1}{2}}[\widehat{\Lambda}_{0j}\{t;\widehat{W},R(\beta_j,\widehat{W})\} - \widehat{\Lambda}_{0j}\{t;W,R(\beta_j,\widehat{W})\}] \quad (16)$$

$$+ n^{\frac{1}{2}}[\widehat{\Lambda}_{0j}\{t;W,R(\beta_j,\widehat{W})\} - \widehat{\Lambda}_{0j}\{t;W,R(\beta_j,W)\}] \quad (17)$$

$$+ n^{\frac{1}{2}}[\widehat{\Lambda}_{0j}\{t;W,R(\beta_j,W)\} - \Lambda_{0j}(t)]. \quad (18)$$

We reorganize (15) as

$$(15) = n^{-\frac{1}{2}}\sum_{i=1}^{n}\int_0^t\{R_j^{(0)}(s;\widehat{\beta}_j,\widehat{W})^{-1} - R_j^{(0)}(s;\beta_j,\widehat{W})^{-1}\}\widehat{W}_{ij}(s)dN_{ij}(s).$$

Further reorganization, followed by a Taylor expansion yields

$$\begin{aligned}(15) &= -\int_0^t \overline{Z}_j^T(s;\beta_j,\widehat{W})d\widehat{\Lambda}_{0j}(s)n^{\frac{1}{2}}(\widehat{\beta}_j - \beta_j) + o_p(1)\\ &= h_j^T(t)n^{\frac{1}{2}}(\widehat{\beta}_j - \beta_j) + o_p(1),\end{aligned}$$

where the last equality follows from the WLLN and Slutsky's Theorem, with

$$h_j(t) = -\int_0^t \overline{z}_j(s;\beta_j,W)d\Lambda_{0j}(s). \quad (19)$$

Expression (16) can be re-written as

$$(16) = n^{-\frac{1}{2}}\sum_{i=1}^{n}\int_0^t \frac{\widehat{W}_{ij}(s) - W_{ij}(s)}{R_j^{(0)}(s;\beta_j,\widehat{W})}dN_{ij}(s).$$

Applying Result 4.4, it follows that

$$(16) = n^{-1}\sum_{i=1}^{n}\int_0^t R_j^{(0)}(s;\beta_j,\widehat{W})^{-1}W_{ij}(s)D_{ij}^T(s)\Omega_j^C(\theta_j)^{-1}n^{-\frac{1}{2}}\sum_{\ell=1}^{n}U_{\ell j}^C(\theta_j)dN_{ij}(s) \quad (20)$$

$$+ n^{-1}\sum_{i=1}^{n}\int_0^t R_j^{(0)}(s;\beta_j,\widehat{W})^{-1}W_{ij}(s)n^{-\frac{1}{2}}\sum_{\ell=1}^{n}J_{i\ell j}^C(s)dN_{ij}(s) + o_p(1). \quad (21)$$

Switching the order of summation, we can re-write

$$(20) \quad = \quad \widehat{B}_j^T(t)\Omega_j^C(\theta_j)^{-1}n^{-\frac{1}{2}}\sum_{i=1}^n U_{ij}^C(\theta_j),$$

where we define

$$\begin{aligned}\widehat{B}_j(t) &= n^{-1}\sum_{i=1}^n \int_0^t \frac{W_{ij}(s)D_{ij}(s)}{R_j^{(0)}(s;\beta_j,\widehat{W})}dN_{ij}(s)\\ &\xrightarrow{p} B_j(t) \equiv E\left[\int_0^t \frac{W_{ij}(s)D_{ij}(s)}{r_j^{(0)}(s;\beta_j,W)}dN_{ij}(s)\right].\end{aligned}$$

Switching the orders of summation and integration, term (21) can be written as

$$\begin{aligned}(21) \quad &= \quad n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^t \widehat{K}_j(s,t)r_{Cj}^{(0)}(s;\theta_j)^{-1}dM_{ij}^C(s) + o_p(1)\\ &= \quad n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^t K_j(s,t)r_{Cj}^{(0)}(s;\theta_j)^{-1}dM_{ij}^C(s) + o_p(1),\end{aligned}$$

where we define

$$\begin{aligned}\widehat{K}_j(t_1,t_2) &= n^{-1}\sum_{i=1}^n e^{\theta_j^T X_i(t_1)}Y_{ij}(t_1)\int_{t_1}^{t_2}\frac{W_{ij}(s)}{R_j^{(0)}(s;\beta_j,\widehat{W})}dN_{ij}(s)\\ K_j(t_1,t_2) &= E\Big\{e^{\theta_j^T X_i(t_1)}Y_{ij}(t_1)\int_{t_1}^{t_2}\frac{W_{ij}(s)}{r_j^{(0)}(s;\beta_j,W)}dN_{ij}(s)\Big\}.\end{aligned}$$

Combining (20) and (21),

$$(16) \quad = \quad B_j^T(t)\Omega_j^C(\theta_j)^{-1}n^{-\frac{1}{2}}\sum_{i=1}^n U_{ij}^C(\theta_j) + n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^t K_j(s,t)r_{Cj}^{(0)}(s;\theta_j)^{-1}dM_{ij}^C(s) + o_p(1).$$

We can re-write expression (17) as follows,

$$(17) \quad = \quad n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^t W_{ij}(s)\{R_j^{(0)}(s;\beta_j,\widehat{W})^{-1} - R_j^{(0)}(s;\beta_j,W)^{-1}\}dN_{ij}(s).$$

Using the Functional Delta Method,

$$
\begin{aligned}
& n^{\frac{1}{2}}\{R_j^{(0)}(s;\beta_j,\widehat{W})^{-1} - R_j^{(0)}(s;\beta_j,W)^{-1}\} \\
= \quad & -R_j^{(0)}(s;\beta_j,W)^{-2} n^{-1}\sum_{\ell=1}^{n} e^{\beta_j^T Z_l} Y_{\ell j}(s) n^{\frac{1}{2}}\{\widehat{W}_{\ell j}(s) - W_{\ell j}(s)\} + o_p(1) \\
= \quad & -R_j^{(0)}(s;\beta_j,W)^{-2} n^{-1}\sum_{\ell=1}^{n} e^{\beta_j^T Z_l} Y_{\ell j}(s) W_{\ell j}(s) n^{-\frac{1}{2}}\sum_{k=1}^{n}\{D_{\ell j}^T(s)\Omega_j^C(\theta_j)^{-1}U_{kj}^C(\theta_j) + J_{\ell kj}^C(s)\} + o_p(1) \\
= \quad & R_j^{(0)}(s;\beta_j,W)^{-2}\widehat{F}_j^T(s)\Omega_j^C(\theta_j)^{-1} n^{-\frac{1}{2}}\sum_{\ell=1}^{n} U_{\ell j}^C(\theta_j) \\
& + R_j^{(0)}(s;\beta_j,W)^{-2} n^{-\frac{1}{2}}\sum_{\ell=1}^{n}\int_0^s \widehat{Q}_j(u,s) r_{Cj}^{(0)}(u;\theta_j)^{-1} dM_{\ell j}^C(u) + o_p(1) \\
= \quad & r_j^{(0)}(s;\beta_j,W)^{-2} F_j^T(s)\Omega_j^C(\theta_j)^{-1} n^{-\frac{1}{2}}\sum_{\ell=1}^{n} U_{\ell j}^C(\theta_j) \\
& + r_j^{(0)}(s;\beta_j,W)^{-2} n^{-\frac{1}{2}}\sum_{\ell=1}^{n}\int_0^s Q_j(u,s) r_{Cj}^{(0)}(u;\theta_j)^{-1} dM_{\ell j}^C(u) + o_p(1),
\end{aligned}
$$

where we define the following quantities:

$$
\begin{aligned}
\widehat{F}_j(s) &= -n^{-1}\sum_{i=1}^{n} e^{\beta_j^T Z_i} Y_{ij}(s) W_{ij}(s) D_{ij}(s), \\
\widehat{Q}_j(t_1,t_2) &= -n^{-1}\sum_{i=1}^{n} e^{\theta_j^T X_i(t_1)} e^{\beta_j^T Z_i} Y_{ij}(t_2) W_{ij}(t_2),
\end{aligned}
$$

with their respective limiting values given by

$$
\begin{aligned}
F_j(s) &= -E\Big\{e^{\beta_j^T Z_i} Y_{ij}(s) W_{ij}(s) D_{ij}(s)\Big\}, \\
Q_j(t_1,t_2) &= -E\Big\{e^{\theta_j^T X_i(t_1)} e^{\beta_j^T Z_i} Y_{ij}(t_2) W_{ij}(t_2)\Big\}.
\end{aligned}
$$

Substituting (22) into (17), we obtain

$$
\begin{aligned}
(17) \quad = \quad & n^{-1}\sum_{i=1}^{n}\int_0^t W_{ij}(s) r_j^{(0)}(s,\beta_j,W)^{-2} F_j^T(s)\Omega_j^C(\theta_j)^{-1} n^{-\frac{1}{2}}\sum_{\ell=1}^{n} U_{\ell j}^C(\theta_j) dN_{ij}(s) \\
& + n^{-1}\sum_{i=1}^{n}\int_0^t W_{ij}(s) r_j^{(0)}(s;\beta_j,W)^{-2} n^{-\frac{1}{2}}\sum_{\ell=1}^{n}\int_0^s Q_j(u,s) r_{Cj}^{(0)}(u;\theta_j)^{-1} dM_{\ell j}^C(u) dN_{ij}(s).
\end{aligned}
$$

Switching the order of summation for the first term, and the orders of summation and integration in the second term, we then have

$$\begin{aligned}
(17) &= \widehat{E}_j^T(t)\Omega_j^C(\theta_j)^{-1}n^{-\frac{1}{2}}\sum_{\ell=1}^n U_{\ell j}^C(\theta_j) + n^{-\frac{1}{2}}\sum_{\ell=1}^n \int_0^t \widehat{P}_j(u,t) r_{Cj}^{(0)}(u;\theta_j)^{-1} dM_{\ell j}^C(u) \\
&= E_j^T(t)\Omega_j^C(\theta_j)^{-1}n^{-\frac{1}{2}}\sum_{\ell=1}^n U_{\ell j}^C(\theta_j) + n^{-\frac{1}{2}}\sum_{\ell=1}^n \int_0^t P_j(u,t) r_{Cj}^{(0)}(u;\theta_j)^{-1} dM_{\ell j}^C(u) + o_p(1),
\end{aligned}$$

where we define

$$\begin{aligned}
\widehat{E}_j(t) &= n^{-1}\sum_{i=1}^n \int_0^t \frac{W_{ij}(s)F_j(s)}{r_j^{(0)}(s;\beta_j,W)^2} dN_{ij}(s) \\
\widehat{P}_j(t_1,t_2) &= n^{-1}\sum_{i=1}^n \int_{t_1}^{t_2} \frac{W_{ij}(s)Q_j(t_1,s)}{r_j^{(0)}(s;\beta_j,W)^2} dN_{ij}(s) \\
E_j(t) &= E\Big\{\int_0^t \frac{W_{ij}(s)F_j(s)}{r_j^{(0)}(s;\beta_j,W)^2} dN_{ij}(s)\Big\} \\
P_j(t_1,t_2) &= E\Big\{\int_{t_1}^{t_2} \frac{W_{ij}(s)Q_j(t_1,s)}{r_j^{(0)}(s;\beta_j,W)^2} dN_{ij}(s)\Big\}.
\end{aligned}$$

Finally, we can express the last term from the original decomposition as

$$\begin{aligned}
(18) &= n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^t \frac{W_{ij}(s)}{R_j^{(0)}(s;\beta_j,W)} dM_{ij}(s) \\
&= n^{-\frac{1}{2}}\sum_{i=1}^n \frac{W_{ij}(s)}{r_j^{(0)}(s;\beta_j,W)} dM_{ij}(s) = o_p(1).
\end{aligned}$$

Combining the re-expressions of (15), (16), (17) and (18), we have

$$n^{\frac{1}{2}}\{\widehat{\Lambda}_{0j}(t) - \Lambda_{0j}(t)\} = n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^t d\Phi_{ij}(u) = n^{-\frac{1}{2}}\sum_{i=1}^n \Phi_{ij}(t),$$

where we set

$$\begin{aligned}
\Phi_{ij}(t) &= h_j^T(t)\Omega_j(\beta_j)^{-1}U_{ij}(\beta_j) \\
&\quad + \{B_j(t) + E_j(t)\}^T\Omega_j^C(\theta_j)^{-1}U_{ij}^C(\theta_j) \\
&\quad + \int_0^t \frac{\{K_j(s,t) + P_j(s,t)\}}{r_{Cj}^{(0)}(s;\theta_j)} dM_{ij}^C(s) \\
&\quad + \int_0^t \frac{W_{ij}(s)}{r_j^{(0)}(s;\beta_j,W)} dM_{ij}(s) \\
&\equiv \int_0^t d\Phi_{ij}(u).
\end{aligned}$$

### 4.7 $n^{\frac{1}{2}}\{\widehat{\Lambda}_{ij}(t) - \Lambda_{ij}(t)\}$

We start with the decomposition,

$$n^{\frac{1}{2}}\{\widehat{\Lambda}_{ij}(t) - \Lambda_{ij}(t)\} = \widehat{\Lambda}_{0j}(t)n^{\frac{1}{2}}\{e^{\widehat{\beta}_j^T Z_i} - e^{\beta_j^T Z_i}\} \quad (22)$$

$$+ e^{\beta_j^T Z_i} n^{\frac{1}{2}}\{\widehat{\Lambda}_{0j}(t) - \Lambda_{0j}(t)\}, \quad (23)$$

which is similar in spirit to that employed for Result 4.3; simpler in the sense that covariates are time-constant, but more complicated in the sense that a weight function is involved. Reorganizing the first term of the decomposition,

$$(22) = \Lambda_{0j}(t)e^{\beta_j^T Z_i} Z_i^T n^{\frac{1}{2}}(\widehat{\beta}_j - \beta_j) + o_p(1).$$

Substituting Result 4.5 into (22) and Result 4.6 into (23), then combining yields

$$n^{\frac{1}{2}}\{\widehat{\Lambda}_{ij}(t) - \Lambda_{ij}(t)\} = \Lambda_{ij}(t)Z_i^T \Omega_j^{-1}(\beta_j)n^{-\frac{1}{2}}\sum_{\ell=1}^{n} U_{\ell j}(\beta_j) + e^{\beta_j^T Z_i} n^{-\frac{1}{2}}\sum_{\ell=1}^{n} \Phi_{\ell j}(t) + o_p(1).$$

### 4.8 $n^{\frac{1}{2}}\{\widehat{S}_{ij}(t) - S_{ij}(t)\}$

Using the Functional Delta Method,

$$n^{\frac{1}{2}}\{\widehat{S}_{ij}(t) - S_{ij}(t)\} = -S_{ij}(t)n^{\frac{1}{2}}\{\widehat{\Lambda}_{ij}(t) - \Lambda_{ij}(t)\} + o_p(1).$$

### 4.9 $n^{\frac{1}{2}}(\widehat{\mu}_{ij} - \mu_{ij})$

Recall that $\mu_{ij}(t) = \int_0^t S_{ij}(s)ds$, with $\mu_{ij} \equiv \mu_{ij}(L)$. By continuity,

$$n^{\frac{1}{2}}(\widehat{\mu}_{ij} - \mu_{ij}) = n^{\frac{1}{2}}\int_0^L \{\widehat{S}_{ij}(t) - S_{ij}(t)\}dt.$$

Using Result 4.8,

$$n^{\frac{1}{2}}(\widehat{\mu}_{ij} - \mu_{ij}) = -\int_0^L S_{ij}(t)n^{\frac{1}{2}}\{\widehat{\Lambda}_{ij}(t) - \Lambda_{ij}(t)\}dt + o_p(1).$$

Then, incorporating Result 4.7, we obtain

$$n^{\frac{1}{2}}(\widehat{\mu}_{ij} - \mu_{ij}) = -\int_0^L S_{ij}(t)\Lambda_{ij}(t)Z_i^T dt\ \Omega_j^{-1}(\beta_j)\ n^{-\frac{1}{2}} \sum_{\ell=1}^n U_{\ell j}(\beta_j) \tag{24}$$

$$-\int_0^L S_{ij}(t)e^{\beta_j^T Z_i} n^{-\frac{1}{2}} \sum_{\ell=1}^n \int_0^t d\Phi_{\ell j}(s)\ dt. \tag{25}$$

For the second term, switching the order of integration,

$$\begin{aligned}(25) &= -n^{-\frac{1}{2}} \sum_{\ell=1}^n \int_0^L \int_s^L S_{ij}(t)e^{\beta_j^T Z_i} dt\ d\Phi_{\ell j}(s) \\ &= -n^{-\frac{1}{2}} \sum_{\ell=1}^n \int_0^L e^{\beta_j^T Z_i}\{\mu_{ij}(L) - \mu_{ij}(t)\} d\Phi_{\ell j}(t).\end{aligned}$$

Combining the re-expressions of (24) and (25),

$$\begin{aligned}n^{\frac{1}{2}}\{\widehat{\mu}_{ij} - \mu_{ij}\} &= -\int_0^L S_{ij}(t)\Lambda_{ij}(t)dt\ Z_i^T \Omega_j^{-1}(\beta_j) n^{-\frac{1}{2}} \sum_{\ell=1}^n U_{\ell j}(\beta_j) \\ &\quad - e^{\beta_j^T Z_i} n^{-\frac{1}{2}} \sum_{\ell=1}^n \int_0^L \{\mu_{ij}(L) - \mu_{ij}(t)\} d\Phi_{\ell j}(t).\end{aligned}$$

## 4.10 $n^{\frac{1}{2}}(\widehat{\mu}_j - \mu_j)$

Recalling that $\widehat{\mu}_j = n^{-1}\sum_{i=1}^n \widehat{\mu}_{ij}$ and $\mu_j = E_{Z_i}\{\mu_{ij}\}$, we can write

$$n^{\frac{1}{2}}(\widehat{\mu}_j - \mu_j) = n^{-\frac{1}{2}} \sum_{i=1}^n (\widehat{\mu}_{ij} - \mu_{ij}) \tag{26}$$

$$+ n^{-\frac{1}{2}} \sum_{i=1}^n (\mu_{ij} - \mu_j). \tag{27}$$

Using Result 4.10, we can re-write the first line as

$$\begin{aligned}(26) &= -n^{-1} \sum_{i=1}^n \int_0^L S_{ij}(t)\Lambda_{ij}(t)dt\ Z_i^T \Omega_j^{-1}(\beta_j) n^{-\frac{1}{2}} \sum_{\ell=1}^n U_{\ell j}(\beta_j) \\ &\quad - n^{-1} \sum_{i=1}^n e^{\beta_j^T Z_i} n^{-\frac{1}{2}} \sum_{\ell=1}^n \int_0^L \{\mu_{ij}(L) - \mu_{ij}(t)\} d\Phi_{\ell j}(t).\end{aligned}$$

Switching the order of summation in the second line yields

$$(26) = -n^{-1} \sum_{i=1}^n \int_0^L S_{ij}(t)\Lambda_{ij}(t)dt\ Z_i^T \Omega_j^{-1}(\beta_j) n^{-\frac{1}{2}} \sum_{\ell=1}^n U_{\ell j}(\beta_j) \tag{28}$$

$$- n^{-\frac{1}{2}} \sum_{\ell=1}^n \int_0^L n^{-1} \sum_{i=1}^n e^{\beta_j^T Z_i}\{\mu_{ij}(L) - \mu_{ij}(t)\} d\Phi_{\ell j}(t). \tag{29}$$

Now, regarding term (28), integration by parts gives

$$\begin{aligned}\int_0^L S_{ij}(t)\Lambda_{ij}(t)dt &= \int_0^L S_{ij}(t)\int_0^t d\Lambda_{ij}(s)dt \\ &= \int_0^L \int_s^L S_{ij}(t)dtd\Lambda_{ij}(s) \\ &= \int_0^L \{\mu_{ij}(L)-\mu_{ij}(s)\}d\Lambda_{ij}(s).\end{aligned}$$

Incorporating this simplification into (26), we have

$$\begin{aligned}(26) &= -n^{-1}\sum_{i=1}^n \int_0^L \{\mu_{ij}(L)-\mu_{ij}(t)\}d\Lambda_{ij}(t)Z_i^T\Omega_j^{-1}(\beta_j)n^{-\frac{1}{2}}\sum_{\ell=1}^n U_{\ell j}(\beta_j) \\ &\quad - n^{-\frac{1}{2}}\sum_{\ell=1}^n \int_0^L n^{-1}\sum_{i=1}^n e^{\beta_j^T Z_i}\{\mu_{ij}(L)-\mu_{ij}(t)\}d\Phi_{\ell j}(t).\end{aligned}$$

Through the WLLN, continuity and Slutsky's Theorem,

$$\begin{aligned}(26) &= -E\Big[Z_i^T\int_0^L \{\mu_{ij}(L)-\mu_{ij}(t)\}d\Lambda_{ij}(t)\Big]\Omega_j^{-1}(\beta_j)n^{-\frac{1}{2}}\sum_{i=1}^n U_{ij}(\beta_j) \\ &\quad - n^{-\frac{1}{2}}\sum_{i=1}^n \int_0^L E\Big[e^{\beta_j^T Z_i}\{\mu_{ij}(L)-\mu_{ij}(t)\}\Big]d\Phi_{ij}(t) + o_p(1).\end{aligned}$$

Combining the re-expression of (26) with (27) gives

$$n^{\frac{1}{2}}\{\widehat{\mu}_j-\mu_j\} = n^{-\frac{1}{2}}\sum_{i=1}^n \phi_{ij} + o_p(1),$$

where we define

$$\begin{aligned}\phi_{ij} &= -E\Big[Z_i^T\int_0^L \{\mu_{ij}(L)-\mu_{ij}(t)\}d\Lambda_{ij}(t)\Big]\Omega_j^{-1}(\beta_j)U_{ij}(\beta_j) \\ &\quad - \int_0^L E\Big[e^{\beta_j^T Z_i}\{\mu_{ij}(L)-\mu_{ij}(t)\}\Big]d\Phi_{ij}(t) + (\mu_{ij}-\mu_j).\end{aligned}$$

## 4.11 $n^{\frac{1}{2}}(\widehat{\delta}-\delta)$

Applying Result 4.10 to $j=1$ and $j=0$, then subtracting,

$$n^{\frac{1}{2}}(\widehat{\delta}-\delta) = n^{-\frac{1}{2}}\sum_{i=1}^n(\phi_{i1}-\phi_{i0}) + o_p(1) = n^{-\frac{1}{2}}\sum_{i=1}^n \phi(O_i) + o_p(1), \qquad (30)$$

where $\phi(O_i)=(\phi_{i1}-\phi_{i0})$ is referred to as the influence function of $\widehat{\delta}$.

# 5 Estimating the Variance of $\widehat{\delta}$

The asymptotic variance of $n^{1/2}(\widehat{\delta}-\delta)$ is equal to $E\{\phi^2(O_i)\}$, which suggests the empirical variance estimator, i.e., $n^{-1}\sum_{i=1}^n \widehat{\phi}^2(O_i)$, where $\widehat{\phi}(O_i)$ is calculated by substituting the sample analogs for the terms in (30). However, the expression for $\phi(O_i)$ in (30) is very complicated and it would be difficult to evaluate it numerically. Moreover, if one uses other weight function instead of the simple inverse weight (e.g., the stabilized weight function)), the expression is going to be even more involved and difficult to implement. Therefore, a practical strategy in estimating the variance of $\widehat{\delta}$ would be to treat the weight function as fixed as opposed to estimated and use the corresponding influence function to estimate the variance as described above. This strategy has been used in similar problems, for example, in Hernán et al. (2000 and 2001) and Lu and Tsiatis (2008).

We list the main results involved in deriving the influence function of $\widehat{\delta}$ when treating the weight function as fixed. Results in this section hold for the general weight function with $W_{ij}(t)=\exp\{\Lambda_{ij}^C(t)\}\kappa(t;Z_i,A_i)$.

## 5.1 $n^{\frac{1}{2}}(\widehat{\beta}_j-\beta_j)$

In the development of Results 4.5, if $\widehat{W}_{ij}$ is replace by $W_{ij}$ (implying that $\widehat{\theta}_j$ and $\widehat{\Lambda}_{0j}^C$ are replaced by $\theta_j$ and $\Lambda_{0j}^C$, respectively), then we arrive at

$$n^{-\frac{1}{2}}(\widehat{\beta}_j-\beta_j) \;=\; \Omega_j(\beta_j)^{-1}n^{-\frac{1}{2}}\sum_{i=1}^n U_{ij}^{\dagger}(\beta_j)+o_p(1),$$

where we introduce

$$U_{ij}^{\dagger}(\beta) \;\equiv\; \int_0^{\tau}\{Z_i-\overline{z}_j(t;\beta)\}W_{ij}(t)dM_{ij}(t).$$

## 5.2 $n^{\frac{1}{2}}\{\widehat{\Lambda}_{0j}(t) - \Lambda_{0j}(t)\}$

Using a decomposition analogous to that for Result 4.6, then incorporating Result 4.1 gives

$$n^{\frac{1}{2}}\{\widehat{\Lambda}_{0j}(t) - \Lambda_{0j}(t)\} = n^{-\frac{1}{2}}\sum_{i=1}^{n}\Phi_{ij}^{\dagger}(t) + o_p(1),$$

where we introduce

$$\Phi_{ij}^{\dagger}(t) = h_j^T(t)\Omega_j(\beta_j)^{-1}U_{ij}^{\dagger}(\beta_j) + \int_0^t \frac{W_{ij}(s)}{r_j^{(0)}(s;\beta_j, W)}dM_{ij}(s) + o_p(1).$$

## 5.3 $n^{\frac{1}{2}}\{\widehat{\Lambda}_{ij}(t) - \Lambda_{ij}(t)\}$

Through a decomposition which parallels that in Result 4.7, then using Results 5.1 and 5.2, we obtain

$$n^{\frac{1}{2}}\{\widehat{\Lambda}_{ij}(t) - \Lambda_{ij}(t)\} = \Lambda_{ij}(t)Z_i^T\Omega_{ij}(\beta_j)^{-1}n^{-\frac{1}{2}}\sum_{\ell=1}^{n}U_{\ell j}^{\dagger}(\beta_j) + e^{\beta_j^T Z_i}\sum_{\ell=1}^{n}\Phi_{\ell j}^{\dagger}(t) + o_p(1).$$

## 5.4 $n^{\frac{1}{2}}\{\widehat{S}_{ij}(t) - S_{ij}(t)\}$

Applying the Functional Delta Method to Result 5.3,

$$n^{\frac{1}{2}}\{\widehat{S}_{ij}(t) - S_{ij}(t)\} = -S_{ij}(t)n^{\frac{1}{2}}\{\widehat{\Lambda}_{ij}(t) - \Lambda_{ij}(t)\} + o_p(1).$$

## 5.5 $n^{\frac{1}{2}}(\widehat{\mu}_{ij} - \mu_{ij})$

Using Result 5.4, then integrating,

$$\begin{aligned} n^{\frac{1}{2}}(\widehat{\mu}_{ij} - \mu_{ij}) &= -\int_0^L S_{ij}(t)\Lambda_{ij}(t)dt\ Z_i^T\Omega_j(\beta_j)^{-1}n^{-\frac{1}{2}}\sum_{\ell=1}^{n}U_{\ell j}^{\dagger}(\beta_j) \\ &\quad - e^{\beta_j^T Z_i}n^{-\frac{1}{2}}\sum_{\ell=1}^{n}\int_0^L\{\mu_{ij}(L) - \mu_{ij}(t)\}d\Phi_{\ell j}^{\dagger}(t) + o_p(1), \end{aligned}$$

upon reorganizing along the lines of Result 4.9.

## 5.6 $n^{\frac{1}{2}}(\widehat{\mu}_j - \mu_j)$

Analogous to the derivation of Result 4.10, we obtain

$$n^{\frac{1}{2}}(\widehat{\mu}_{ij} - \mu_{ij}) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \phi_{ij}^{\dagger} + o_p(1),$$

where we define

$$\begin{aligned} \phi_{ij}^{\dagger} &= -E\Big[Z_i^T \int_0^L \{\mu_{ij}(L) - \mu_{ij}(t)\} d\Lambda_{ij}(t)\Big] \Omega_j(\beta_j)^{-1} U_{ij}^{\dagger}(\beta_j) \\ &\quad - \int_0^L E\Big[e^{\beta_j^T Z_i}\{\mu_{ij}(L) - \mu_{ij}(t)\}\Big] d\Phi_{ij}^{\dagger}(t) + (\mu_{ij} - \mu_j). \end{aligned}$$

## 5.7 $n^{\frac{1}{2}}(\widehat{\delta} - \delta)$

Finally, applying Result 5.6 twice ($j$=0, $j$=1) then subtracting, we obtain $n^{\frac{1}{2}}(\widehat{\delta} - \delta) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \{\phi_{i1}^{\dagger} - \phi_{i0}^{\dagger}\} + o_p(1)$, such that the asymptotic variance of is approximated by $n^{-1} \sum_{i=1}^{n} (\widehat{\phi}^{\dagger}{}_{i1} - \widehat{\phi}^{\dagger}{}_{i0})^2$, where $\widehat{\phi}^{\dagger}{}_{ij}$ is calculated as by substituting the sample averages in place of limiting values in $\phi_{ij}^{\dagger}$.

### Additional References


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